Studies in Logic, Vol. 19, No. 3 (2026): 92–105
PII: 1674-3202(2026)-03-0092-14

Intuitionistic Epistemic Logic **IEL** with Justification*

Youan Su

Abstract. In Artemov and Protopopescu (2016), an intuitionistic epistemic logic **IEL** is introduced, in which the axiom $A \to KA$ is accepted whereas the axiom $KA \to A$ is rejected. This paper explores the relation between knowledge and justification on the basis of **IEL**. The expressions for the knowledge of a proposition "A" denoted as "KA" and the justification "t" of a proposition "A" denoted as "$t{:}A$" are included. The proposed systems contain the axioms of **IEL** and the axioms of the logic of proof **LP** (or **JT4**). It also includes a connection axiom $t{:}A \to KA$. This paper gives Hilbert systems and Kripke-Fitting semantics for the newly introduced systems and shows that they are sound and complete for the intended semantics.

1 Introduction

In Artemov and Protopopescu ([3]), systems of intuitionistic epistemic logic **IEL** are introduced. Different from the well-known Hintikka-style classical epistemic logic, **IEL** provides the conception of intuitionistic knowledge based on the BHK interpretation, in which "KA" (A is known) is explained as "conclusive evidence of verification that A has a proof". As a consequence, a formula in the form of $A \to KA$ is accepted as an axiom (co-reflection axiom), but the T-axiom in the form of $KA \to A$ is no longer valid. The intuitionistic propositional system with the K-axiom $K(A \to B) \to (KA \to KB)$ and the above co-reflection axiom is named as **IEL**$^-$, which is designed to express the intuitionistic belief. Adding the intuitionistic reflection axiom in the form of $KA \to \neg\neg A$ into **IEL**$^-$, we can obtain the system **IEL** of intuitionistic knowledge. The proof-theoretic properties of **IEL** have been studied in [7, 16, 25], and a first-order expansion in [24].

Justification logic is introduced first in the name of the logic of proof **LP** in Artemov. ([4]) In **LP**, Artemov revived a project begun by Gödel to formalize the intuitionistic truth in the classical provability. He succeeded in developing an explicit proof-terms for provability in **S4**. After that, it is realized that proofs were only one kind of justification, **LP** is renamed as **JT4** and became a member of a big family. The family of justification logic has gained more and more attention.

In justification logic, a formula in the form of $t{:}A$ is read as "t is a justification for A". The K-axiom of modal logic is named as $Application$-axiom in justification

Received 2024-11-15

Youan Su Department of Philosophy, Liaoning University
su.youan@lnu.edu.cn

*I am grateful to the anonymous reviewers for their helpful comments.

logic, given as $s{:}(A \to B) \to (t{:}A \to s \cdot t{:}B)$. Similar to the modal logic, justification logic is a family containing, for example, **JD**, **JT**, **J4**, **JD4**, etc. ([5]) Many studies have been conducted on this topic, for example, self-referentiality of justification terms is studied in [6, 27, 28], and justification logic over relevant logic is studied in [21, 23].

As an extension of the studies in justification logic, the efforts to keep both epistemic modal operators and justification operators in one system have been made in Artemov and Nogina ([1]), Fitting ([9, 10]). This system is named as **LPS4**. By putting an axiom in the form of $t{:}A \to KA$, it provides a bridge to connect the explicit/implicit knowledge in one epistemic logic system.

This paper proposes bimodal **IEL** systems expanded with justification. The new introduced systems are named as **JIEL**$^-$ and **JIEL**. The modal axioms and rules from justification logic **JT4**, namely the K-axiom, T-axiom and 4-axiom and necessitation rule are added into **IEL** systems. All original axioms of **IEL** and **IEL**$^-$ remain. Furthermore, a connection axiomatic schema in the form of $t{:}A \to KA$ and a negation introspection schema $\neg(t{:}A) \to K\neg(t{:}A)$ are also valid in the new systems. The internalization property of the system is built. Kripke-Fitting semantics is proposed. Soundness and Completeness theorems are shown.

2 Hilbert Systems of JIEL

A justification term t is defined as follows:

$$t ::= c \,|v\, |t \cdot t\, |t + t\, |!t$$

where, c is called a justification constant and v is called a justification variable. Operator "$\cdot$ " is called "application", "$+$" is called "sum", "!" is called "positive introspection".

A formula A in the language $\mathcal{L}$ is defined as follows:

$$A ::= p \,|\bot\, |A \wedge A\, |A \vee A\, |A \to A\, |KA\, |t{:}A.$$

The negation of a formula A, $\neg A$ is defined as $A \to \bot$.

The systems **JIEL**$^-$ and **JIEL** are introduced in the following table.

We have the following two axioms in our new systems: negative introspection: $\neg(t{:}A) \to K\neg(t{:}A)$; connection axiom: $t{:}A \to KA$. These two axioms are provided for the convenience. The latter can be derived from T-axiom (it is called explicit reflexivity) in the form $t{:}A \to A$ and CR-axiom (co-reflection axiom) in the form of $A \to KA$. The Negation introspection is nothing but a special case of CR-axiom.

