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On Definable J-sets

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Abstract. We study definable J-sets for definable groups and compare them with weakly generic sets. We show that J-property, the property that J-sets coincide with weakly generic sets, is invariant on enough saturated models, and hence a model-theoretical property. We have a positive result for superstable commutative groups. We also give an example without J-property.

1 Introduction

In [4], J-sets are studied in combinatorics for semigroups and are regarded as "large" sets. We can study them among definable subsets of a definable group G. The method for Stone-Cěch compactification can be replaced by the type space of externally definable subsets of G in [5]. It is natural to compare J-sets with weakly generic sets given in [5] which are also about largeness.

In Section 2, we give some basic properties on J-sets. Let X be a definable subset of G. We can study whether X is a J-set of G on different models. Like Lemma 2.1 [5] for weakly generic sets, the question has the same answer on $\aleph_0$-saturated models. Hence, whether J-sets coincide with weakly generic sets is invariant for $\aleph_0$-saturated models. Thus, the property that J-sets coincide with weakly generic sets on enough saturated models (for convenience, named by **J-property**) is for the theory of G. Then we focus on commutative groups.

In Section 3, we show that J-property is related to the classification theory. We show that superstable commutative groups have J-property.

In Section 4, we study the group $(\mathbb{Z}, +)$. This is the group where the classical combination theory arose (but for all subsets, not only for the definable ones). Following [3], we give an example which is an expansion of $(\mathbb{Z}, +)$ by a unary predicate and does not have J-property. However, the theory is not stable. So we ask whether there exist stable examples.

For notation, we always let M be a structure and G a group definable over M. We usually work in a monster model $\mathbb{M}$ of $T = \mathrm{Th}(M)$, the theory of M. We say that a set X is defined in M if $X = X(M)$ is definable in the structure M. If we do not say in which a definable set X is defined, it is considered in $\mathbb{M}$ and $X = X(\mathbb{M})$. We always write $G = G(\mathbb{M})$. We say that G is stable/superstable if its induced theory

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from T is, and for convenience, we assume that T is stable/superstable in this case. Sometimes, we assume that $0 \in \mathbb{N}$, while sometimes, to start from 1, we assume that $0 \notin \mathbb{N}$.

2 Basic Properties

We recall the definition of J-sets from Definition 14.14.1 [4]:

Definition 1. A subset X of $G(M)$ is called a **J-set of G in M**, if for every $F \in \mathcal{P}_{\text{fin}}(G(M)^{\mathbb{N}}) := \{F \subset G(M)^{\mathbb{N}} : |F| < \aleph_0\}$, there are $m \in \mathbb{N}$, $t \in ȷ_m := \{s \in \mathbb{N}^m : s(j) < s(j+1) \text{ for all } j\}$ and $a \in G(M)^{m+1}$ such that for every $f \in F$,

$$\chi(m, a, t, f) := a(1)f(t(1))a(2)f(t(2)) \cdots a(m)f(t(m))a(m+1) \in X$$

We call a type p on G over M is a **J-type**, if every $X \in p$ is a J-set in M.

Remark 1. It is easy to see that if X is a J-set of G in M and $g \in G(M)$, then $gX = \{gx : x \in X\}$ and $Xg := \{xg : x \in X\}$ are also J-sets of G in M.

Note that (see Lemma 14.14.6[4]) if $X \cup Y$ is a J-set of G in M, then at least one of X and Y is a J-set of G in M. Hence, every F.I.P (finite intersection property) family of definable J-sets of G in M can be extended to a J-type over M and J-types always exist.

Let $S_G^{\text{ext}}(M)$ be the type space of externally definable subsets of $G(M)$ given in [5] Section 4. It is known that $(S_G^{\text{ext}}(M), *)$ is a compact right topological semigroup where for $p, q \in S_G^{\text{ext}}(M)$ and externally definable $X \subset G(M)$,

$$X \in p * q \quad \text{iff} \quad \{g \in G(M) : g^{-1}X \in q\} \in p.$$

Moreover, $G(M)$ naturally (left) acts on $S_G^{\text{ext}}(M)$ and $(S_G^{\text{ext}}(M), *)$ is the envelope semigroup of the action. Then every $G(M)$-subflow (i.e. a closed space which is closed under the $G(M)$-action) is a left ideal. Let $\text{AP}_G(M)$ be the minimal two-side ideal of $S_G^{\text{ext}}(M)$ whose elements are called **almost periodic** in [5]. It is the union of all minimal $G(M)$-subflows. Let $\text{WGen}_G(M)$ be the closure of $\text{AP}_G(M)$ whose elements are called **weakly generic** in [5]. Here, the definition of weak genericity is not original but from Corollary 1.8 [5].

