Studies in Logic, Vol. 18, No. 3 (2025): 1–24
PII: 1674-3202(2025)-03-0001-24

On Modal Logics of Subset Spaces*

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Abstract. In modal logic, topological semantics is an intuitive and natural special case of neighbourhood semantics. This paper stems from the observation that the satisfaction relation of topological semantics applies to subset spaces which are more general than topological spaces. The minimal modal logic which is strongly sound and complete with respect to the class of subset spaces is found. Soundness and completeness results of some famous modal logics (e.g. **S4**, **S5** and **Tr**) with respect to various important classes of subset spaces (e.g. intersection structures and complete fields of sets) are also proved. In the meantime, some known results, e.g. the soundness and completeness of **Tr** with respect to the class of discrete topological spaces, are proved directly using some modifications of the method of canonical model, without a detour via neighbourhood semantics or relational semantics.

1 Introduction

Neighbourhood semantics is a general semantics of modal logic. ([4, 10]) In a neighbourhood structure, to each possible world[1] a set of neighbourhoods is assigned, where a neighbourhood of the world is a set of possible worlds. And the only modal law in the minimal modal logic of neighbourhood semantics is replacement of logical equivalence. Neighbourhood semantics is a fine-grained tool in studying modal logic; however, arguably it is also too abstract. There are two famous natural and intuitive special cases of neighbourhood semantics. One is relational semantics. ([8]) In a relational structure, to each possible world a singleton consisting of a set of possible worlds is assigned. Hence a neighbourhood assignment boils down to a binary relation between possible worlds which admits many intuitive interpretations from alethic, epistemic, deontic and many other perspectives. The other is topological semantics.

Received 2023-02-16

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*The results in this paper are all from my Master's thesis ([12]) finished in 2011 at Institute of Logic and Cognition, Department of Philosophy, Sun Yat-sen University. In this paper, the term "subset frame" is changed into "subset space" in standard terminology, and many proofs are streamlined and made much more succinct. I'm very grateful to my supervisor, Prof. Hu Liu, for his supervision and help. I also thank very much the committee and the audience in the defense, especially Prof. Xuefeng Wen, for the helpful discussion. I'm very grateful to the reviewer for the helpful comments. The writing of this paper is supported by the National Social Science Fund of China (No. 20CZX048).

[1]Strictly speaking, the set underlying a mathematical structure can be any non-empty set, regardless of what are its elements. Here, to be intuitive, we call the elements of this set possible worlds.

([9]) In a topological structure, to each possible world the set of its neighbourhoods has to satisfy some constraints, one of which is that a possible world must be in each one of its neighbourhoods. (For the details, please refer to Definition 1.28 in [10].) These constraints make a neighbourhood assignment correspond to a topology on the set of possible worlds. While relational semantics is widely used in philosophical logic and computer science, topological semantics is mainly used in spatial logic and epistemic logic. ([2])

In topological semantics, a formula $\Box\varphi$ is satisfied at a possible world, if and only if there is a neighbourhood of the possible world such that φ is satisfied at each possible world in the neighbourhood. We observe that this definition of the satisfaction relation has nothing to do with the constraints on neighbourhood assignments except for the one explicitly mentioned above.[2] Hence it works for mathematical structures more general than topological spaces, which turn out to be subset spaces. A subset space consists of a non-empty set and a set of subsets of the set which here I call open sets borrowing the terminology from topology. A topological space is a subset space where the set of open sets includes the non-empty set itself and is closed under finite intersection and arbitrary union. A subset space can also be considered as a neighbourhood structure where the neighbourhoods of a possible world are the open sets containing the world. Therefore, using the definition of the satisfaction relation in topological semantics, modal language can in fact describe subset spaces; and the resulting semantics is more general than topological semantics but is still a special case of neighbourhood semantics.