The notion of $t{:}F$ according to [2], can be explained as *"t is a justification of F", " t is accepted by agent as a justification of F", "t is a sufficient resource for F".*

Hilbert system **JIEL**$^-$	
($\wedge$-Ax)	$A_1 \wedge A_2 \to A_i \ \ (i = 1 \text{ or } i = 2)$
	$A \to (B \to (A \wedge B))$
($\vee$-Ax)	$A_i \to A_1 \vee A_2 \ \ (i = 1 \text{ or } i = 2)$
	$(A \to C) \to ((B \to C) \to (A \vee B \to C))$
($\to$-Ax)	$A \to (B \to A)$
	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
($\bot$-Ax)	$\bot \to A$
(K)	$K(A \to B) \to (KA \to KB)$
(CR)	$A \to KA$
(Application)	$s{:}(A \to B) \to (t{:}A \to s \cdot t{:}B)$
(Union)	$t_i{:}A \to (t_1 + t_2{:}A) \ \ (i = 1 \text{ or } i = 2)$
(Explicit reflexivity)	$t{:}A \to A$
(Proof checker)	$t{:}A \to !t{:}t{:}A$
(Explicit-implicit connection)	$t{:}A \to KA$
(Negation introspection)	$\neg(t{:}A) \to K\neg(t{:}A)$
(MP)	From A and $A \to B$, infer B.
(AN)	$\vdash c{:}A$ where A is an axiom and c is a constant
Hilbert system **JIEL**	
All the axioms and rules of **JIEL**$^-$ plus:	
(IR)	$KA \to \neg\neg A$

Artemov has mentioned, "Justification Logic provides a novel, evidence-based mechanism of truth-tracking which seems to be a key ingredient of the analysis of knowledge."([2]) By adding justification terms into **IEL**, we obtain a method to do truth-tracking in a deduction. For example, a formula $t{:}(KA \to KB) \to (s{:}KA \to s{\cdot}t{:}KB)$ states that "given a justification t of knowing A implies knowing B, combing a justification of knowledge of A we can derive that a justification $s \cdot t$ of knowledge of B." Justification term $s \cdot t$ can be seen as a track of the validity of a knowledge of B.

As Artemov has pointed out, we can have factive justifications or partial justifications. ([2]) A factive justification always entails truth, i.e., the axiomatic schema $t{:}A \to A$ is valid, while a partial justification does not. Here, we only consider a justification as a factive justification on **IEL**. The reason is that, according to Brouwer, an intuitionistic truth is seen as the existence of a proof. ([26, p. 4]) It is very natural to see a genuine intuitionistic justification as a proof. That is to say, a justification always entails a truth.

However, when we consider an intuitionistic knowledge, it cannot be factive. That is because, according to Artemov and Protopopescu ([3, p. 273]), an intuitionistic knowledge is the conclusive verification of the existence of a proof. This verification can be given by zero-knowledge protocols, the testimony of an authority,

classified sources, and existential generalization. ([3]) Based on this explanation, $KA \to A$ is not valid. Because "it is possible to have a provably verified proposition A without possessing a specific proof of A itself" ([3]), a verification does not always provide a proof. If we follow this notion, a factive justification can be thought as a source of conclusive verification. As a result, whenever given a justification t of a proposition A, we can derive a conclusive evidence that A has a proof. That is to say, $t{:}A \to KA$ is valid. On the other hand, $KA \to t{:}A$ is not valid, since a verification cannot yield a factive justification.

To conclude it, we can check the following expression. Give an arbitrary formula A, we can have the following chain in **JIEL**. The left formula always implies the right formula.

$$t{:}A \hookrightarrow A \hookrightarrow KA \hookrightarrow \neg\neg A$$

Here, $t{:}A$ stands for an intuitionistic truth A with justification t. A stands for an intuitionistic truth, namely a proof of A. KA stands for an intuitionistic knowledge, namely a conclusive verification of the existence of a proof of A. Finally, $\neg\neg A$, according to Artemov and Protopopescu ([3]), is a classical truth.

Definition 1. A *constant specification* $\mathcal{CS}$ is a set of formulas in the form of $c : A$, where A is an axiom and c is a justification constant. Let $\mathbf{JIEL}^-{}_{\mathcal{CS}}$ and $\mathbf{JIEL}_{\mathcal{CS}}$ be the fragments of the systems with only formulas from $\mathcal{CS}$ being allowed by the rule (AN). Especially, $\mathbf{JIEL}^-{}_{\varnothing}$ and $\mathbf{JIEL}_{\varnothing}$ are the systems without any constants specification.

A $\mathcal{CS}$ has the following property:

- axiomatically appropriate: For each axiom A, there is a constant c such that $c{:}A \in \mathcal{CS}$.
- schematic: If $c{:}A \in \mathcal{CS}$ and B is an instance of the same axiom schema with A then $c{:}B \in \mathcal{CS}$.