Originally, from Definition 1.1 [5], a subset of X of $G(M)$ is **generic** if $G(M)$ can be covered by finitely many (left) translates of X, and a subset of X of $G(M)$ is **weakly generic** if $X \cup Y$ is generic for some non-generic (definable, see Remark 1.2 [5]) Y. A type on G over M is (weakly) generic if every its element is (weakly) generic. Let $\text{Gen}_G(M)$ be the subspace of generic types in $S_G^{\text{ext}}(M)$. Note that generic types always exist and play an important role in stable groups. But in general, generic types may not exist, while weakly generic types always exist. Moreover, when $\mathrm{Gen}_G(M) \neq \varnothing$, by Corollary 1.9 [5], $\mathrm{WGen}_G(M)$ is the only minimal $G(M)$-subflow and then $\mathrm{AP}_G(M) = \mathrm{WGen}_G(M) = \mathrm{Gen}_G(M)$.

We denote the subspace of J-types in $S_G^{\mathrm{ext}}(M)$ by $\mathrm{J}_G(M)$. Then like Theorem 14.14.4 [4], we have

Proposition 1. $\mathrm{J}_G(M)$ is a compact two-sided ideal of $S_G^{\mathrm{ext}}(M)$.

Proof. Clearly, $\mathrm{J}_G(M)$ is the intersection of $[X] := \{p \in S_G^{\mathrm{ext}}(M) : X \in p\}$ for all externally definable X with $G(M)\backslash X$ not a J-set for G in M. Hence, $\mathrm{J}_G(M)$ is closed in $S_G^{\mathrm{ext}}(M)$ and is compact. As $\mathrm{J}_G(M)$ is closed by the $G(M)$-action, it is a left ideal of $S_G^{\mathrm{ext}}(M)$.

Let $X \in p * q$. We show that X is a J-set in M. Let $F \in \mathcal{P}_{\mathrm{fin}}(G(M)^{\mathbb{N}})$. As $\{g \in G(M) : g^{-1}X \in q\}$ is in p and hence a J-set of G in M, there are $m \in \mathbb{N}$, $t \in \mathrm{j}_m$ and $a \in G(M)^{m+1}$ such that for every $f \in F$, $\chi(m,a,t,f)^{-1} \cdot X \in q$. Then $\bigcap_{f\in F} \chi(m,a,t,f)^{-1} \cdot X \in q$ is nonempty and we take b from it. Then $\chi(m,a,t,f)b \in X$ for every $f \in F$. We let $c \in G(M)^{m+1}$ with $c(i) = a(i)$ for $i \leq m$ and $c(m+1) = a(m+1)b$. Then $\chi(m,c,t,f) \in X$ for every $f \in F$. $\square$

As $\mathrm{J}_G(M)$ is a two-sided ideal of $S_G^{\mathrm{ext}}(M)$, $\mathrm{J}_G(M) \supset \mathrm{AP}_G(M)$. Then by closeness of $\mathrm{J}_G(M)$ in $S_G^{\mathrm{ext}}(M)$, we have $\mathrm{J}_G(M) \supset \mathrm{WGen}_G(M)$. In summary, we have that

Corollary 1. Every definable weakly generic sets of G in M is a J-set of G in M.

Now we study the J-sets in different models:

Proposition 2. Assume that $M \prec N$ and $X \subset G$ is definable in M. If $X(N)$ is a J-set of G in N, then X is a J-set of G in M. Moreover, if M is $\aleph_0$-saturated, the converse holds.