In this paper, starting from the above observation, we investigate two basic questions about modal logics of subset spaces. First, we determine the minimal logic that is strongly sound and complete with respect to the class of subset spaces. Without surprise, this logic is stronger than the minimal logic of neighbourhood semantics and weaker than that of topological semantics, and it is incomparable with that of relational semantics. Second, we establish soundness and completeness results between some important classes of subset spaces (e.g. topological spaces, complete fields of sets and power set algebra) and some famous modal logics (e.g. **S4**, **S5** and **Tr**). Some of these results are already known. However, here we adapt the method of canonical model in topological semantics and devise some modifications of it which are natural and tailored to our semantics to directly prove the results in detail, instead of taking detours to neighbourhood semantics or relational semantics. As a result, this paper can be considered as a technical remark on topological semantics of modal logic.

To be more precise, first, based on the notion of (subset space) models, the semantics notions of the satisfaction relation, semantics consequence and (subset space) bisimulations are literally the same as those in topological semantics. What is impor-

[2] Arguably, this constraint should be satisfied, because a possible world should be “closed to” itself and thus in any “neighbourhood” of it.

tant here is that, after the notion of models behind is generalized, these semantics notions still make sense. Second, the frame correspondence results in this paper are in principle but not straightforward consequences of those in neighbourhood semantics. Their specializations in topological semantics are also mentioned in the literature. ([2]) Here we prove them from scratch for the convenience of the readers to get a feeling of how the semantics describes subset spaces. Third, the definition of the canonical model is adapted from that in the topological semantics in such a way that, for **S4**, the canonical model in the literature (e.g. [1]) is the union closure of the canonical model here. Here is a point worth mentioning: from our abstract framework, we see that union closure, which is the operation that generates a topology from a basis of it, preserves the satisfaction relation. This point is bypassed in topological semantics because topological spaces are defined to be closed under arbitrary union. Moreover, we find two modal logics and prove that they are strongly sound and complete with respect to the class of subset spaces and that of intersection structures, respectively, which are both stronger than the minimal logic of neighbourhood semantics and weaker than **S4**, the modal logic of topological spaces. Finally, the method of using intersection-closed canonical model in proving completeness is not widely mentioned and discussed in the literature. Here it is used to prove the completeness results for some important classes of mathematical structures like intersection structures, complete lattices of sets and complete fields of sets. In a word, we think that the satisfaction relation in topological semantics is apt to describe subset spaces, i.e. sets each equipped with a collection of its subsets, and we study this per se in this paper. In contrast, neighbourhood semantics is too general to see exactly how this satisfaction relation describes subset spaces, while topological semantics is too specialized and thus bypasses some subtleties.

Recently modal logics of hypergraphs have been investigated by Ding, Liu and Wang in their conference paper ([6]) and its extended journal version ([7]). A hypergraph is just a subset space where the empty subset is always excluded, and the satisfaction relation for the unary modal operator in their paper is the same as that in this paper. Hence their work is closely related to ours. However, there are also some differences between the two works. The first difference is about motivation. Their motivation is from weak aggregative modal logic and epistemic logic concerning local reasoning, evidence or "someone knows". Thus they use the modal language to describe some combinatorial properties of hypergraphs such as n-boundedness and being non-n-colorable. In this paper, what we hope to describe is some important subset spaces such as intersection structures and complete fields of sets. The second difference is about formal language. The language in this paper is not strong enough to express the combinatorial properties in their paper. Thus the formal language in their paper has in addition the universal modality. The third difference is about the definition of canonical models. Due to the strong expressive power of their language and the combinatorial properties under study, canonical models in their paper are defined in a much more subtle way. Roughly speaking, in their completeness proofs they hope that canonical models do not contain too many subsets of the underlying sets. In contrast, in this paper the subset spaces under consideration satisfy many closure conditions, so what we do here is to add enough subsets to canonical models while preserving the satisfaction relation. In particular, in their papers they prove that the logic $\mathrm{MT4}$ is strongly sound and complete with respect to the class of hypergraphs, but the language has the universal modality, the logic has axioms characterizing the universal modality and the canonical model contains much less subsets than that in this paper.