Proposition 1. $\mathbf{JIEL}^-{}_{\mathcal{CS}}, \mathbf{JIEL}_{\mathcal{CS}}$ *are closed under substitutions of justification terms for justification variables and formulas for propositional variables.*

Proof. Proof is standard. In the proof of substitutions, $\mathcal{CS}$ is required to be schematic in the case of (AN)-rule. □

Lemma 1 (Internalization)**.** *Let* $\Lambda \in \{\mathbf{JIEL}^-, \mathbf{JIEL}\}$*, if* $A_1, \cdots, A_k \vdash_{\Lambda_{\mathcal{CS}}} B$ *then there exists a term* $t(x_1, ...x_n)$ *such that*

$$x_1{:}A_1, \cdots x_n{:}A_n \vdash_{\Lambda_{\mathcal{CS}}} t(x_1, ...x_n){:}B.$$

Proof. We proceed by induction on the derivation. Suppose B is an instance of the axioms, then there is a constant c such that $c : B \in \mathcal{CS}$. Then, we can apply the (AN) rule. Suppose B is obtained from (AN), namely, $B \equiv t{:}C \in \mathcal{CS}$, we show

that there is a term e such that $\vdash_\Lambda e{:}(t{:}C)$. Since we always have $\vdash_\Lambda t{:}C \to !t{:}(t{:}C)$, we can derive that $\vdash_\Lambda !t{:}(t{:}C)$. In the case that B is in one of the $A_1, \cdots A_k$, the proof is trivial. When B is derived from MP rule, we need to apply induction hypothesis and axiom of application. □

By internalization theorem, we can see that whenever there is proof in the systems **JIEL**$^-$ and **JIEL**, we can always give it a justification term. In that case, we say this proof is being internalized. By an argument similar to the above proof we can have the following result.

Proposition 2 (Constructive Necessitation)**.** *Let* $\Lambda \in \{\mathbf{JIEL}^-, \mathbf{JIEL}\}$*, if* $\Lambda_{\mathcal{CS}} \vdash A$ *then there exists a variables-free term* t *such that* $\Lambda_{\mathcal{CS}} \vdash t{:}A$*.*

3 Kripke-Fitting Semantics

The semantics of justification logic is originated from the following two studies on **LP**:

- Mkrtychev gives a proof-theorem assignment function from justification terms to the set of formulas. ([18]) This function is required to be closed under the term operators. Associating with the standard propositional truth-assignment, the **JT4** (or, **LP**) is shown to be complete and sound.
- Fitting gives an evidence function on the setting of Kripke model. ([10]) The evidence function is required to satisfy the monotonicity condition on the Kripke binary relation.

Since one-state semantics from Mkrtyche is difficult to be used on the intuitionistic logic ([18]), this paper adopts the semantics from Fitting ([10]), especially from Fitting ([9]).

Definition 2. A *Kripke-Fitting model* is a tuple $M = (W, \leq, R, \epsilon, V)$, where:

- $(W, \leq)$ is a preorder i.e., $\leq$ satisfies reflexive and transitive conditions.
- R is a binary relation on W such that
  1. $R \subseteq \leq$, i.e., if wRv, then $w \leq v$, and
  2. $\leq \circ R \subseteq R$, i.e., if $w \leq v$ and vRu, then wRu.
- V is a mapping from propositional variables to subsets of W, satisfying:
  - if $w \leq v$ and $w \in V(p)$ then $v \in V(p)$.
- ϵ is an evidence function from states and terms to sets of formulas on $(W, \leq)$ satisfying:
  - **(App)** if $A \to B \in \epsilon(w, s)$ and $A \in \epsilon(w, t)$ then, $B \in \epsilon(w, s \cdot t)$
  - **(Mono)** if $w \leq v$ then $\epsilon(w, t) \subseteq \epsilon(v, t)$.

– **(P.C.)** if $A \in \epsilon(w,t)$ then $t:A \in \epsilon(w,!t)$.
– **(Sum)** $\epsilon(w,s) \cup \epsilon(w,t) \subseteq \epsilon(w,s+t)$.

Definition 3. The class of all Kripke-Fitting models is denoted as $\mathbb{M}_{\text{all}}$. Furthermore, $\mathbb{M}_{\text{ser}}$ is used to mean the class of all models in which R is serial; that is, for any $w \in W$, there is a $v \in W$ such that wRv.

Next, we move to the truth definition. Generally, in the study of intuitionistic modal logic or intuitionistic epistemic logic, we need to add a new binary relation into the frame, like in [15, 19, 22]. However, since our preorder in the frame is already reflexive and transitive, we can directly use the $\leq$ relation as an epistemic accessibility relation. We say that a justification, for example $t{:}A$, holds in the state w if and only if A is true in any $\leq$-accessible state from w, and A is assigned to a pair of state w and term t by an evidence function.

Definition 4. Given a closed formula A in $\mathcal{L}$, we inductively define the *satisfaction relation* $M, w \models A$ as follows:

$$
\begin{array}{lll}
M,w \models p & \text{iff} & w \in V(p). \\
M,w \not\models \bot. & & \\
M,w \models A \wedge B & \text{iff} & M,w \models A \text{ and } M,w \models B. \\
M,w \models A \vee B & \text{iff} & M,w \models A \text{ or } M,w \models B. \\
M,w \models A \to B & \text{iff} & \text{for all } v \in W, w \leq v \text{ and } M,v \models A \\
 & & \text{imply } M,v \models B. \\
M,w \models KA & \text{iff} & \text{for all } v \in W, wRv \text{ implies } M,v \models A. \\
M,w \models t{:}A & \text{iff} & \text{for all } v \in W, w \leq v \text{ implies } M,v \models A, \\
 & & \text{and } A \in \epsilon(w,t).
\end{array}
$$

Definition 5. For a set $\Gamma \cup \{A\}$ of formulas and a class $\mathbb{M}$ of models, $\Gamma \models_{\mathbb{M}} A$ means that for all models $M \in \mathbb{M}$ and all states w in M, for every formula $B \in \Gamma$, $M,w \models B$ implies $M,w \models A$. When Γ is an emptyset, we simply write $\models_{\mathbb{M}} A$ to say that A is *valid* in $\mathbb{M}$.