Proof. Let $F \in \mathcal{P}_{\mathrm{fin}}(G(M)^{\mathbb{N}})$ be arbitrary. Then there are $m \in \mathbb{N}$, $a \in G(N)^{m+1}$ and $t \in \mathrm{j}_m$ such that $\chi(m,a,t,f) \in X$ for every $f \in F$. Then

$$N \models \exists y_1 \ldots y_{m+1} \bigwedge_{f\in F} \text{“}\chi(m,y_1,\ldots,y_{m+1},t,f) \in X\text{”}$$

As $M \prec N$ and the statement is over M, we have

$$M \models \exists y_1 \ldots y_{m+1} \bigwedge_{f\in F} \text{“}\chi(m,y_1,\ldots,y_{m+1},t,f) \in X\text{”}$$

Then there is $a' \in G(M)^{m+1}$ such that $\chi(m,a',t,f) \in X$ for every $f \in F$.

Now we show the "moreover" part. If $X(N)$ is not a J-set of G in N, then there are $f_1, \dots, f_n \in G(N)^{\mathbb{N}}$ such that for every $m \in \mathbb{N}$ and $t \in \mathfrak{j}_m$,

$$N \models \forall y_1 \dots y_{m+1} \bigvee_{i=1}^{n} \text{“}\chi(m, y_1, \dots, y_{m+1}, t, f_i) \notin X\text{”}$$

Let C be a finite set of M over which X and G are defined. As M is $\aleph_0$-saturated, after reordering the index set of $f_{i,j}$ by $(\mathbb{N}, <)$, we can find $g_{i,j} \in G(M)$ inductively such that

$$\mathrm{tp}(g_{i,j} : 1 \leq i \leq n, j \in \mathbb{N}/C) = \mathrm{tp}(f_{i,j} : 1 \leq i \leq n, j \in \mathbb{N}/C)$$

Let $g_i \in G(M)^{\mathbb{N}}$ with $g_i(j) = g_{i,j}$. Then for every $m \in \mathbb{N}$ and $t \in \mathfrak{j}_m$,

$$M \models \forall y_1 \dots y_{m+1} \bigvee_{i=1}^{n} \text{“}\chi(m, y_1, \dots, y_{m+1}, t, g_i) \notin X\text{”}$$

which is a contradiction. □

And, for weakly generic sets, we have

Fact 1 (Lemma 2.1 [5]). Assume that $M \prec N$ and $X \subset G$ is definable in M. If $X(M)$ is weakly generic for $G(M)$, then $X(N)$ is weakly generic for $G(N)$. Moreover, if M is $\aleph_0$-saturated, the converse holds.

Then we have

Proposition 3. Assume that $M \prec N$ and they are both $\aleph_0$-saturated. Then J-sets of $G(M)$ coincide with weakly generic sets of $G(M)$ iff J-sets of $G(N)$ coincide with weakly generic sets of $G(N)$.

Proof. The direction $\Leftarrow$ is obvious from the above results.

For the other direction, we assume that J-sets of $G(M)$ coincide with weakly generic sets of $G(M)$. Since the direction $\Leftarrow$ is known, we may assume that $N = \mathbb{M}$. For convenience, we assume that G is $\varnothing$-definable. If the result fails, there are a formula $\varphi(x; z)$ without parameters and a tuple c in $\mathbb{M}$ such that $X = \varphi(\mathbb{M}, c) \subset G(\mathbb{M})$ is, in $\mathbb{M}$, a J-set but not weakly generic. As M is $\aleph_0$-saturated, there is a tuple c' in M such that $\mathrm{tp}(c') = \mathrm{tp}(c)$. Then there is an isomorphism σ of $\mathbb{M}$ such that $c' = \sigma(c)$. Obviously, $X' := \varphi(\mathbb{M}, c')$ is, in $\mathbb{M}$, a J-set but not weakly generic. Then $X'(M) = \varphi(M, c')$ is, in M, a J-set but not weakly generic, which is a contradiction. □

Since whether J-sets coincide with weakly generic sets is invariant on $\aleph_0$-saturated models, we have

Definition 2. We say that a definable group G has **J-property** if every weakly generic set of G in M is a J-set of G in M for some (any) $\aleph_0$-saturated model M.

In particular, we can study the J-property in the monster $\mathbb{M}$. And for convenience, if we do not say in which model a definable set of G is a J-set, then it is considered in $\mathbb{M}$.

3 Positive Results for Commutative Groups

As $\mathrm{AP}_G(M) = \mathrm{WGen}_G(M) = \mathrm{Gen}_G(M)$ for stable groups, we study the relation between the properties on $S_G^{\mathrm{ext}}(M)$ and the place of the induced theory of G in $T = \mathrm{Th}(M)$ in the classification theory.

