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Lewisian Connexive Logics*

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Abstract. In connexive logic, two fundamental ideas are observed: first, no proposition implies or is implied by its own negation; second, if a proposition implies φ then it will not imply the negation of φ. In classical logic, neither of the ideas holds, which makes it difficult to give a natural semantics for connexive logic. By combining Kleene's three valued logic and Lewis' conditional logic, we propose a new natural semantics for connexive logic. We give four axiomatic systems characterizing different classes of selection models in the new semantics. We prove soundness and completeness of these logics and compare them with some connexive logics in the literature.

1 Introduction

Unlike most nonclassical logics, which are either sublogics or extensions of classical logic, connexive logic is contra-classical. ([3]) It not only lacks some axioms of classical logic, but also has some axioms that are new to classical logic. This makes it difficult to propose a natural semantics for connexive logic. Based on Lewisian semantics for conditionals, this paper aims to propose a new natural semantics for connexive logic.

Connexive logic has two basic ideas. One is that no proposition implies or is implied by its own negation, which is arguably attributed to Aristotle. The other is that if a proposition implies φ then it will not imply the negation of φ, which is attributed to the medieval logician Boethius. Aristotle's theses are usually formulated by $\neg(\varphi \rightarrow \neg\varphi)$ and $\neg(\neg\varphi \rightarrow \varphi)$. Boethius' theses are usually formulated by $(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)$ and $(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)$. It is easily seen that neither Aristotle's theses nor Boethius' theses are valid in classical logic. As classical logic is Post-complete, a consistent connexive logic has to give up some classical tautologies. This is why connexive logic is contra-classical.

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Both Aristotle's theses and Boethius' theses have several variants, which makes it obscure what is counted as a connexive logic. We list the theses in connexive logic to be discussed for reference in Table 1. As far as we know, wAT and wAbT in Table 1 have not been proposed before. According to Wansing ([17]), a logic is connexive if it validates both AT and BT; logics validating only BTr are called weakly connexive. According to Kapsner ([4]), connexive logics satisfying Unsat1 and Unsat2 are called strongly connexive.

Table 1: Common theses in connexive logic and conditional logic

Label	Theses	name
AT	$\neg(\varphi \to \neg\varphi)$[1]	Aristotle's Theses
wAT	$(\varphi \to \psi) \vee \neg(\varphi \to \psi) \vDash \neg(\varphi \to \neg\varphi)$	Weak Aristotle's Theses
BT	$(\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$	Strong Boethius' Theses
BTr	$\varphi \to \psi \vDash \neg(\varphi \to \neg\psi)$	Boethius' Theses in rule form
Unsat1	$\varphi \to \neg\varphi$ is unsatisfiable	Aristotle's Theses via satisfiability
Unsat2	$(\varphi \to \psi) \wedge (\varphi \to \neg\psi)$ is unsatisfiable	Boethius's Theses via satisfiability
AbT	$\neg((\varphi \to \psi) \wedge (\varphi \to \neg\psi))$	Abelard's First Principle
wAbT	$(\varphi \to \chi) \vee \neg(\varphi \to \chi) \vDash$ $\neg((\varphi \to \psi) \wedge (\varphi \to \neg\psi))$	Weak Abelard's First Principle
EFQ	$\varphi, \neg\varphi \vDash \psi$	Ex Falso Quodlibet
LEM	$\varphi \vee \neg\varphi$	Law of Excluded Middle
ID	$\varphi \to \varphi$	Identity
MP	$\varphi, \varphi \to \psi \vDash \psi$	Modus Ponens
CEM	$(\varphi \to \psi) \vee (\varphi \to \neg\psi)$	Conditional Excluded Middle
wCEM	$\neg(\varphi \to \psi) \vDash \varphi \to \neg\psi$	Weak Conditional Excluded Middle

2 Lewisian Semantics for Connexivity

Our language $\mathcal{L}$ is formed from a given set At of atoms by the usual logical connectives $\neg, \wedge, \vee$, and $\to$. We stipulate that $\wedge$ and $\vee$ have priority of combination over $\to$.

We modify Lewis' semantics for conditionals to obtain connexive logic. Among several equivalent Lewisian semantics, we choose the framework of selection models,

[1] The thesis AT is often listed with its variant AT′: $\neg(\neg\varphi \to \varphi)$, and similarly for BT and BTr. As most connexive logics, including ours, have the rule of replacing equivalent consequents and double negation, we will omit these variants for simplicity.

which is technically most convenient. In a selection model, a selection function f selects for each world w and each formula φ a set $f(w, \varphi)$ of worlds that are meant to closest to w making φ true. In Lewisian semantics, a conditional $\varphi \to \psi$ is true at w if and only if ψ is true at all the worlds in $f(w, \varphi)$. We modify Lewisian semantics in two aspects. First, our framework is three-valued, allowing sentences to be neither true nor false. Second, we require $f(w, \varphi)$ to be nonempty for a conditional $\varphi \to \psi$ to be true at w, whereas in standard conditional logic, it not only lacks this requirement but also makes all conditionals $\varphi \to \psi$ true at w if $f(w, \varphi)$ is empty.

We will not impose additional model conditions at first, to obtain a minimal logic. Then we impose additional model conditions to obtain stronger logics.

Definition 1 (Selection models)**.** A (three-valued) selection model is a tuple $\mathfrak{M} = (W, f, V^+, V^-)$, where

- $W \neq \varnothing$ consists of worlds,
- $f : W \times \mathcal{L} \to \wp(W)$ is a selection function, and
- $V^+ : At \to \wp(W)$ and $V^- : At \to \wp(W)$ are truth and falsity valuations, respectively, such that for all $p \in At$, $V^+(p) \cap V^-(p) = \varnothing$.

Let LC be the class of all selection models. The truth and falsity conditions for $\mathcal{L}$ are given below. The result is a combination of Kleene's three-valued logic $\mathbf{K_3}$ (cf. [6]) and a tweak of Lewis' conditional logic.

Definition 2 (Truth and falsity conditions)**.** Given a selection model $\mathfrak{M} = (W, f, V^+, V^-)$, the truth and falsity conditions of any formula φ at any $w \in W$ in $\mathfrak{M}$ is inductively defined as follows.

- $\mathfrak{M}, w \Vdash^+ p$ iff $w \in V^+(p)$
- $\mathfrak{M}, w \Vdash^- p$ iff $w \in V^-(p)$
- $\mathfrak{M}, w \Vdash^+ \neg\varphi$ iff $\mathfrak{M}, w \Vdash^- \varphi$
- $\mathfrak{M}, w \Vdash^- \neg\varphi$ iff $\mathfrak{M}, w \Vdash^+ \varphi$
- $\mathfrak{M}, w \Vdash^+ \varphi \wedge \psi$ iff $\mathfrak{M}, w \Vdash^+ \varphi$ and $\mathfrak{M}, w \Vdash^+ \psi$
- $\mathfrak{M}, w \Vdash^- \varphi \wedge \psi$ iff $\mathfrak{M}, w \Vdash^- \varphi$ or $\mathfrak{M}, w \Vdash^- \psi$
- $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi$ iff $\mathfrak{M}, w \Vdash^+ \varphi$ or $\mathfrak{M}, w \Vdash^+ \psi$
- $\mathfrak{M}, w \Vdash^- \varphi \vee \psi$ iff $\mathfrak{M}, w \Vdash^- \varphi$ and $\mathfrak{M}, w \Vdash^- \psi$
- $\mathfrak{M}, w \Vdash^+ \varphi \to \psi$ iff $f(w, \varphi) \neq \varnothing$ and for all $v \in f(w, \varphi)$, $\mathfrak{M}, v \Vdash^+ \psi$
- $\mathfrak{M}, w \Vdash^- \varphi \to \psi$ iff $\exists v \in f(w, \varphi)$ s.t. $\mathfrak{M}, v \nVdash^+ \psi$

We write $\mathfrak{M}, w \Vdash^+ \Gamma$ if $\mathfrak{M}, w \Vdash^+ \gamma$ for all $\gamma \in \Gamma$. The set of worlds making φ true in $\mathfrak{M}$ is denoted by $[\![\varphi]\!]^{\mathfrak{M}}$, where the superscript $\mathfrak{M}$ is often omitted if no confusion occurs.