Definition 6. Given a $\mathcal{CS}$, we say that a model M *is meeting with* $\mathcal{CS}$, noted as $M^{\mathcal{CS}}$ if for each $c : A \in \mathcal{CS}$, given an arbitrary $w \in W$, we have $A \in \epsilon(w,c)$. In the same line, we say a class $\mathbb{M}$ of models is meeting with a $\mathcal{CS}$, noted as $\mathbb{M}^{\mathcal{CS}}$ if for each model $M \in \mathbb{M}$, M is meeting with $\mathcal{CS}$.

Proposition 3 (Persistency)**.** *For any model M, for any $w, v \in W$, if $w \leq v$ then $M,w \models A$ implies $M,v \models A$.*

Proof. We proceed by induction on formula A. Propositional cases are standard. In the case where A is in the form of KB, we need to apply $\leq \circ R \subseteq R$ condition.

In the case where A is in the form of $t{:}B$, we assume $w \leq v$ and $M, w \models t{:}B$. We will show that $M, v \models t{:}B$. Fix an arbitrary u such that $v \leq u$. We can derive that $w \leq u$ from the transitivity of $\leq$ in W. Then $M, w \models t{:}B$ gives $M, u \models B$. Furthermore, $M, w \models t{:}B$ gives $B \in \epsilon(w, t)$. Combining with the **(Mono)** condition and $w \leq v$, we can derive that $B \in \epsilon(v, t)$. Then we can have that $M, v \models t{:}B$. $\square$

Next, based on an argument about the connection between Kripke semantics and BHK-interpretation in [24], this part explains the notion of justification terms in our semantics.

As was explained by van Dalen, the $\leq$-relation in semantics represents the course of time, along which the idealized mathematician (this notion can be attributed to Brouwer) extends his or her knowledge . ([8, p. 164]) That is to say, the set of proved propositions held by the idealized mathematician always increases toward the future, and never decrease.

According to [24], the idealized mathematician can have not only proved propositions but also verified propositions at an arbitrary state. Let $\mathrm{Proved}_{\mathrm{w}}$ denote the set of proved propositions in the state w, $\mathrm{Verified}_w$ denote the set of verified formulas in w. Then, the axiom $A \to KA$ states that a proof always implies a verification, that is, $\mathrm{Proved}_w \subseteq \mathrm{Verified}_w$ for any w. From this approach, we can interpret the binary relations $\leq$ and R as:

$$\begin{array}{rcl} w \leq v & \text{iff} & \mathrm{Proved}_w \subseteq \mathrm{Proved}_v \\ wRv & \text{iff} & \mathrm{Verified}_w \subseteq \mathrm{Proved}_v \end{array}$$

Based on this interpretation, we can justify the conditions $R \subseteq \leq$ and $\leq \circ R \subseteq R$. ([24, p. 623])

Artemov and Protopopescu ([3, p. 280]), state that if a $\leq$-successor of a stage s represents the "principally (logically) possible" state from s, then an R-successor can be thought of as a "possible" state of verification. Combined with the above explanation, the possibility of the verification in the form of R-successor can be naturally thought as a kind of "promise" that the verified formulas are all proved in that future state. ([24, p. 623])

Now, we are ready to put justification terms into this explanation. Artemov suggests that "Evidence operators $t : F$ model real knowledge of the agent which provides a justification that F is true in all situations...An assertion $t : F$ if true in a world holds in all worlds of a model, the present, future, or past ones." ([1]) However, in the setting of intuitionistic logic, as we mentioned above, an idealized mathematician is said to increase his proved contents along the flow of time. In this case, we only need to consider the future, which is reflected by the $\leq$ relation.

Then, the idealized mathematician at a moment, holds not only proved propositions and verified propositions but also justification terms. The set of these terms is also increasing in the course of time (according to **(Mono)**) and is closed under the

operation **(App)**, **(P.C.)** and **(Sum)**. Thus, it describes the closed theory-like information that becomes the explicit resources supporting some of the proved propositions. However, not all proved propositions have some terms as resources. When this is compared to the set of verified propositions, as we observe above, all verified propositions must be proved in some future. However, not all verified propositions can be justified in the future. That is to say, every justified proposition is verified, however, not every verified proposition can be justified.