Now we show that every superstable commutative group has J-property. Note that as every type over M is definable, every externally definable set is in fact definable and $S_G^{\mathrm{ext}}(M) = S_G(M)$. For $p \in S_G(M)$ and $N \prec M$, by $p|N$ we denote the restriction of p on N.

Theorem 2. *Assume that G is commutative and superstable. Then G has J-property.*

Proof. Assume that G is superstable and $X = X(\mathbb{M})$ is a definable J-set of G. For convenience, we assume that they are $\varnothing$-definable.

Let $p \in S_G(\mathbb{M})$ be a generic global type with $p * p = p$. Such idempotent p exists by the famous Ellis's Lemma. Let $(M_i)_{i\in\mathbb{N}}$ be an elementary increasing chain where M_{i+1} is $|M_i|^+$-saturated for each i. Let $f_1 \in G(M_1)^{\mathbb{N}}$ be a Morley sequence of $p|M_0$ and inductively, $f_{i+1} \in G(M_{i+1})^{\mathbb{N}}$ be a Morley sequence of $p|M_i$.

As X is a J-set of G, for each $k \in \mathbb{N}$ and $\{f_i : 1 \le i \le k\}$, there are $m_k \in \mathbb{N}$, $a_k \in G$ and $t_k \in \mathfrak{j}_{m_k}$ such that for every $i \le k$,

$$a_k + \sum_{j=1}^{m_k} f_i(t_k(j)) \in X$$

Let $b_{i,k} = \sum_{j=1}^{m_k} f_i(t_k(j))$ for $i, k \in \mathbb{N}$ and $\varphi(x; y) :=$ "$x, y \in G$ and $x + y \in X$". Then $\mathbb{M} \models \varphi(a_k, b_{i,k})$ when $i \le k$.

We claim that for each $k \in \mathbb{N}$, the sequence $(b_{i,k})_{i\in\mathbb{N}}$ is a Morley sequence of p.

Note that $b_{i,k} \in G(M_i)$ for every i, k. As $p*p = p$, we have $p|M_i*p|M_i = p|M_i$ and consequently $\mathrm{tp}(b_{i+1,k}/M_i) = p|_{M_i}$ for every $i \in \mathbb{N}$. It completes the proof of the claim because G is stable (see Fact 7.3[6]).

Then as G is stable, by Proposition 7.6 [6], there is $l \in \mathbb{N}$ which depends only on $\varphi(x; y)$ such that for every $k \in \mathbb{N}$, either $|\{i \in \mathbb{N} : \mathbb{M} \models \varphi(a_k; b_{i,k})\}| < l$ or $|\{i \in \mathbb{N} : \mathbb{M} \models \neg\varphi(a_k; b_{i,k})\}| < l$. We let $a = a_l$ and $b_i = b_{i,l}$ for $i \in \mathbb{N}$. Then $|\{i \in \mathbb{N} : \mathbb{M} \models \neg\varphi(a; b_i)\}| < l$ and there is $l_1 \in \mathbb{N}$ such that $\mathbb{M} \models \varphi(a; b_i)$ for every $i \ge l_1$.

Then as G is superstable, by Proposition 5.7 [6], the extension of tp$(a/M_{i+1}) \supset$ tp(a/M_i) forks only for finitely many i, and there is $l_2 \in \mathbb{N}$ such that tp(a/M_{i+1}) is nonforking over M_i for $i \geq l_2$.

Take $l' > \max l_1, l_2$. Let $M = M_{l'}$ and $b = b_{l'+1}$. Then a and b are independent over M. Hence,

$$\text{tp}(a+b/M) = \text{tp}(a/M) * \text{tp}(b/M) = \text{tp}(a/M) * p|M$$

which is generic. As $\mathbb{M} \models \varphi(a;b)$, we have $X(M) \in \text{tp}(a+b/M)$. Hence, X is generic. $\square$

As a corollary, every algebraic group in an algebraically closed field has J-property. In addition, it is reasonable to study the J-property for definable groups in geometric fields in further research.

4 An Example for Expansions of $(\mathbb{Z}, +)$

Now we study expansions of $(\mathbb{Z}, +)$. We give an example without J-property.