Definition 3 (Validity)**.** Given a class of selection models S, an inference from Γ to

φ is valid in S, denoted $\Gamma \vDash_{\mathrm{S}} \varphi$, iff for all $\mathfrak{M}$ in S with w in $\mathfrak{M}$, if $\mathfrak{M}, w \Vdash^{+} \Gamma$ then $\mathfrak{M}, w \Vdash^{+} \varphi$.

According to our semantics, $\varphi \rightarrow \psi$ is neither true nor false at w if $f(w, \varphi)$ is empty, which is more intuitive than standard semantics for conditionals. For example, we would take neither π_o nor π_e below to be true (or false).

(π_o) If π were an integer then it would be odd.

(π_e) If π were an integer then it would be even.

One may propose an alternative to the falsity condition for conditionals as follows.

- $\mathfrak{M}, w \Vdash^{-} \varphi \rightarrow \psi$ iff $\exists v \in f(w, \varphi)$, s.t. $\mathfrak{M}, v \Vdash^{-} \psi$

In a two-valued setting with the law of excluded middle, the alternative is equivalent to what we proposed in Definition 2. In a three-valued logic, however, $\mathfrak{M}, v \nVdash^{+} \psi$ does not imply $\mathfrak{M}, v \Vdash^{-} \psi$, as ψ can be neither true nor false at v. Intuitively, we will deny $\varphi \rightarrow \psi$ as long as ψ is not true at some closest φ-world to w. For example, let φ be $2+2=4$ and ψ be "π is female". Intuitively, ψ is neither true nor false. But we will accept that it is not the case that if $2+2=4$ then π is female, which means that $\varphi \rightarrow \psi$ is false, whereas according to the alternative falsity condition, $\varphi \rightarrow \psi$ is neither true nor false.

The following lemma says that a formula cannot be both true and false at the same world, whose proof is straightforward and omitted.

Lemma 1. *For all selection models $\mathfrak{M}$, for all $\varphi \in \mathcal{L}$, there is no w in $\mathfrak{M}$ such that both $\mathfrak{M}, w \Vdash^{+} \varphi$ and $\mathfrak{M}, w \Vdash^{-} \varphi$.*

3 Proofs Systems and Completeness

In this section, we define four logics **LC**, **LC1**,**LC2** and **LC3** by different classes of selection models and give axiomatization of them.

3.1 LC: a weakly connexive logic

The logic **LC** is semantically defined by $\vDash_{\mathrm{LC}}$ (see Definition 3). Recall that LC is the class of all selection models.

Proposition 1. *The theses* BTr, Unsat2, wAbT *and* EFQ *in Table 1 hold in* **LC**.

Proof. Let $\mathfrak{M} = (W, f, V^{+}, V^{-})$ be any selection model and w any world in W.

For BTr, suppose $\mathfrak{M}, w \Vdash^{+} \varphi \rightarrow \psi$. Then $f(w, \varphi)$ is not empty, and for all $v \in f(w, \varphi)$ we have $\mathfrak{M}, v \Vdash^{+} \psi$. By Lemma 1, there exists a world $v \in f(w, \varphi)$, s.t. $\mathfrak{M}, v \nVdash^{+} \neg\psi$, whence $\mathfrak{M}, w \Vdash^{-} \varphi \rightarrow \neg\psi$, i.e., $\mathfrak{M}, w \Vdash^{+} \neg(\varphi \rightarrow \neg\psi)$.

For Unsat2, suppose both $\mathfrak{M}, w \Vdash^+ \varphi \to \psi$ and $\mathfrak{M}, w \Vdash^+ \varphi \to \neg\psi$. Then $f(w, \varphi)$ is not empty and for all $v \in f(w, \varphi)$ we have both $\mathfrak{M}, v \Vdash^+ \psi$ and $\mathfrak{M}, v \Vdash^+ \neg\psi$, contradicting Lemma 1.

For wAbT, suppose $\mathfrak{M}, w \Vdash^+ (\varphi \to \chi) \vee \neg(\varphi \to \chi)$. Then $f(w, \varphi)$ is not empty. We just need to prove $\mathfrak{M}, w \Vdash^- \varphi \to \psi$ or $\mathfrak{M}, w \Vdash^- \varphi \to \neg\psi$. If $f(w, \varphi) \subseteq [\![\psi]\!]$, then we have $\mathfrak{M}, w \Vdash^- \varphi \to \neg\psi$. If $f(w, \varphi) \nsubseteq [\![\psi]\!]$, then we have $\mathfrak{M}, w \Vdash^- \varphi \to \psi$. Given that we have either $f(w, \varphi) \subseteq [\![\psi]\!]$ or $f(w, \varphi) \nsubseteq [\![\psi]\!]$, it follows that $\mathfrak{M}, w \Vdash^- \varphi \to \psi$ or $\mathfrak{M}, w \Vdash^- \varphi \to \neg\psi$.

The validity of EFQ is obvious, since φ and $\neg\varphi$ cannot both be true. □

The logic **LC** is axiomatized by Table 2, in which REF, CE, CI, DI, DM1, DM2, DN, EFQ, CC, BTr, and ECW are axioms, and MON, CUT, DIL, and RW are rules. Among these axioms and rules, BTr is just the one that makes **LC** weakly connexive. The axiom ECW says that if $f(w, \varphi)$ is nonempty then any conditional with the antecedent φ is either true or false. All the other axioms and rules are common in classical logic. The reason why we use $\vdash$ for axiomatization is because **LC** does not have any valid formulas, as in Kleene's three-valued logic $\mathbf{K_3}$. Nonetheless, just like Kleene's logic, **LC** has valid inferences.

Table 2: Axiomatic System **LC**

REF	$\varphi \vdash \varphi$	MON	$\dfrac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi}$
CUT	$\dfrac{\Gamma \vdash \varphi \quad \varphi, \Delta \vdash \psi}{\Gamma, \Delta \vdash \psi}$	RW	$\dfrac{\alpha \vdash \beta}{\varphi \to \alpha \vdash \varphi \to \beta}$
CE	$\varphi \wedge \psi \vdash \varphi \quad \varphi \wedge \psi \vdash \psi$	CI	$\varphi, \psi \vdash \varphi \wedge \psi$
DI	$\varphi \vdash \varphi \vee \psi \quad \psi \vdash \varphi \vee \psi$	DIL	$\dfrac{\Gamma, \alpha \vdash \varphi \quad \Gamma, \beta \vdash \varphi}{\Gamma, \alpha \vee \beta \vdash \varphi}$
DM1	$\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$	DM2	$\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$
DN	$\neg\neg\varphi \dashv\vdash \varphi$	EFQ	$\varphi, \neg\varphi \vdash \psi$
CC	$(\varphi \to \alpha) \wedge (\varphi \to \beta) \vdash \varphi \to \alpha \wedge \beta$	BTr	$\varphi \to \psi \vdash \neg(\varphi \to \neg\psi)$
ECW	$(\varphi \to \psi) \vee \neg(\varphi \to \psi) \vdash (\varphi \to \chi) \vee \neg(\varphi \to \chi)$		

Definition 4 (Syntactic consequence of **S**). Given an axiomatic system **S**, we call φ a syntactic consequence of Γ, denoted $\Gamma \vdash_{\mathbf{S}} \varphi$, if there exists a finite set $\Delta \subseteq \Gamma$

such that $\Delta \vdash \varphi$ is a theorem of **S**, i.e., there exists a sequence $\sigma_1, \ldots, \sigma_n$ with $\sigma_n = \Delta \vdash \varphi$ such that for all $1 \leq i \leq n$, either σ_i is an axiom of **S** or it can be derived from $\sigma_{j_1}, \ldots, \sigma_{j_k}$ with $j_1, \ldots, j_k < i$ using the rules of **S**.