Theorem 4 (Soundness)**.** *Let A be a formula in $\mathcal{L}$, given a $\mathcal{CS}$,*

1 if $\mathbf{JIEL}^{-}{}_{\mathcal{CS}} \vdash A$ *then* $\mathbb{M}^{\mathcal{CS}}_{\text{all}} \models A$
2 if $\mathbf{JIEL}_{\mathcal{CS}} \vdash A$ *then* $\mathbb{M}^{\mathcal{CS}}_{\text{ser}} \models A$

Proof. We proceed by induction on the length of the proof. The validity of the propositional axioms and rules can be shown in a standard way. To prove the validity of the K axiom, we need to apply reflexivity of $\leq$ and persistency proved in Proposition 3. The validity of the (CR) axiom can be given by $R \subseteq \leq$ condition and persistency proved in Proposition 3. The validity of the (IR) axiom can be given by seriality of R.

The validiy of the (Application) axiom can be shown as follows. Fix any $w, u \in W$, suppose $w \leq u$ and $M, u \models s{:}(A \to B)$, we show $M, u \models t{:}A \to s \cdot t{:}B$. Fix any $v \in W$, suppose $u \leq v$ and $M, v \models t{:}A$, we show $M, v \models s \cdot t{:}B$. Fix any $x \in W$, suppose $v \leq x$, we aim to that $M, x \models B$ and $B \in \epsilon(v, s \cdot t)$. From the persistenty by proposition 3, we can derive that $M, v \models s{:}(A \to B)$. Since $v \leq x$ and $M, v \models t{:}A$, we can see that $M, x \models B$. From definition, we can have that $A \to B \in \epsilon(v, s)$ and $A \in \epsilon(v, t)$. According to the **(App)** condition, we can derive $B \in \epsilon(v, s \cdot t)$.

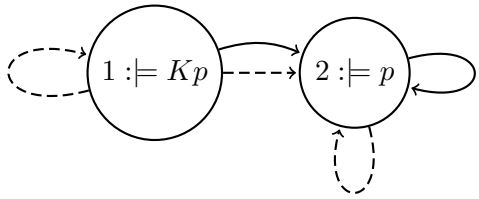
The validity of $(Union)$ can be easily obtained from **(Sum)** condition and transitivity of $\leq$. The validity of $(Explicitly\text{-}reflexivity)$ can be obtained from the reflexivity of $\leq$. To see the validity of $(Proof\ checker)$, we need the transitivity of $\leq$, persistency in proposition 3 and **(P.C)**. In the case of $(Explicit\text{-}implicit\ connection)$ and $(Negation\ introspection)$, as we have mentioned their validity are covered in the cases of (CR) and $(Explicit\ \ reflexivity)$.

The validity of (AN) can be seen from the fact that we are handling the class of models meeting with the $\mathcal{CS}$. □

Then let us see an example of the invalidity. As is shown in Artemov and Protopopescu, the usual T-axiom in the form of $KA \to A$ is not derivable. ([3]) This to say, a conclusive evidence of the existence of a proof does not always give a proof. In the **IEL** with justifications, this amounts to suggest that an implicit knowledge does not yield an explicit knowledge. Next, we show that the converse explicit-implicit connection in the form of $KA \to t{:}A$ is not derivable.

Proposition 5. *Given a propositional variable p, given a $\mathcal{CS}$, $Kp \to t{:}p$ is not derivable in* **JIEL**$_{\mathcal{CS}}$.

Proof. We difine a model M meeting with $\mathcal{CS}$. Let $W = \{1, 2\}$, $\leq = \{(1, 2)\} \cup \{(x, x) | x \in W\}$, $\epsilon(1, t) = \epsilon(2, t) = \{p\}$, $R = \{(1, 2), (2, 2)\}$, $V(p) = \{2\}$, some irrelevant conditions concerning ϵ are omitted. In the following picture, we use dotted lines to represent $\leq$, lines to represent R.



To see that $M^{\mathcal{CS}}, 1 \nvDash Kp \to t{:}p$, we show that $M^{\mathcal{CS}}, 1 \models Kp$ and $M^{\mathcal{CS}}, 1 \nvDash t{:}p$. We can observe that $M^{\mathcal{CS}}, 1 \models Kp$, because 2 is the only state R-accessible from 1 and $M^{\mathcal{CS}}, 2 \models p$. Even though we have $p \in \epsilon(1, t)$, we cannot have $t{:}p$ satisfied in state 1. It is because, $1 \notin V(p)$ and 1 is $\leq$-accessible from 1. Then, we apply the soundness in Theorem 4. □

4 Completeness

This section provides a proof to show that the completeness of **JIEL**$^-$ and **JIEL**.

Definition 7. Given a $\mathcal{CS}$, let $\Lambda \in \{$**JIEL**$^-{}_{\mathcal{CS}}$, **JIEL**$_{\mathcal{CS}}\}$, we call a set Γ of formulas Λ-*consistent prime theory*,

- $\Gamma \nvdash_\Lambda \bot$;
- for any formula A, $\Gamma \vdash_\Lambda A$ implies $A \in \Gamma$;
- for any formulas A and B, $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$.