The rest of this paper is organized as follows. In Section 2 we set up the formal language, modal logics and formal semantics used in this paper. Section 3 introduces the notion of subset space bisimulations by adapting the notion of topological bisimulations, which is a useful tool in proving completeness. In Section 4 we prove four correspondence results to exemplify the expressive power of our semantics and to facilitate the proofs in the following section. Section 5 proves some soundness theorems using the results in the previous section. In Section 6 we prove some completeness theorems using the canonical models and some simple transformations of them. Section 7 introduces intersection-closed canonical models and use them to prove some completeness theorems of modal logics containing the axiom $\Box p \land \Box q \to \Box(p \land q)$. Section 8 summarizes the results proved in this paper.

2 Basic Definitions

2.1 Syntax

In this paper, we fix a countable set $\mathbf{P} = \{p_i \mid i \in \omega\}$ as the set of propositional letters. We will consider only one formal language, that is, the standard modal language with exactly one unary modal operator $\Box$.

Definition 1 (Formula)**.** The notion of *formulas* is defined in the Backus-Naur form as follows:

$$\varphi ::= p_i \mid \neg\varphi \mid (\varphi \land \varphi) \mid \Box\varphi,\ i \in \omega$$

Denote the set of formulas by $Form$.

The propositional connectives $\bot$, $\lor$, $\to$ and $\leftrightarrow$ and the modal operator $\Diamond$ are defined as usual. We use p, q, etc. as metavariables of propositional letters. Moreover, we may omit the parentheses according to the usual conventions.

The following are the modal formulas studied in this paper; they and their names are common in the literature:

(M)	$\Box(p \wedge q) \to \Box p \wedge \Box q$	(C)	$\Box p \wedge \Box q \to \Box(p \wedge q)$
(T)	$\Box p \to p$	(Tr)	$p \to \Box p$
(4)	$\Box p \to \Box\Box p$	(5)	$\Diamond p \to \Box\Diamond p$

The following are the rules studied in this paper; they and their names are common in the literature:

(MP) given φ and $\varphi \to \psi$, prove ψ;
(US) given φ, prove φ^σ, where φ^σ is the (uniform) substitution instance of φ under the substitution $\sigma : \mathbf{P} \to Form$;
(RM) given $\varphi \to \psi$, prove $\Box\varphi \to \Box\psi$;
(RN) given φ, prove $\Box\varphi$.

Definition 2 (Modal Logic)**.** A *modal logic* is a subset of $Form$ containing all propositional tautologies and closed under (MP) and (US).

We will focus on the following five modal logics, where the latter three are common in the literature while the former two may not:

1. **S** is the smallest modal logic containing the modal formulas (M), (T) and (4) and closed under (RM);
2. **S**$^+$ is the smallest modal logic containing the modal formulas (M), (T) and (4), as well as (C), and closed under (RM);
3. **S4** is the smallest modal logic containing the modal formulas (M), (T) and (4), as well as (C), and closed under (RM), as well as (RN);
4. **S5** is the smallest modal logic containing the modal formulas (M), (T) and (4), as well as (C) and (5), and closed under (RM), as well as (RN);
5. **Tr** is the smallest modal logic containing the modal formulas (M), (T) and (4), as well as (C), (5) and (Tr), and closed under (RM), as well as (RN).

Since any (set-theoretic) intersection of modal logics is a modal logic, the word "smallest" above means the same as "intersection of".

Our definitions of **S4**, **S5** and **Tr** are different but equivalent to the usual definitions in the literature. An analysis can be found in Sections 1.2 and 1.3 in [11].

Definition 3 (Extensions of Modal Logics)**.** Let Λ and Λ' be two modal logics defined by characteristic formulas and rules. Λ' is an *extension* of Λ, denoted by $\Lambda \sqsubseteq \Lambda'$, if every characteristic formula in Λ is in Λ' and Λ' is closed under every characteristic rule under which Λ is closed.