Theorem 2 (Soundness of **LC**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$*, if* $\Gamma \vdash_{\mathbf{LC}} \varphi$ *then* $\Gamma \vDash_{\mathrm{LC}} \varphi$.

Proof. It suffices to show that all the axioms are valid and the rules preserve validity. The validity of REF, CE, CI, DI, DM1, DM2, DN, and EFQ are immediate. The proof of BTr has been established in Proposition1. That MON, CUT and DIL preserve validity is also obvious. For brevity, we write $\vDash$ for $\vDash_{\mathrm{LC}}$ in the following proof. Let $\mathfrak{M} = (W, f, V^+, V^-)$ be any selection model and w any world in W.

For CC, suppose $\mathfrak{M}, w \Vdash^+ (\varphi \to \alpha) \wedge (\varphi \to \beta)$. Then we have $\mathfrak{M}, w \Vdash^+ \varphi \to \alpha$ and $\mathfrak{M}, w \Vdash^+ \varphi \to \beta$. It follows that $\varnothing \neq f(w, \varphi) \subseteq [\![\alpha]\!]$ and $\varnothing \neq f(w, \varphi) \subseteq [\![\beta]\!]$. Hence, $\varnothing \neq f(w, \varphi) \subseteq [\![\alpha]\!] \cap [\![\beta]\!] = [\![\alpha \wedge \beta]\!]$, whence $\mathfrak{M}, w \Vdash^+ \varphi \to \alpha \wedge \beta$.

For ECW, suppose $\mathfrak{M}, w \Vdash^+ (\varphi \to \psi) \vee \neg(\varphi \to \psi)$. Then $f(w, \varphi)$ is not empty. Note that for any formula χ, we have $f(w, \varphi) \subseteq [\![\chi]\!]$ or $f(w, \varphi) \nsubseteq [\![\chi]\!]$. Hence, $\mathfrak{M}, w \Vdash^+ (\varphi \to \chi) \vee \neg(\varphi \to \chi)$, as required.

For RW, suppose $\alpha \vDash \beta$. We prove that $\varphi \to \alpha \vDash \varphi \to \beta$. Suppose $\mathfrak{M}, w \Vdash^+ \varphi \to \alpha$. Then $\varnothing \neq f(w, \varphi) \subseteq [\![\alpha]\!]$. By $\alpha \vDash \beta$ we have $[\![\alpha]\!] \subseteq [\![\beta]\!]$. It follows that $\varnothing \neq f(w, \varphi) \subseteq [\![\beta]\!]$. Hence, $\mathfrak{M}, w \Vdash^+ \varphi \to \beta$. □

For the proof of completeness, we combine the proof techniques in three-valued logic ([15]) and conditional logic ([7, 14]).

Definition 5 (Saturated sets in **LC**). A set Γ is saturated in **LC**, if Γ is non-trivial, namely, $\Gamma \neq \mathcal{L}$, and the following conditions are satisfied. For all $\alpha, \beta \in \mathcal{L}$,

(cc) if $\alpha \in \Gamma$ and $\beta \in \Gamma$ then $\alpha \wedge \beta \in \Gamma$;

(dc) if $\alpha \vee \beta \in \Gamma$ then $\alpha \in \Gamma$ or $\beta \in \Gamma$;

(rw) if $\alpha \vdash_{\mathbf{LC}} \beta$ and $\alpha \in \Gamma$ then $\beta \in \Gamma$.

Lemma 2. *Let* Γ *be a saturated set in* **LC**. *Then for all* $\alpha, \beta \in \mathcal{L}$,

(re) *if* $\alpha \dashv\vdash \beta$ *then* $\alpha \in \Gamma$ *iff* $\beta \in \Gamma$;

(cc1) $\alpha \wedge \beta \in \Gamma$ *iff* $\alpha \in \Gamma$ *and* $\beta \in \Gamma$;

(dc1) $\alpha \vee \beta \in \Gamma$ *iff* $\alpha \in \Gamma$ *or* $\beta \in \Gamma$;

(cc2) $\neg(\alpha \wedge \beta) \in \Gamma$ *iff* $\neg\alpha \in \Gamma$ *or* $\neg\beta \in \Gamma$;

(dc2) $\neg(\alpha \vee \beta) \in \Gamma$ *iff* $\neg\alpha \in \Gamma$ *and* $\neg\beta \in \Gamma$.

Proof. It is obvious that (re) follows from (rw). And (cc1) and (dc1) follow from (rw) together with Axioms CE and DI. Finally, (cc2) and (dc2) follow from (re) and (cc1), together with Axioms DM1 and DM2. □

With REF, MON, CUT, CI and DIL, the following lemma can be proved in the standard way, which is omitted here.

Lemma 3 (Lindenbaum's lemma for **LC**). *If* $\Gamma \nvdash_{\mathbf{LC}} \varphi$*, then there exists a saturated set* $\Gamma' \supseteq \Gamma$ *in* **LC** *such that* $\varphi \notin \Gamma'$.

Now we define the canonical model of **LC**. We modify the constructions in [7] to make them function in our modified semantics.

Definition 6 (Canonical model of **LC**). The canonical model of **LC** is given by $\mathfrak{M}^c = (W^c, f^c, V^{c+}, V^{c-})$, where

- W^c is the set of all saturated sets in **LC**;
- $f^c(\Gamma, \varphi) = \begin{cases} \{w \in W^c \mid \Gamma_\varphi \subseteq w\} & \text{if there exists } \chi \in \mathcal{L} \text{ s.t. } (\varphi \to \chi) \vee \neg(\varphi \to \chi) \in \Gamma \\ \varnothing & \text{otherwise} \end{cases}$
- $\Gamma \in V^{c+}(p)$ iff $p \in \Gamma$
- $\Gamma \in V^{c-}(p)$ iff $\neg p \in \Gamma$

where $\Gamma_\varphi := \{\psi \mid \varphi \to \psi \in \Gamma\}$.

The following lemmas will be used for the proof of the truth lemma.

Lemma 4. *The canonical model* $\mathfrak{M}^c$ *is a selection model in* LC.

Proof. It suffices to show that the valuation functions V^{c+} and V^{c-} are well defined, i.e., for all $p \in At$, $V^{c+}(p) \cap V^{c-}(p) = \varnothing$. Suppose that there is a $\Gamma \in V^{c+}(p) \cap V^{c-}(p)$. Then $p \in \Gamma$ and $\neg p \in \Gamma$. By EFQ and (rw), it follows that $\Gamma = \mathcal{L}$, contradicting that Γ is non-trivial. □

Lemma 5. *If* $\Gamma \in W^c$*, then for all* $\varphi \in \mathcal{L}$*, the set* Γ_φ *is closed under* $\vdash_{\mathbf{LC}}$*, i.e., for all* $\psi \in \mathcal{L}$*, if* $\Gamma_\varphi \vdash_{\mathbf{LC}} \psi$ *then* $\psi \in \Gamma_\varphi$.