Lemma 2. *Given a $\mathcal{CS}$, let $\Lambda \in \{$***JIEL**$^-{}_{\mathcal{CS}}$, **JIEL**$_{\mathcal{CS}}\}$*, given a set $\Gamma \cup \{A\}$ of formulas such that $\Gamma \nvdash_\Lambda A$, there exists a Λ-consistent prime theory Δ such that, $\Gamma \subseteq \Delta$ and $A \notin \Delta$.*

This can be established by a standard construction. (see **Lemma** 6.3.8 in [8])

Definition 8. Given a $\mathcal{CS}$, let $\Lambda \in \{$**JIEL**$^-{}_{\mathcal{CS}}$, **JIEL**$_{\mathcal{CS}}\}$, we define the *canonical model* M^Λ as follows:

- W is the set of all Λ-consistent prime theories.
- for any $\Gamma, \Delta \in W$, $\Gamma \leq \Delta$ iff $\Gamma \subseteq \Delta$.
- for any $\Gamma, \Delta \in W$, $\Gamma R \Delta$ iff $K^-(\Gamma) \subseteq \Delta$, where $K^-(\Gamma) = \{A | KA \in \Gamma\}$.
- for any $\Gamma \in W$, term t, $\epsilon(\Gamma, t) = \{A | t{:}A \in \Gamma\}$.

- for any $\Gamma \in W$, propositional variable p, $\Gamma \in V(p)$ iff $p \in \Gamma$.

Proposition 6. *Given a $\mathcal{CS}$. Let $\Lambda \in \{\mathbf{JIEL}^{-}{}_{\mathcal{CS}}, \mathbf{JIEL}_{\mathcal{CS}}\}$, the model M^{Λ} is a Kripke-Fitting model.*

Proof. When we consider $R \subseteq \leq$ condition, for any formula $A \in \Gamma$, from $\Gamma R \Delta$ and $A \to KA \in \Gamma$ we can derive that $A \in \Delta$. When we consider $\leq \circ R \subseteq R$ condition, fix any Γ, Δ such that $\Gamma \leq \circ R\ \Delta$, we show that $\Gamma R \Delta$. Take a Π such that $\Gamma \subseteq \Pi$ and $K^{-}(\Pi) \subseteq \Delta$.Take an $A \in K^{-}(\Gamma)$, then $KA \in \Gamma \subseteq \Pi$. From $K^{-}(\Pi) \subseteq \Delta$, we derive $A \in \Delta$. In the case of seriality of R in $M^{\mathbf{JIEL}_{\mathcal{CS}}}$, we need to apply the (IR)-axiom. Other frame properties can be easily checked.

In the case of ϵ function, we only show the following conditions.

- **(App)** Fix a state Γ, formulas A and B, terms t and s, suppose $A \to B \in \epsilon(\Gamma, s)$ and $A \in \epsilon(\Gamma, t)$. Then, according to the definition, $s{:}(A \to B) \in \Gamma$ and $t{:}A \in \Gamma$. Since $(s{:}A \to B) \to (t{:}A \to (s \cdot t){:}B) \in \Gamma$, we have $(s \cdot t){:}B \in \Gamma$. Then we have $B \in \epsilon(\Gamma, s \cdot t)$.
- **(Mono)** Fix a state Γ, Δ, a term t. Suppose $\Gamma \leq \Delta$. We show $\epsilon(\Gamma, t) \subseteq \epsilon(\Delta, t)$. Fix any formula A, suppose $A \in \epsilon(\Gamma, t)$. Then, we have $t{:}A \in \Gamma \subseteq \Delta$. That gives $t{:}A \in \Delta$, i.e., $A \in \epsilon(\Delta, t)$.
- **(P.C.)** Fix a state Γ, a formula A, a term t. Suppose $A \in \epsilon(\Gamma, t)$. We show $t{:}A \in \epsilon(\Gamma, !t)$. This can be easily given by the fact that $t{:}A \to !t{:}t{:}A \in \Gamma$.
- **(Sum)** Fix a state Γ, terms s and t. The condition $\epsilon(\Gamma, s) \cup \epsilon(\Gamma, t) \subseteq \epsilon(\Gamma, s+t)$ can be confirmed by observing that $s{:}A \to (s+t){:}A \in \Gamma$ and $t : A \to (s+t){:}A \in \Gamma$ for an arbitrary formula A.
- (**Meeting** $\mathcal{CS}$) Fix a formula $c{:}A \in \mathcal{CS}$, fix an arbitrary $\Gamma \in W$. We can see that $\mathcal{CS} \subseteq \Gamma$. Then $A \in \epsilon(\Gamma, c) = \{B | c{:}B \in \Gamma\}$. □

Lemma 3 (Truth Lemma)**.** *Given a $\mathcal{CS}$. Let $\Lambda \in \{\mathbf{JIEL}^{-}{}_{\mathcal{CS}}, \mathbf{JIEL}_{\mathcal{CS}}\}$, let A be a formula in $\mathcal{L}$. For any $\Gamma \in W$ of M^{Λ},*

$$M^{\Lambda}, \Gamma \models A \textit{ iff } A \in \Gamma.$$

Proof. Propositional cases are standard. ([8])

In the case where A is in the form of KB, we show that $M^{\Lambda}, \Gamma \models KB$ if and only if $KB \in \Gamma$. Suppose $KB \in \Gamma$, we show that $M^{\Lambda}, \Gamma \models KB$. Fix any Δ, suppose $\Gamma R \Delta$, we show $M^{\Lambda}, \Delta \models B$. From $\Gamma R \Delta$ we have $K^{-}(\Gamma) \subseteq \Delta$. Since $B \in K^{-}(\Gamma)$, we have $B \in \Delta$. Then we have $M^{\Lambda}, \Delta \models B$ by induction hypothesis.