We follow [3]. For $a \in \mathbb{N}$ with $a \geq 1$, it has a unique binary expansion $a = \sum_i \epsilon(i)2^i$. We call $\{i \in \mathbb{N} : \epsilon(i) = 1\}$ the **support** of a and denote it by supp(a). Let $B_k := \{2^k, 2^k + 1, \ldots, 2^{k+1} - 1\}$ for $k \in \mathbb{N}$ and

$$A = \{n \in \mathbb{N} : B_k \backslash \text{supp}(n) \neq \varnothing \text{ for every } k\}$$

In [3], $A \subset \mathbb{N}$ is shown to be a J-set (in the semigroup $\mathbb{N}$) but not piecewise syndetic. We do not give the original definition of piecewise syndetic here, but by Theorem 3.2 [3], a subset X of a (discrete) group S is **piecewise syndetic** iff there is p in the minimal two-sided ideal of βS with $X \in p$. Here βS is the Stone-Cěch compactification of the discrete space S and it can be considered as the collection of all ultrafilters on S. The product on βS is given by

$$X \in p * q \quad \text{iff} \quad \{s \in S : s^{-1}X \in q\} \in p$$

for all $p, q \in \beta S$ and $X \subset S$.

As we study groups, we transfer $(\mathbb{N}, +)$ into a group $(\mathbb{Z}, +)$. Obviously, $\beta\mathbb{Z} = \beta\mathbb{N} \cup (-\beta\mathbb{N})$ where $-\beta\mathbb{N} := \{-p : p \in \beta\mathbb{N}\}$ and $-p := \{-X : X \in p\}$. We denote the minimal two-sided ideal of $\beta\mathbb{N}$ by K. Then clearly, $\hat{K} = K \cup (-K)$ is the minimal two-sided ideal of $\beta\mathbb{Z}$. Hence, A is not piecewise syndetic in $(\mathbb{Z}, +)$. Note that $\beta\mathbb{Z}$ can be regard as the type space over $\mathbb{Z}$ in the theory that every subset of $\mathbb{Z}$ is definable, and in this theory, the notions of piecewise syndetic sets and weakly generic sets are the same.

We let $\mathbb{A}$ be a unary predicate, T the theory of $M := (\mathbb{Z}, +, A)$ with $A = \mathbb{A}(M)$, $\mathbb{M} = (\mathbb{M}, +, \mathbb{A})$ a monster model and $G := (\mathbb{M}, +)$. By restricting $p \in \beta\mathbb{Z}$ to externally definable sets in M, we have a surjective semigroup morphism $\pi : \beta\mathbb{Z} \to S_G^{\text{ext}}(M)$. It is also a map of $\mathbb{Z}$-actions and by Lemma 1.4 [5], A is not weakly generic in $G(M)$. Then we have

Proposition 4. A is a J-set of G in M which is not weakly generic in M.

For things on $\mathbb{M}$, we should study the proof of Lemma 5.2 [3] carefully. Note that from Remark 4.46 [4], a subset X of a semigroup S is piecewise syndetic iff $\bigcup_{t\in H}\{s \in S : tx \in X\}$ is thick for some finite subset H of S. Here, a subset X of a semigroup S is called **thick** if for every finite subset H of S, there is $s \in S$ such that $Hs \subset X$ (Definition 4.45 [4]). Hence, to show that A is not piecewise syndetic in $(\mathbb{N}, +)$, it is equivalent to showing that for every $c_1, \ldots, c_m \in \mathbb{N}$, there are $b_1, \ldots, b_n$ such that for every $x \in \mathbb{N}$, there is $y \in x + \{b_1, \ldots, b_n\}$ with $(y + \{c_1, \ldots, c_m\}) \bigcap A = \varnothing$. We prove a better form that n depends only on m.

Lemma 1. *For every $m \in \mathbb{N}$. Let $n = 2^{2^{m+1}}$. Then for every $c_1, \ldots, c_m \in \mathbb{N}$ and $x \in \mathbb{N}$, there is $y \in x + \{1, \ldots, n\}$ such that $(y + \{c_1, \ldots, c_n\}) \bigcap A = \varnothing$.*

Proof. Let $c_1, \ldots, c_m \in \mathbb{N}$ and $x \in \mathbb{N}$ be arbitrary.