Proof. First we show that Γ_φ satisfies (rw). Suppose $\alpha \vdash_{\mathbf{LC}} \beta$ and $\alpha \in \Gamma_\varphi$. Then $\varphi \to \alpha \in \Gamma$. By RW and (rw) for Γ, we have $\varphi \to \beta \in \Gamma$, whence $\beta \in \Gamma_\varphi$.

Now suppose $\Gamma_\varphi \vdash_{\mathbf{LC}} \psi$. Then there exists $\psi_1, \dots, \psi_n \in \Gamma_\varphi$ such that $\psi_1, \dots, \psi_n \vdash \psi$ is a theorem of **LC**. By CE and CUT, $\psi_1 \wedge \cdots \wedge \psi_n \vdash \psi$ is also a theorem of **LC**. Note that for all $1 \leq i \leq n$, we have $\varphi \to \psi_i \in \Gamma$. It follows by (cc), CC, RW, and (rw) that $\varphi \to \psi_1 \wedge \cdots \wedge \psi_n \in \Gamma$, whence $\psi_1 \wedge \cdots \wedge \psi_n \in \Gamma_\varphi$. Then by (rw) again, we have $\psi \in \Gamma_\varphi$, as required. □

Lemma 6. *For any* $\psi \in \mathcal{L}$ *we have* $(\varphi \to \psi) \vee \neg(\varphi \to \psi) \in \Gamma$ *iff* $f^c(\Gamma, \varphi) \neq \varnothing$.

Proof. Take an arbitrary $\psi \in \mathcal{L}$.

$\Rightarrow$) Suppose $(\varphi \to \psi) \vee \neg(\varphi \to \psi) \in \Gamma$. Then $f^c(\Gamma, \varphi) = \{w \in W^c \mid \Gamma_\varphi \subseteq w\}$. It can be verified that Γ_φ is consistent. Then there exists $\chi \in \mathcal{L}$ such that $\Gamma_\varphi \nvdash_{\mathbf{LC}} \chi$. Then by Lemma 3, we have $f^c(\Gamma, \varphi) \neq \varnothing$.

$\Leftarrow$) Suppose $f^c(\Gamma, \varphi) \neq \varnothing$. Then by definition there exists $\chi \in \mathcal{L}$ such that $(\varphi \to \chi) \vee \neg(\varphi \to \chi) \in \Gamma$. Then by ECW and (rw), we have $(\varphi \to \psi) \vee \neg(\varphi \to \psi) \in \Gamma$. □

Lemma 7. *Let* $\mathfrak{M}^c = (W^c, f^c, V^{c+}, V^{c-})$ *be a canonical model of* **LC** *and* $\Gamma \in W^c$. *If* $f^c(\Gamma, \varphi) \neq \varnothing$ *then* $\bigcap f^c(\Gamma, \varphi) = \Gamma_\varphi$.

Proof. The direction $\Gamma_\varphi \subseteq \bigcap f^c(\Gamma, \varphi)$ is immediate by the definition. For the other direction, suppose $\psi \notin \Gamma_\varphi$. By Lemma 5, we have $\Gamma_\varphi \nvdash_{\mathbf{LC}} \psi$. Then by Lemma 3 and $f^c(\Gamma, \varphi) \neq \varnothing$, there exits $\Lambda \in f^c(\Gamma, \varphi)$ such that $\psi \notin \Lambda$. Hence, $\psi \notin \bigcap f^c(\Gamma, \varphi)$. □

Lemma 8 (Truth lemma for **LC**)**.** *Let* $\mathfrak{M}^c$ *be the canonical model of* **LC**. *Then for all* $\varphi \in \mathcal{L}$ *and* $\Gamma \in \mathfrak{M}^c$,

1. $\mathfrak{M}^c, \Gamma \Vdash^+ \varphi$ iff $\varphi \in \Gamma$,
2. $\mathfrak{M}^c, \Gamma \Vdash^- \varphi$ iff $\neg\varphi \in \Gamma$.

Proof. By induction on φ. The only interesting case is $\varphi = \alpha \to \beta$. We have

$\mathfrak{M}, \Gamma \Vdash^+ \alpha \to \beta$
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\mathfrak{M}^c, \Delta \Vdash^+ \beta$ for all $\Delta \in f^c(\Gamma, \alpha)$
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\beta \in \Delta$ for all $\Delta \in f^c(\Gamma, \alpha)$ (by induction hypothesis)
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\beta \in \bigcap f^c(\Gamma, \alpha)$
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\beta \in \Gamma_\alpha$ (by Lemma 7)
iff $\alpha \to \beta \in \Gamma$.

The direction from left to right of the last 'iff' is by the definition of Γ_α. For the other direction, suppose $\alpha \to \beta \in \Gamma$. By DI and (rw), we have $(\alpha \to \beta) \vee \neg(\alpha \to \beta) \in \Gamma$. Then by Lemma 6, we have $f^c(\Gamma, \alpha) \neq \varnothing$, as required.

Similarly, we have

$\mathfrak{M}, \Gamma \Vdash^- \alpha \to \beta$
iff there exists $\Delta \in f^c(\Gamma, \alpha)$ such that $\mathfrak{M}^c, \Delta \nVdash^+ \beta$
iff there exists $\Delta \in f^c(\Gamma, \alpha)$ such that $\beta \notin \Delta$ (by induction hypothesis)
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\beta \notin \bigcap f^c(\Gamma, \alpha)$
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\beta \notin \Gamma_\alpha$ (by Lemma 3)
iff $f^c(\Gamma, \alpha) \neq \varnothing$ and $\alpha \to \beta \notin \Gamma$ (by Def. of Γ_α)
iff $\neg(\alpha \to \beta) \in \Gamma$

The direction from left to right of the last 'iff' is verified as follows. Suppose $f^c(\Gamma, \alpha) \neq \varnothing$ and $\alpha \to \beta \notin \Gamma$. By the former and Lemma 6, we have $(\alpha \to$

$\beta) \vee \neg(\alpha \to \beta) \in \Gamma$. Then by $\alpha \to \beta \notin \Gamma$ and (dc), it follows that $\neg(\alpha \to \beta) \in \Gamma$. For the other direction, suppose $\neg(\alpha \to \beta) \in \Gamma$. Then by DI and (rw), $(\alpha \to \beta) \vee \neg(\alpha \to \beta) \in \Gamma$. It follows by Lemma 6 that $f^c(\Gamma, \alpha) \neq \varnothing$. By EFQ and $\Gamma \neq \mathcal{L}$, we have $\alpha \to \beta \notin \Gamma$. □

Theorem 3 (Completeness of **LC**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, *if* $\Gamma \vDash_{\mathrm{LC}} \varphi$ *then* $\Gamma \vdash_{\mathbf{LC}} \varphi$.

Proof. We prove by contrapositive. Suppose $\Gamma \nvdash_{\mathbf{LC}} \varphi$. By Lemma 3, there is a saturated set $\Gamma' \supseteq \Gamma$ in **LC** such that $\varphi \notin \Gamma'$. By the truth lemma, $\mathfrak{M}^c, \Gamma' \Vdash^+ \psi$ for all $\psi \in \Gamma$, but $\mathfrak{M}^c, \Gamma' \nVdash^+ \varphi$. Hence, $\Gamma \nvDash_{\mathrm{LC}} \varphi$. □

3.2 LC1: a proper extension with more connexivity

Now we consider more natural models, in which $f(w, \varphi)$ are required to be φ-worlds.

Definition 7 (Models for **LC1**). The class LC1 of proper selection models are those $\mathfrak{M} = (W, f, V^+, V^-)$ in LC constrained by the following condition.

(id) $\qquad f(w, \varphi) \subseteq [\![\varphi]\!]$.

The logic **LC1** is semantically defined by $\vDash_{\mathrm{LC1}}$ (see Definition 3).