On the other direction. Suppose $KB \notin \Gamma$, we show $M^{\Lambda}, \Gamma \nvDash KB$. From $KB \notin \Gamma$, we can find that $\Gamma \nvdash_{\Lambda} KB$, then $K^{-}(\Gamma) \nvdash_{\Lambda} B$ holds. Then by Henkin construction in Lemma 2, we get a consistent prime theory $\Delta \in W$ such that $B \notin \Delta$ and $K^{-}(\Gamma) \subseteq \Delta$, i.e., $\Gamma R \Delta$. Besides, from $B \notin \Delta$, we obtain $M^{\Lambda}, \Delta \nvDash B$ by induction hypothesis.

In the case where A is in the form of $t{:}B$, we show that $M^{\Lambda}, \Gamma \models t{:}B$ if and only if $t{:}B \in \Gamma$. Suppose $M^{\Lambda}, \Gamma \models t{:}B$. We can see that $B \in \epsilon(\Gamma, t)$ from the truth definition. Since $\epsilon(\Gamma, t) = \{C | t{:}C \in \Gamma\}$, we derive that $t{:}B \in \Gamma$.

On the other direction, assume that $t{:}B \in \Gamma$. We show $M^{\Lambda}, \Gamma \models t{:}B$. Fix an arbitrary $\Delta \in W$ s.t., $\Gamma \leq \Delta$. We show that $M^{\Lambda}, \Delta \models B$. Since $t{:}B \in \Gamma$ and $\Gamma \subseteq \Delta$, we can derive that $t{:}B \in \Delta$. We know that Δ is a prime theory, which implies $t{:}B \to B \in \Delta$. It is obvious that $B \in \Delta$. By induction hypothesis, we have $M^{\Lambda}, \Delta \models B$. Furthermore, from $t{:}B \in \Gamma$ we see that $B \in \epsilon(\Gamma, t)$ by the definition of ϵ function. Thus, $M^{\Lambda}, \Gamma \models t{:}B$ holds. □

Theorem 7 (Completeness)**.** *Let A be a formula in $\mathcal{L}$, given a $\mathcal{CS}$,*

1 if $\mathbb{M}^{\mathcal{CS}}_{\text{all}} \models A$, *then* $\mathbf{JIEL}^{-}{}_{\mathcal{CS}} \vdash A$,
2 if $\mathbb{M}^{\mathcal{CS}}_{\text{ser}} \models A$, *then* $\mathbf{JIEL}_{\mathcal{CS}} \vdash A$.

Proof. Let $\Lambda \in \{\mathbf{JIEL}^{-}{}_{\mathcal{CS}}, \mathbf{JIEL}_{\mathcal{CS}}\}$. Let $\mathbb{M}^{\mathbf{JIEL}^{-}{}_{\mathcal{CS}}}$ be $\mathbb{M}^{\mathcal{CS}}_{\text{all}}$, $\mathbb{M}^{\mathbf{JIEL}_{\mathcal{CS}}}$ be $\mathbb{M}^{\mathcal{CS}}_{\text{ser}}$. Suppose $\Lambda \nvdash A$. According to proposition 2, there is a Λ-consistent prime theory Δ such that $A \notin \Delta$. By truth lemma 3, $M^{\Lambda}, \Delta \nvDash A$ for $M^{\Lambda} \in \mathbb{M}^{\Lambda}$. □

5 Future Studies

In Artemov ([4]), he gave a proof to show that embedding theorem between **LP** and **S4**, namely the famous *Realization Theorem*. Artemov also suggests that when combine with the famous embedding theorem from **S4** to **Int** (propositional intuitionistic logic) introduced by Gödel ([14]), McKinsey and Tarski ([17]), the realization theorem can be considered as a BHK-style provability semantics. Furthermore, Artemov raises a question as to whether **LPS4** (a classical system with both implicit and explicit knowledge) enjoys the realization property. ([1]) This question is answered in Fitting ([11]). He provides a proof for *Local Realization Property* and claims it to be a part of a project to understand the deeper aspects of Artemov's Realization Theorem.

In Protopopescu ([20]), the realization theorem for **IEL** is studied, where the justification systems are classical and separated from the implicit epistemic systems. However, the local realization property of **IEL** has not been studied. In the next step, it is necessary to study whether the newly introduced **JIEL** satisfies the realization property, especially with respect to the notion of locality in [11].

Also, when considering the proof theoretic properties, the Hilbert system alone seems not enough. Justification logic has been studied in sequent calculi ([4, 12]), and in tableaux ([9, 13]). It is interesting to study the **JIEL** in a different proof system.

References

[1] S. Artemov and E. Nogina, 2004, "Logic of knowledge with justifications from the provability perspective", *Technical Report TR-2004011*, CUNY Academic Works.

[2] S. Artemov, 2008, "The logic of justification", *The Review of Symbolic Logic*, **1(4)**: 477–513.

[3] S. Artemov and T. Protopopescu, 2016, "Intuitionistic epistemic logic", *The Review of Symbolic Logic*, **9(2)**: 266–298.