Inductively, we give c'_k and d_k for $1 \leq k \leq m$. Let $c'_1 = c_1$ and assume that $c'_1 = \sum_j \epsilon_{1,j}2^j$ where $\epsilon_{1,j} \in \{0, 1\}$. Let $d_1 = \sum_{j\in B_1}(1 - \epsilon_{1,j})2^j$. Given c'_k and d_k, we let $c'_{k+1} = c_k + d_k$. Assume that $c'_k = \sum_j \epsilon_{k,j}2^j$ where $\epsilon_{k,j} \in \{0, 1\}$. Let $d_{k+1} = (\sum_{1\leq j\leq k} d_k) + (\sum_{j\in B_{k+1}}(1 - \epsilon_{k,j})2^j)$.

Let $d = d_m$. It is easy to see that supp$(d + c_k) \supset B_k$ for every $1 \leq k \leq m$. Clearly, there always exists $y \in x + \{1, \ldots, n\}$ such that $y \equiv d$ (mod $n = 2^{2^{m+1}}$). Then $y + c_k \equiv d + c_k$ (mod n) and supp$(y + c_k) \bigcap\{1, \ldots, 2^{m+1} - 1\} =$ supp$(d + c_k) \bigcap\{1, \ldots, 2^{m+1} - 1\} \supset B_k$ for each k. Then $(y + \{c_1, \ldots, c_n\}) \bigcap A = \varnothing$. $\square$

The advantage of the above from is that for given m and $n = 2^{2^{m+1}}$, the statement is first order and can be transferred. Then we have

Proposition 5. $\mathbb{A}$ is not weakly generic for G in $\mathbb{M}$.

Proof. To make use of the structure $(\mathbb{N}, +)$, we add $<$ into the language of T. Then we may assume that $(\mathbb{M}^{>0}, +, \mathbb{A})$ is a monster model of $(\mathbb{N}, +, \mathbb{A})$. Then by transferring the above lemma for each m to $(\mathbb{M}^{>0}, +)$, we have that $\mathbb{A}$ is not piecewise syndetic in $(\mathbb{M}^{>0}, +)$. Like the above argument from $(\mathbb{N}, +)$ to $(\mathbb{Z}, +)$, we have that $\mathbb{A}$ is not piecewise syndetic in $(\mathbb{M}, +)$. Also by the same argument above, we have that $\mathbb{A}$ is not weakly generic for G in $\mathbb{M}$. $\square$

Now we show that $\mathbb{A}$ is a J-set of G in $\mathbb{M}$. Like Lemma 5.1 [4], we have

Lemma 2. *Let $f_1, \ldots, f_n \in \mathbb{M}^{\mathbb{N}}$. Then there is $m \in \mathbb{N}$ and $t \in \jmath_m$ such that $\sum_{j\in t} f_i(j) \in 2^{n+1}\mathbb{M}$ for each i.*

Proof. Note that $\mathbb{Z}/2^{n+1}\mathbb{Z}$ is finite, and we have $\mathbb{M}/2^{n+1}\mathbb{M} = \mathbb{Z}/2^{n+1}\mathbb{Z}$. Let $I_0 = \mathbb{N}$. Inductively, we can find $I_i \subset I_{i-1}$ for $i = 1, \ldots, n$ such that for every $j_1, j_2 \in I_i$, $f_i(j_1) - f_i(j_2) \in 2^{n+1}\mathbb{M}$. Let $m = 2^{n+1}$ and $t \in \mathfrak{j}_m$ with $t \subset I_n$. It is easy to check that these m and t are what we want. □

Then we modify the proof of Lemma 5.2 [3] which is original on $(\mathbb{N}, +)$ to $(\mathbb{Z}, +)$.

Lemma 3. *Let $b_1, \ldots, b_n \in 2^{n+1}\mathbb{Z}$. Then there is $a \in \mathbb{Z}$ such that $a + b_i \in A$ for every i.*

Proof. Let $b \in 2^{n+1}\mathbb{N}$ such that $b > |b_i|$ for each i. Let $c_{0,i} = b + b_i$ for each i. Note that $2^{n+1} \le c_{0,i} \in 2^{n+1}\mathbb{N}$ for each i. The next paragraph is from the proof of Lemma 5.2 [3].