Proposition 4. *The theses* wAT*,* BTr*,* Unsat1*,* Unsat2*,* wAbt *and* EFQ *in Table 1 hold in* **LC1**.

Proof. By Proposition 1, it suffices to show that wAT and Unstat1 hold in **LC1**.

For wAT, let $\mathfrak{M} = (W, f, V^+, V^-)$ be in LC1 with $w \in W$ such that $\mathfrak{M}, w \Vdash^+ (\varphi \to \psi) \vee \neg(\varphi \to \psi)$. Then $f^c(w, \varphi) \neq \varnothing$. By (id) we have $f(w, \varphi) \subseteq [\varphi]$. Hence, $\mathfrak{M}, w \Vdash^+ \varphi \to \varphi$.

For Unsat1, suppose there is a model $\mathfrak{M} = (W, f, V^+, V^-)$ in LC1 with $w \in W$ such that $\mathfrak{M}, w \Vdash^+ \varphi \to \neg\varphi$. Then $f(w, \varphi) \neq \varnothing$ and $f(w, \varphi) \subseteq [\![\neg\varphi]\!]$, contradicting (id) by Lemma 1. □

The axiomatic system of **LC1** is given by Table 3.

Table 3: Axiomatic System **LC1**

All the axioms and rules in **LC**	
wID	$(\varphi \to \psi) \vee \neg(\varphi \to \psi) \vdash \varphi \to \varphi$

As we have argued, the reason why $\varphi \to \varphi$ is not valid is that $f(w, \varphi)$ may be empty. But if $\varphi \to \psi$ or $\neg(\varphi \to \psi)$ is already true at w then $f(w, \varphi)$ is not empty.

Thus $\varphi \to \varphi$ is true at w, thanks to the model condition (id). This completes the proof of the soundness of **LC1**.

Theorem 5 (Soundness of **LC1**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$*, if* $\Gamma \vdash_{\mathbf{LC1}} \varphi$ *then* $\Gamma \vDash_{\mathrm{LC1}} \varphi$.

The canonical model of **LC1** is the same as that of **LC**, except that W^c is the set of all saturated sets in **LC1** instead of **LC**. Note that we can also define the selection function in the canonical model of **LC1** as follows, thanks to the axiom wID. We leave the verification to the reader.

$$f^c(\Gamma, \varphi) = \begin{cases} \{w \in W^c \mid \Gamma_\varphi \subseteq w\} & \text{if } \varphi \to \varphi \in \Gamma \\ \varnothing & \text{otherwise} \end{cases}$$

It is easily seen that Lemma 7 and Lemma 8 still holds for **LC1**. Now the completeness of **LC1** follows immediately from the following lemma.

Lemma 9. *The canonical model* $\mathfrak{M}^c$ *of* **LC1** *is in* LC1*, i.e.,* $\mathfrak{M}^c$ *satisfies (id).*

Proof. Obviously when $f^c(\Gamma, \varphi) = \varnothing$, (id) is satisfied. Suppose $f^c(\Gamma, \varphi) \neq \varnothing$. Then $\varphi \to \varphi \in \Gamma$, i.e., $\varphi \in \Gamma_\varphi$. It follows that for all $\Delta \in f^c(\Gamma, \varphi)$ we have $\varphi \in \Delta$. By the truth lemma for **LC1**, for all $\Delta \in f^c(\Gamma, \varphi)$ we have $\mathfrak{M}^c, \Delta \Vdash^+ \varphi$. Hence, $f^c(\Gamma, \varphi) \subseteq [\![\varphi]\!]$. □

Theorem 6 (Completeness of **LC1**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$*, if* $\Gamma \vDash_{\mathrm{LC1}} \varphi$ *then* $\Gamma \vdash_{\mathbf{LC1}} \varphi$.

3.3 LC2: a semi-Lewisian extension without DAE and CS

In this subsection, we add two more model conditions proposed by Lewis.

Definition 8 (Models for **LC2**). The class LC2 of Semi-Lewisian selection models are those $\mathfrak{M} = (W, f, V^+, V^-)$ in LC constrained by the following conditions.

(id) $f(w, \varphi) \subseteq [\varphi]$;

(wwt) if $f(w, \varphi) \neq \varnothing$, $f(w, \psi) \neq \varnothing$, $f(w, \varphi) \subseteq [\![\psi]\!]$ and $f(w, \psi) \subseteq [\![\varphi]\!]$, then $f(w, \varphi) = f(w, \psi)$;

(cmp) if $w \in [\![\varphi]\!]$ then $w \in f(w, \varphi)$.

Proposition 7. *The theses* wAT*,* BTr*,* Unsat1*,* Unsat2*,* wAbT*,* EFQ *and* MP *in Table 1 hold in* **LC2**.

Proof. By Proposition 2, it suffices to show that MP is valid. Suppose $\mathfrak{M}, w \Vdash^+ \varphi \to \psi$ and $\mathfrak{M}, w \Vdash^+ \varphi$. Then we have $f(w, \varphi) \neq \varnothing$, $f(w, \varphi) \subseteq [\![\psi]\!]$ and $w \in [\![\varphi]\!]$. By (cmp), we have $w \in f(w, \varphi)$. It follows from $f(w, \varphi) \subseteq [\![\psi]\!]$ that $w \in [\![\psi]\!]$, whence $\mathfrak{M}, w \Vdash^+ \psi$. □

Table 4: Axiomatic System **LC2**

All the axioms and rules in **LC**	
wID	$(\varphi \to \psi) \vee \neg(\varphi \to \psi) \vdash \varphi \to \varphi$
tID	$\varphi \vdash \varphi \to \varphi$
WT	$\varphi \to \psi, \psi \to \varphi, \varphi \to \chi \vdash \psi \to \chi$
MP	$\varphi, \varphi \to \psi \vdash \psi$

The axiomatic system of **LC2** is given by Table 4.

Theorem 8 (Soundness of **LC2**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, *if* $\Gamma \vdash_{\mathbf{LC2}} \varphi$ *then* $\Gamma \vDash_{\mathrm{LC2}} \varphi$.

Proof. The validity of wID and MP have already been proved in Theorem 5 and Proposition 7, respectively. Let $\mathfrak{M} = (W, f, V^+, V^-)$ be a model in LC2 and w in W.

For tID, suppose $\mathfrak{M}, w \Vdash^+ \varphi$. By (cmp), we have $f(w, \varphi) \neq \varnothing$. It follows from (id) that $f(w, \varphi) \subseteq [\![\varphi]\!]$, whence $\mathfrak{M}, w \Vdash^+ \varphi \to \varphi$.

For WT, suppose $\mathfrak{M}, w \Vdash^+ \varphi \to \psi$, $\mathfrak{M}, w \Vdash^+ \psi \to \varphi$ and $\mathfrak{M}, w \Vdash^+ \varphi \to \chi$. Then we have $f(w, \varphi) \neq \varnothing$, $f(w, \psi) \neq \varnothing$, $f(w, \varphi) \subseteq [\![\psi]\!]$, and $f(w, \psi) \subseteq [\![\varphi]\!]$. By (wwt), we have $f(w, \varphi) = f(w, \psi)$. It follows that $f(w, \psi) \subseteq [\![\chi]\!]$, whence $\mathfrak{M}, w \Vdash^+ \psi \to \chi$. □

The canonical model $\mathfrak{M}^c = (W^c, f^c, V^{c+}, V^{c-})$ of **LC2** is the same as that of **LC**, except that W^c is the set of all saturated sets in **LC2** instead of **LC**. The proof of the truth lemma for **LC2** is the same as that for **LC**. Now the completeness of **LC2** follows immediately from the following lemma.

Lemma 10. *The canonical model* $\mathfrak{M}^c$ *of* **LC2** *is in* LC2.