[4] S. Artemov, 2001, "Explicit provability and constructive semantics", *Bulletin of Symbolic Logic*, **7(1)**: 1–36.

[5] V. Brezhnev, 2000, "On explicit counterparts of modal logics", *Technical Report CFIS 2000-05*, Ithaca: Cornell University.

[6] V. Brezhnev and R. Kuznets, 2006, "Making knowledge explicit: How hard it is", *Theoretical Computer Science*, **357(1)**: 23–34.

[7] C. P. Brogi, 2021, "Curry-Howard-Lambek correspondence for intuitionistic belief", *Studia Logica*, **109(6)**: 1441–1461.

[8] D. van Dalen, 2013, *Logic and Structure, 5th edition*, London: Springer London.

[9] M. Fitting, 2004, "Semantics and tableaus for LPS4", *Technical Report TR-2004016*, CUNY Academic Works.

[10] M. Fitting, 2005, "The logic of proofs, semantically", *Annals of Pure and Applied Logic*, **132(1)**: 1–25.

[11] M. Fitting, 2008, "S4LP and local realizability", in E. A. Hirsch, A. A. Razborov, A. Semenov and A. Slissenko (eds.), *Computer Science — Theory and Applications*, pp. 168–179, Berlin, Heidelberg: Springer Berlin Heidelberg.

[12] M. Ghari, 2017, "Labeled sequent calculus for justification logics", *Annals of Pure and Applied Logic*, **168(1)**: 72–111.

[13] M. Ghari, 2024, "Tableaux and interpolation for propositional justification logics", *Notre Dame Journal of Formal Logic*, **65(1)**: 81–112.

[14] K. Gödel, 1933, "Eine interpretation des intuitionistischen aussagenkalküls", *Ergebnisse Eines Mathematischen Kolloquiums*, **4**: 39–40.

[15] K. Kojima, 2012, "Relational and neighborhood semantics for intuitionistic modal logic", *Reports on Mathematical Logic*: 87–113.

[16] V. N. Krupski and A. Yatmanov, 2016, "Sequent calculus for intuitionistic epistemic logic IEL", *International Symposium on Logical Foundations of Computer Science*, pp. 187–201, Springer.

[17] J. C. C. McKinsey and A. Tarski, 1948, "Some theorems about the sentential calculi of Lewis and Heyting", *The Journal of Symbolic Logic*, **13(1)**: 1–15.

[18] A. Mkrtychev, 1997, "Models for the logic of proofs", in S. Adian and A. Nerode (eds.), *Logical Foundations of Computer Science*, pp. 266–275, Berlin, Heidelberg: Springer.

[19] C. Proietti, 2012, "Intuitionistic epistemic logic, Kripke models and Fitch's paradox", *Journal of Philosophical Logic*, **41**: 877–900.

[20] T. Protopopescu, 2020, "An arithmetic interpretation of intuitionistic verification", *Journal of Logic and Computation*, **30(1)**: 381–402.

[21] N. Savic and T. Studer, 2019, "Relevant justification logic", *Journal of Applied Logics*, **6(2)**: 395–410.

[22] A. K. Simpson, 1994, *The proof theory and semantics of intuitionistic modal logic*, PhD thesis, University of Edinburgh.

[23] S. Standefer, 2022, "Weak relevant justification logics", *Journal of Logic and Computation*, **33(7)**: 1665–1683.

[24] Y. Su and K. Sano, 2023, "A first-order expansion of Artemov and Protopopescu's intuitionistic epistemic logic", *Studia Logica*, **111(2)**: 615–652.

[25] Y. Su and K. Sano, 2020, "Cut-free and analytic sequent calculus of intuitionistic epistemic logic", *The Logica Yearbook 2019*, pp. 179–192, London: College Publications.

[26] A. Troelstra and D. van Dalen, 1988, *Constructivism in Mathematics*, Amsterdam: Elsevier Science.

[27] J. Yu, 2014, "Self-referentiality of Brouwer-Heyting-Kolmogorov semantics", *Annals of Pure and Applied Logic*, **165(1)**: 371–388.

[28] J. Yu, 2017, "On non-self-referential fragments of modal logics", *Annals of Pure and Applied Logic*, **168(4)**: 776–803.

带核证的直觉主义认知逻辑 **IEL**

苏有安

摘　要

在阿尔捷莫夫与普罗托波佩斯库（2016）的研究中，一种直觉主义认知逻辑 IEL 被提出，该逻辑接受公理 $A \to KA$，但拒斥公理 $KA \to A$。本文在直觉主义认知逻辑 IEL 的基础上对知识和核证的关系展开研究。本文包含了命题“A”是知识（即“KA”），“t”是命题“A”的核证（“$t : A$”）两种表达。新构建的逻辑系统的公理体系融合了直觉主义认知逻辑 **IEL** 的公理与证明逻辑 **LP**（或称 **JT4**）的公理，同时还引入了一条关联公理 $t : A \to KA$。本文为上述新提出的逻辑系统构建了希尔伯特演算系统与克里普克–菲廷语义模型，并证明了这些系统相对于其目标语义具备可靠性与完备性。

苏有安　辽宁大学哲学院
su.youan@lnu.edu.cn