Let $l \in \mathbb{N}$ be the least such that $2^l > n$. As $2^{l-1} < n + 1$, we have $2^{l-1} \in B_{l-1} \backslash \operatorname{supp}(c_{0,i})$. As $|B_l| = 2^l > n$, there is $r_0 \in B_l$ such that $B_l \backslash \operatorname{supp}(2^{r_0} + c_{0,i}) \neq \varnothing$ for each i. Let $c_{1,i} = 2^{r_0} + c_{0,i}$. Inductively, given $c_{j,i}$, as $|B_{l+j}| = 2^{l+j} > n$, there is $r_j \in B_{l+j}$ such that $B_{l+j} \backslash \operatorname{supp}(2^{r_j} + c_{j,i}) \neq \varnothing$ for each i. We can stop at $j = k$ when $2^{2^{k+i}} > \max_i c_{0,i}$. Let $c = \sum_{j=0}^{k} 2^{r_j}$. It is easy to check that $c + c_{0,i} \in A$ for each i.

Let $a = b + c$. Then $a + b_i \in A$ for each i. □

It is clear that the statement of Lemma 3 is first order on each n and hence holds in $(\mathbb{M}, +, \mathbb{A})$. Combining with Lemma 2, we have

Proposition 6. $\mathbb{A}$ is a J-set of G in $\mathbb{M}$.

Hence, witnessed by $\mathbb{A}$, we have

Proposition 7. The definable group G in T does not have J-property.

However, this theory T is not stable:

Proposition 8. The upper Banach density δ of A in $\mathbb{N}$ is > 0. Hence, by Theorem D [2], the theory T is not stable.

Proof. It is easy to see that for $d \ge 1$,

$$\frac{|A \cap \{0, 1, \ldots, 2^{2^{d+1}} - 1\}|}{2^{2^{d+1}}} = \frac{2\prod_{k=0}^{d}(2^{2^k} - 1)}{2^{2^{d+1}}} = \prod_{k=0}^{d}(1 - 2^{-2^k})$$

Clearly, the limit $\prod_{k=0}^{\infty}(1-2^{-2^k})$ exists and $\delta \ge \prod_{k=0}^{\infty}(1-2^{-2^k})$. To show $\prod_{k=0}^{\infty}(1-2^{-2^k}) > 0$, it suffices to show $\sum_{k=0}^{\infty} \log(1-2^{-2^k})$ converges. Note that $\log(1+x) =$

$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ when $|x| < 1$. Then $|\log(1+x)| \leq |x| + |x|^2 + |x|^3 + \cdots = \frac{|x|}{1-|x|}$ for $|x| < 1$. Then for each k,

$$|\log(1 - 2^{-2^k})| \leq \frac{2^{-2^k}}{1 - 2^{-2^k}} = \frac{1}{2^{2^k} - 1} \leq 2^{-2^k - 1}$$

Hence, $\sum_{k=0}^{\infty} \log(1 - 2^{-2^k})$ converges absolutely. □

As we have shown that every superstable commutative group has J-property and it is well-known that $(\mathbb{Z}, +)$ is superstable, it is natural to ask:

Question 1. Is there a stable expansion of the superstable group $(\mathbb{Z}, +)$ by a unary predicate for $A \subset \mathbb{N}$ such that $\mathbb{A} := A(\mathbb{M})$ avoids J-property?

More weakly, we have the following natural question:

Question 2. Is there a stable commutative group without J-property?

Remark 2. No example for Question 2 can be found within the pure group language. To see this, let G be a commutative group. It is easy to check that for a definable subgroup H of G, it is a J-set iff it is of finite index iff it is generic. In the pure group language, every definable set is a Boolean combination of pp-formulas (see [1], Chapter 1, Theorem 4.12), and hence a Boolean combination of cosets of definable subgroups. Thus, one can easily verify that G has J-property in the pure group language.

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可定义 J-集

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摘　要

本文研究可定义群上的可定义 J-集，并将其与可定义弱脱殊集进行对比。本文证明 J-性质（即可定义 J-集与可定义弱脱殊集等价的性质）在充分饱和的模型上具有不变性，因此是模型论性质。本文证明了超稳定交换群具有 J-性质；另一方面，也给出了一个不满足 J-性质的例子。

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