Proof. It suffices to verity that (id), (wwt), and (cmp) for $\mathfrak{M}^c$ are satisfied. The verification of (id) is the same as that for Lemma 9.

For (wwt), suppose $f^c(\Gamma, \varphi) \neq \varnothing$, $f^c(\Gamma, \psi) \neq \varnothing$, $f^c(\Gamma, \varphi) \subseteq [\![\psi]\!]$ and $f^c(\Gamma, \psi) \subseteq [\![\varphi]\!]$. We prove by contrapositive. Suppose $f^c(\Gamma, \varphi) \neq f^c(\Gamma, \psi)$. W.l.o.g. suppose $\Delta \notin f^c(\Gamma, \varphi)$ but $\Delta \in f^c(\Gamma, \psi)$. It follows from the former that there exists $\chi \in \Gamma_\varphi$ such that $\chi \notin \Delta$. Then $\mathfrak{M}^c, \Delta \nVdash^+ \chi$, whence $\mathfrak{M}^c, \Gamma \nVdash^+ \psi \to \chi$. It follows from the truth lemma for **LC2** that $\varphi \to \psi, \psi \to \varphi, \varphi \to \chi \in \Gamma$. By WT and (rw), we have $\psi \to \chi \in \Gamma$, contradicting $\mathfrak{M}^c, \Gamma \nVdash^+ \psi \to \chi$.

For (cmp), suppose $\Gamma \in [\![\varphi]\!]$. Then $\mathfrak{M}^c, \Gamma \Vdash^+ \varphi$. By tID, we have $\mathfrak{M}^c, \Gamma \Vdash^+ \varphi \to \varphi$. It follows that $f^c(\Gamma, \varphi) \neq \varnothing$. For all $\psi \in \Gamma_\varphi$, we have $\varphi \to \psi \in \Gamma$. By MP and (rw), we have $\psi \in \Gamma$ for all $\psi \in \Gamma_\varphi$. Then we have $\Gamma_\varphi \subseteq \Gamma$. It follows from the definition of $f^c(\Gamma, \varphi)$ that $\Gamma \in f^c(\Gamma, \varphi)$. □

Theorem 9 (Completeness of **LC2**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, *if* $\Gamma \vDash_{\mathrm{LC2}} \varphi$ *then* $\Gamma \vdash_{\mathbf{LC2}} \varphi$.

3.4 LC3: an attempt to add DAE

In contrast to (wt) and (cmp), we are unable to add Leswis' model condition (dae) directly. This disparity arises from the difference in handling $f(w, \varphi)$ when it is empty in our logic compared to Lewis' conditional logic. As a resolution, we can add (em) and (ar) for the proof of completeness.

Definition 9 (Models for **LC3**). The class LC3 of selection models are those $\mathfrak{M} = (W, f, V^+, V^-)$ in LC constrained by the following condition.

(id) $f(w, \varphi) \subseteq [\![\varphi]\!]$;

(wwt) if $f(w, \varphi) \neq \varnothing$, $f(w, \psi) \neq \varnothing$, $f(w, \varphi) \subseteq [\![\psi]\!]$, and $f(w, \psi) \subseteq [\![\varphi]\!]$, then $f(w, \varphi) = f(w, \psi)$;

(cmp) if $w \in [\![\varphi]\!]$ then $w \in f(w, \varphi)$;

(dae) $f(w, \varphi \vee \psi) \subseteq [\![\varphi]\!]$ or $f(w, \varphi \vee \psi) \subseteq [\![\psi]\!]$ or $f(w, \varphi \vee \psi) = f(w, \varphi) \cup f(w, \psi)$;

(em) $f(w, \varphi) = f(w, \psi) = \varnothing$ iff $f(w, \varphi \vee \psi) = \varnothing$;

(ar) if $f(w, \varphi) = \varnothing$ then $f(w, \varphi \vee \psi) = f(w, \psi)$.

Note that by (em) and (wwt), we have $f(w, \varphi \vee \psi) = f(w, \psi \vee \varphi)$. Hence, an variant of (ar) also holds, i.e., if $f(w, \psi) = \varnothing$ then $f(w, \varphi \vee \psi) = f(w, \varphi)$. Lewis' conditional logic does not require the conditions (em) and (ar), as they can be derived by the other conditions together with the limit assumption, i.e., if $f(w, \varphi) = \varnothing$ then $[\![\varphi]\!] = \varnothing$. Our canonical model, however, does not satisfy the limit assumption. We have to add (em) and (ar) to incorporate (dae).

The axiomatic system of **LC3** is given by Table 5.

The connective $\equiv$ above is defined as in Kleene's three-valued logic, i.e., $\varphi \equiv \psi =_{df} (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$. The original DAE axiom in Lewis' conditional logic does not hold in **LC3**, as $f(w, \varphi \vee \psi)$ can be empty. We replace DAE with its weakened form wDAE, which says that if $f(w, \varphi \vee \psi)$ is not empty then DAE holds.

Theorem 10 (Soundness of **LC3**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, *if* $\Gamma \vdash_{\mathbf{LC3}} \varphi$ *then* $\Gamma \vDash_{\mathrm{LC3}} \varphi$.

Table 5: Axiomatic System **LC3**

All the axioms and rules in **LC**	
wID	$(\varphi \to \psi) \vee \neg(\varphi \to \psi) \vdash \varphi \to \varphi$
tID	$\varphi \vdash \varphi \to \varphi$
WT	$\varphi \to \psi, \psi \to \varphi, \varphi \to \chi \vdash \psi \to \chi$
MP	$\varphi, \varphi \to \psi \vdash \psi$
EM	$(\varphi \vee \psi) \to (\varphi \vee \psi) \dashv\vdash (\varphi \to \varphi) \vee (\psi \to \psi)$
AR1	$\varphi \vee \psi \to \chi, \neg(\psi \to \chi) \vdash \varphi \to \varphi$
AR2	$\neg(\varphi \vee \psi \to \chi), \psi \to \chi \vdash \varphi \to \varphi$
wDAE	$\varphi \vee \psi \to \varphi \vee \psi \vdash$
	$(\varphi \vee \psi \to \varphi) \vee (\varphi \vee \psi \to \psi) \vee ((\varphi \vee \psi \to \chi) \equiv ((\varphi \to \chi) \wedge (\psi \to \chi)))$

Proof. The validity of wID, tID, WT, and MP have already been proved in Theorem 8. Let $\mathfrak{M} = (W, f, V^+, V^-)$ be a model in LC3 and w in W.

For EM, we have $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi \to \varphi \vee \psi$

iff $f(w, \varphi \vee \psi) \neq \varnothing$ (by (id))

iff $f(w, \varphi) \neq \varnothing$ or $f(w, \psi) \neq \varnothing$ (by (em))

iff $\mathfrak{M}, w \Vdash^+ (\varphi \to \varphi) \vee (\psi \to \psi)$ (by (id))

Hence, $(\varphi \vee \psi \to \chi) \vDash ((\varphi \to \chi) \wedge (\psi \to \chi))$ and $((\varphi \to \chi) \wedge (\psi \to \chi)) \vDash (\varphi \vee \psi \to \chi)$, as required.

For AR1, suppose $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi \to \chi$ and $\mathfrak{M}, w \Vdash^+ \neg(\psi \to \chi)$. Then we have $f(w, \varphi \vee \psi) \neq \varnothing$, $f(w, \psi) \neq \varnothing$, and $f(w, \varphi \vee \psi) \neq f(w, \psi)$. By (ar), we have $f(w, \varphi) \neq \varnothing$. It follows from (id) that $\mathfrak{M}, w \Vdash^+ \varphi \to \varphi$. The same reasoning can be applied to AR2.