Remark 4. $\Lambda \sqsubseteq \Lambda'$ implies $\Lambda \subseteq \Lambda'$, but not vice versa.

Definition 5. Let Λ be a modal logic and $\Gamma \cup \{\varphi\} \subseteq Form$.

1. φ is a *syntactic consequence* of Γ in Λ, denoted by $\Gamma \vdash_\Lambda \varphi$, if there is a *finite* set $\Gamma' \subseteq \Gamma$ such that $\bigwedge \Gamma' \to \varphi \in \Lambda$.
   We write $\vdash_\Lambda \varphi$ for $\varnothing \vdash_\Lambda \varphi$.
2. Γ is Λ-*consistent*, if $\Gamma \nvdash_\Lambda \bot$.
3. Γ is a *maximal* Λ-*consistent set (*Λ-***MCS****)*, if Γ is Λ-consistent and any proper superset of Γ is not Λ-consistent.

For a modal logic Λ, the notions of syntactic consequence in Λ, Λ-consistent and maximal Λ-consistent set are the same as usual, e.g. Definitions 4.4 and 4.15 in [3]. Hence Proposition 4.16 and Lemma 4.17 in [3] can be applied in this paper, since they do not involve modal reasoning at all.

2.2 Semantics

We use subset spaces to interpret our formal language.

Definition 6 (Subset Space)**.** A *subset space* is an ordered pair $\mathcal{H} = (W, \mu)$, where

1. W is a non-empty set;
2. $\mu \subseteq \wp(W)$, where $\wp$ is the power set operator.

Then we can define the notion of a (subset space) model.

Definition 7 ((Subset Space) Model)**.** A *(subset space) model* is an ordered pair $\mathcal{M} = (\mathcal{H}, V)$, where

1. $\mathcal{H} = (W, \mu)$ is a subset space;
2. V is a function from **P** to $\wp(W)$.

Next we define the relation of satisfaction.

Definition 8 (Satisfaction)**.** Let $\mathcal{M} = (\mathcal{H}, V)$ be a model, where $\mathcal{H} = (W, \mu)$, $w \in W$ and $\varphi \in Form$, we define the *satisfaction relation*, denoted by $\mathcal{M}, w \Vdash \varphi$, by recursion as follows:

- $\mathcal{M}, w \Vdash p_i$, if and only if $w \in V(p_i)$, for each $i \in \omega$;
- $\mathcal{M}, w \Vdash \neg\varphi$, if and only if $\mathcal{M}, w \nVdash \varphi$;
- $\mathcal{M}, w \Vdash \varphi \wedge \psi$, if and only if $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$;
- $\mathcal{M}, w \Vdash \Box\varphi$, if and only if there is a $U \in \mu$ such that $w \in U$ and, for each $u \in U$, $\mathcal{M}, u \Vdash \varphi$.

For $\Gamma \subseteq Form$, we write $\mathcal{M}, w \Vdash \Gamma$, if $\mathcal{M}, w \Vdash \varphi$ is true for each $\varphi \in \Gamma$.

Moreover, we denote by $\|\varphi\|_\mathcal{M}$ the set $\{w \in W \mid \mathcal{M}, w \Vdash \varphi\}$. When $\mathcal{M}$ is clear from the context, we may omit the subscript. Then $\mathcal{M}, w \Vdash \Box\varphi$, if and only if there is a $U \in \mu$ such that $w \in U$ and $U \subseteq \|\varphi\|$.

Remark 9. The satisfaction relation defined above is the same as that in the topological semantics of modal logic. Behind this coincidence lies the observation that the definition of this relation still make sense even if the mathematical structures involved are not topological spaces.

Remark 10. By definition $\mathcal{M}, w \Vdash \Diamond\varphi$, if and only if, for each $U \in \mu$, if $w \in U$, then there is a $u \in U$ such that $\mathcal{M}, u \Vdash \varphi$; in other words, if and only if, for each $U \in \mu$, if $w \in U$, then $U \cap \|\varphi\| \neq \varnothing$.