For wDAE, suppose $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi \to \varphi \vee \psi$, $\mathfrak{M}, w \nVdash^+ \varphi \vee \psi \to \varphi$ and $\mathfrak{M}, w \nVdash^+ \varphi \vee \psi \to \psi$. By the former two, there exists $v \in f(w, \varphi \vee \psi)$ such that $v \notin [\![\varphi]\!]$, whence $f(w, \varphi \vee \psi) \nsubseteq [\![\varphi]\!]$. Moreover, we have $f(w, \varphi \vee \psi) \neq f(\varphi)$ (Otherwise, we would have $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi \to \varphi$ by (id)). Then by (ar), we have $f(w, \psi) \neq \varnothing$. Similarly, we have $f(w, \varphi \vee \psi) \nsubseteq [\![\psi]\!]$ and $f(w, \varphi) \neq \varnothing$. By (dae), we have $f(w, \varphi \vee \psi) = f(w, \varphi) \cup f(w, \psi)$.

Then for any formula χ, we have $\mathfrak{M}, w \Vdash^+ \varphi \vee \psi \to \chi$

iff $f(w, \varphi \vee \psi) \subseteq [\![\chi]\!]$ (as $f(w, \varphi \vee \psi) \neq \varnothing$)

iff $f(w,\varphi) \subseteq [\![\chi]\!]$ and $f(w,\psi) \subseteq [\![\chi]\!]$ (by $f(w,\varphi \vee \psi) = f(w,\varphi) \cup f(w,\psi)$)
iff $\mathfrak{M}, w \Vdash^{+} (\varphi \to \chi) \wedge (\psi \to \chi)$.
For any formula χ, we also have $\mathfrak{M}, w \Vdash^{+} \neg(\varphi \vee \psi \to \chi)$
iff $\mathfrak{M}, w \Vdash^{-} \varphi \vee \psi \to \chi$
iff $f(w,\varphi \vee \psi) \nsubseteq [\![\chi]\!]$ (as $f(w,\varphi \vee \psi) \neq \varnothing$)
iff $f(w,\varphi) \nsubseteq [\![\chi]\!]$ or $f(w,\psi) \nsubseteq [\![\chi]\!]$ (by $f(w,\varphi \vee \psi) = f(w,\varphi) \cup f(w,\psi)$)
iff $\mathfrak{M}, w \Vdash^{+} \neg(\varphi \to \chi)$ or $\mathfrak{M}, w \Vdash^{+} \neg(\psi \to \chi)$. □

The canonical model $\mathfrak{M}^c = (W^c, f^c, V^{c+}, V^{c-})$ of **LC3** is the same as that of **LC**, except that W^c is the set of all saturated sets in **LC3** instead of **LC**. The proof the truth lemma for **LC3** is the same as that for **LC**. Now the completeness of **LC3** follows immediately from the following lemma.

Lemma 11. *The canonical model* $\mathfrak{M}^c$ *of* **LC3** *is in* LC3.

Proof. It suffices to verity that (id), (wwt), (cmp), (em), (ar), and (dae) for $\mathfrak{M}^c$ are satisfied. The verification of (id), (wwt), and (cmp) is the same as that for Lemma 10.

For (em), we have $f^c(\Gamma, \varphi \vee \psi) \neq \varnothing$
iff $\varphi \vee \psi \to \varphi \vee \psi \in \Gamma$
iff $(\varphi \to \varphi) \vee (\psi \to \psi) \in \Gamma$ (by EM and (rw))
iff $\varphi \to \varphi \in \Gamma$ or $\psi \to \psi \in \Gamma$ (by (dc1))
iff $f^c(\Gamma, \varphi) \neq \varnothing$ or $f^c(\Gamma, \psi) \neq \varnothing$.

For (ar), suppose $f^c(\Gamma, \varphi \vee \psi) \neq f^c(\Gamma, \psi)$, we prove $f^c(\Gamma, \varphi) \neq \varnothing$. By (em) we have $f^c(\Gamma, \varphi \vee \psi) \neq \varnothing$. If $f^c(\Gamma, \psi) = \varnothing$, by (em) we have $f^c(\Gamma, \varphi) \neq \varnothing$. If $f^c(\Gamma, \psi) \neq \varnothing$, by $f^c(\Gamma, \varphi \vee \psi) \neq f^c(\Gamma, \psi)$ we have $\Gamma_{\varphi \vee \psi} \neq \Gamma_{\psi}$. Suppose there exists $\chi \in \Gamma_{\varphi \vee \psi}$ but $\chi \notin \Gamma_{\psi}$. It follows that $\varphi \vee \psi \to \gamma \in \Gamma$ and $\psi \to \gamma \notin \Gamma$. By $f^c(\Gamma, \psi) \neq \varnothing$, we have $(\psi \to \gamma) \vee \neg(\psi \to \gamma) \in \Gamma$. By (dc) and $\psi \to \gamma \notin \Gamma$, we have $\neg(\psi \to \gamma) \in \Gamma$. It follows from AR1 that $\varphi \to \varphi \in \Gamma$. Hence, $f^c(\Gamma, \varphi) \neq \varnothing$. The case for $\chi \in \Gamma_{\psi}$ and $\chi \notin \Gamma_{\varphi \vee \psi}$ can be obtained by AR2 in the same way.

For (dae), if $f^c(\Gamma, \varphi \vee \psi) = \varnothing$, then obviously $f^c(\Gamma, \varphi \vee \psi) \subseteq [\![\varphi]\!]$. Suppose $f^c(\Gamma, \varphi \vee \psi) \neq \varnothing$, $f^c(\Gamma, \varphi \vee \psi) \nsubseteq [\![\varphi]\!]$, and $f^c(\Gamma, \varphi \vee \psi) \nsubseteq [\![\psi]\!]$, then we have $\varphi \vee \psi \to \varphi \notin \Gamma$ and $\varphi \vee \psi \to \psi \notin \Gamma$. It follows from wDAE and (rw) that $(\varphi \vee \psi \to \chi) \equiv ((\varphi \to \chi) \wedge (\psi \to \chi)) \in \Gamma$ for any formula χ. If $f^c(\Gamma, \varphi)$ were empty, we would have $f^c(\Gamma, \varphi \vee \psi) = f^c(\Gamma, \psi)$ by (ar). Then we have $f^c(\Gamma, \varphi \vee \psi) \subseteq [\![\psi]\!]$, contrary to our claim. Hence, both $f^c(\Gamma, \varphi)$ and $f^c(\Gamma, \psi)$ are not empty.

To prove $f^c(\Gamma, \varphi \vee \psi) = f^c(\Gamma, \varphi) \cup f^c(\Gamma, \psi)$, suppose $\Delta \in f^c(\Gamma, \varphi) \cup f^c(\Gamma, \psi)$. Then $\Gamma_{\varphi} \subseteq \Delta$ or $\Gamma_{\psi} \subseteq \Delta$. Hence, for all $\chi \in \mathcal{L}$, if $\varphi \to \chi \in \Gamma$ then $\chi \in \Delta$, or if $\psi \to \chi \in \Gamma$ then $\chi \in \Delta$ (using the inference from $\forall x \alpha \vee \forall x \beta$ to $\forall x(\alpha \vee \beta)$). Then for all $\chi \in \mathcal{L}$, if $\varphi \to \chi \in \Gamma$ and $\psi \to \chi \in \Gamma$ then $\chi \in \Delta$ (using the equivalence between $\alpha \wedge \beta \supset \gamma$ and $(\alpha \supset \gamma) \vee (\beta \supset \gamma)$, where $\supset$ is material implication). Since $(\varphi \vee \psi \to \chi) \equiv ((\varphi \to \chi) \wedge (\psi \to \chi)) \in \Gamma$, it follows that for all $\chi \in \mathcal{L}$, if

$\varphi \vee \psi \to \chi \in \Gamma$ then $\chi \in \Delta$, i.e., for all $\chi \in \Gamma_{\varphi\vee\psi}$ we have $\chi \in \Delta$. Then $\Gamma_{\varphi\vee\psi} \subseteq \Delta$ and thus $\Delta \in f^c(\Gamma, \varphi \vee \psi)$.

For the other direction, suppose $\Delta \notin f^c(\Gamma, \varphi) \cup f^c(\Gamma, \psi)$. Then $\Gamma_\varphi \nsubseteq \Delta$ and $\Gamma_\psi \nsubseteq \Delta$, i.e., there exist $\alpha \in \Gamma_\varphi$ and $\beta \in \Gamma_\psi$ such that, $\alpha \notin \Delta$ and $\beta \notin \Delta$. Then we have $\varphi \to \alpha \in \Gamma$, $\psi \to \beta \in \Gamma$, $\alpha \notin \Delta$, and $\beta \notin \Delta$. Note that Δ is a saturated set. Then we have $\varphi \to \alpha \vee \beta \in \Gamma$, $\psi \to \alpha \vee \beta \in \Gamma$, and $\alpha \vee \beta \notin \Delta$, using the rule RW. Since $(\varphi \vee \psi \to \chi) \equiv ((\varphi \to \chi) \wedge (\psi \to \chi)) \in \Gamma$, it follows that $\varphi \vee \psi \to \alpha \vee \beta \in \Gamma$ and $\alpha \vee \beta \notin \Delta$. Thus $\Gamma_{\varphi\vee\psi} \nsubseteq \Delta$, i.e., $\Delta \notin f^c(\Gamma, \varphi \vee \psi)$, as required. □

Theorem 11 (Completeness of **LC3**). *For all* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, *if* $\Gamma \vDash_{\mathrm{LC3}} \varphi$ *then* $\Gamma \vdash_{\mathbf{LC3}} \varphi$.

4 Related Works

There have been a lot of connexive logics in the literature. Before [16], connexive logics are usually obtained by adding more constraints on the truth condition of standard implications, including material implication, strict implication, and relevant implication (e.g., [1, 11, 12]), which makes the semantics rather cumbersome and unbalanced between truth and falsity conditions. Wansing's logic **C** ([16]), based on strict implication, tweaks instead the falsity condition of conditionals. In **C**, a conditional $\varphi \to \psi$ is false iff all accessible φ-worlds falsify ψ, whereas the standard falsity condition is that some accessible φ-worlds falsify ψ. The connexive logics proposed in [9], [5], and [18] are based on this falsity tweaking strategy. Apart from these two strategies, Wen ([19]) proposed a semantics for connexive logic based on Stalnaker's semantics using partial selection functions, which does not change the truth or falsity condition of conditionals but only puts some preconditions for conditionals. Our logic mainly follows the truth tweaking strategy and in some sense generalizes the semantics in [19].

4.1 Lewis' doctored counterfactuals

Apart from the standard semantics for conditionals, Lewis ([8, p. 438]) had considered defining a counterfactual $\varphi \to \psi$ by $\Diamond\varphi \wedge (\varphi > \psi)$, where $>$ is standard Lewisian conditional implication. The semantics was later discussed in [10] and rediscovered in [2]. For convenience, we denote by **DC** this new logic for conditionals. It is easily seen that **DC** belongs to the truth tweaking strategy. As $\Diamond\varphi$ in basic conditional logics can be defined by $\neg(\varphi > \neg\varphi)$, it follows that $\varphi \to \psi$ is just $\neg(\varphi > \neg\varphi) \wedge (\varphi > \psi)$. So Aristotle's thesis is already "written" in **DC**. More precisely, in **DC** $\varphi \to \psi$ is true at w iff all the closest φ-worlds are ψ-worlds and φ is possible at w. The latter requirement in the framework of selection models means that $f(w, \varphi)$ is not empty. So the truth condition of conditionals in our logics is essentially the same as that in **DC**. The falsity condition of conditionals in our logics, however, is different from **DC**. When $f(w, \varphi)$ is empty, our semantics does not make $\varphi \to \psi$ for any ψ to be false at w, whereas in **DC** all such conditionals are false, as it is a two-valued logic. As a result, even $\varphi \to \varphi$ could be false in **DC**, whereas in our logics, though $\varphi \to \varphi$ is not valid, it cannot be false as long as the model condition (id) is imposed. This makes our semantics more intuitive than **DC**.

4.2 Wansing and Unterhuber's connexive conditional logics

Following the falsity tweaking strategy, Wansing and Unterhuber ([18]) constructed several connexive logics by combing FDE with Chellas-Segerberg semantics for conditionals. In their semantics, a conditional is true if its consequent is true at all accessible worlds, and false if its consequent is false at *all* accessible worlds. To obtain connexivity, the falsity condition of conditionals was modified from an existential condition in standard conditional logics to a universal one. Our logics are quite different from their logics, which validate AT and BT, whereas our logics only validate weakened forms of AT and BT. On the other hand, their logics do not validate Unsat1, Unsat 2, and (w)AbT, whereas our logics except **LC** validate both Unsat1 and Unsat2, and a weakened form of AbT. The difference may be expected, as our logics follow the truth tweaking strategy, whereas theirs follow the falsity tweaking strategy. Another difference is that their logics are both paracomplete (invalidating LEM) and paraconsistent (invalidating EFQ), whereas ours are only paracomplete. This is also expected, as their logics are based on four-valued setting, whereas ours are based on three-valued setting.

4.3 Stalnakerian connexive logics

Combing Kleene's three-valued logic and Stalnaker's semantics for conditionals, Wen ([19]) proposed a natural semantics for connexive logic. In the new semantics, the selection function for selecting the closest world for evaluating conditionals can be undefined. Truth and falsity conditions for conditionals are then supplemented with a precondition that the selection function is defined. This partial function plus precondition strategy not only balances truth and falsity conditions but also renders the change of semantics of standard implication as minimal as possible. The prominent difference between Wen's logics and ours is that the former validate both BTr and its converse wCEM, whereas our logics does not validate wCEM. Though wCEM has been supported recently (see, e.g., [20] and [13]), it is not without controversy. On the other hand, Wen's logics lack ECW, as it requires the consequent to be false in the closest world for a conditional to be false. If we take the alternative falsity condition for conditionals discussed above, then our new semantics can be regarded as a generalization of Wen's semantics.

5 Conclusion

We propose a new natural semantics for connexive logic by combing Kleene's three-valued logic and Lewis' semantics for conditionals. In the new semantics for a conditional $\varphi \to \psi$ to be true, we require not only the closest φ-worlds to be ψ-worlds but also the closest φ-worlds to exist. We give four axiomatic systems for difference classes of models in the semantics and prove soundness and completeness of them. Two axioms for weakened forms of AT and AbT we proposed are supposed to be new in the literature. Our semantics mainly follows the truth tweaking strategy but is more intuitive and in some sense generalizes the semantics in [19].

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Lewisian 连接逻辑

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摘　要

在连接逻辑中，有两个核心观点：第一，任何命题既不蕴涵其否定，也不被其否定所蕴涵；第二，如果一个命题蕴涵 φ，则它不会蕴涵 φ 的否定。而在经典逻辑中，这两个观点均不成立，这使得为连接逻辑提供一种自然的语义面临困难。通过结合 Kleene 的三值逻辑与 Lewis 的条件句逻辑，我们提出了一种自然的连接逻辑语义。我们给出了四个公理系统，用于刻画新语义中不同类别的选择模型。我们证明了这些逻辑的可靠性与完全性，并与一些常见的连接逻辑进行了比较。

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