Now the notions of validity and logical consequence can be defined in the usual pattern.

Definition 11 (Validity)**.** Let $\mathcal{H} = (W, \mu)$ be a subset space, $\mathbb{H}$ a class of subset spaces and $\varphi \in Form$.

1. φ is *valid* on $\mathcal{H}$, denoted as $\mathcal{H} \Vdash \varphi$, if $(\mathcal{H}, V), w \Vdash \varphi$ is true for any $w \in W$ and function V from **P** to $\wp(W)$.
2. φ is *valid* on $\mathbb{H}$, if $\mathcal{H} \Vdash \varphi$ is true for each $\mathcal{H} \in \mathbb{H}$.

Definition 12 (Logical Consequence)**.** Let $\mathbb{H}$ be a class of subset spaces and $\Gamma \cup \{\varphi\} \subseteq Form$. φ is a *logical consequence* of Γ with respect to $\mathbb{H}$, denoted by $\Gamma \Vdash_{\mathbb{H}} \varphi$, if, for any model $\mathcal{M} = ((W, \mu), V)$ with $(W, \mu) \in \mathbb{H}$ and $w \in W$, $\mathcal{M}, w \Vdash \Gamma$ implies $\mathcal{M}, w \Vdash \varphi$.

Classes of subset spaces are usually specified by properties of subset spaces. We mainly focus on the following properties of a subset space $\mathcal{H} = (W, \mu)$:

$(Bound)$	$\varnothing \in \mu$ and $W \in \mu$
$(2\bigcup)$	for any $U, V \in \mu, U \cup V \in \mu$
$(\omega\bigcup)$	for any $\{U_i \mid i \in \omega\} \subseteq \mu, \bigcup_{i\in\omega} U_i \in \mu$
$(\bigcup)$	for any set I and $\{U_i \mid i \in I\} \subseteq \mu, \bigcup_{i\in I} U_i \in \mu$
$(2\bigcap)$	for any $U, V \in \mu, U \cap V \in \mu$
$(\omega\bigcap)$	for any $\{U_i \mid i \in \omega\} \subseteq \mu, \bigcap_{i\in\omega} U_i \in \mu$
$(\bigcap)$	for any non-empty set I and $\{U_i \mid i \in I\} \subseteq \mu$, $\bigcap_{i\in I} U_i \in \mu$
(Com)	for any $U \in \mu, W \setminus U \in \mu$
(Up)	for any $U, V \in \mu$, if $U \in \mu$ and $U \subseteq V \subseteq W$, then $V \in \mu$
$(Down)$	for any $U, V \in \mu$, if $U \in \mu$ and $V \subseteq U \subseteq W$, then$V \in \mu$

$(Pow) \quad \mu = \wp(W)$

Some classes of subset spaces are well-known in the literature ([5]):

Definition 13. Let $\mathcal{H} = (W, \mu)$ be a subset space.

1. $\mathcal{H}$ is an *intersection structure*, if μ satisfies $(\bigcap)$.
2. $\mathcal{H}$ is a *topped intersection structure*, if μ contains W and satisfies $(\bigcap)$.
3. $\mathcal{H}$ is a *topological space*, if μ satisfies $(Bound)$, $(2\bigcap)$ and $(\bigcup)$.
4. $\mathcal{H}$ is a *discrete topological space*, if $\mu = \wp(W)$.

Moreover, here we abuse the terminologies a bit; we call a subset space $\mathcal{H} = (W, \mu)$ a kind of algebraic structure, if μ is the underlying set of an algebraic structure of this kind.

1. $\mathcal{H}$ is a *lattice of sets*, if μ satisfies $(Bound)$, $(2\bigcap)$ and $(2\bigcup)$.
2. $\mathcal{H}$ is a *σ-lattice of sets*, if μ satisfies $(Bound)$, $(\omega\bigcap)$ and $(\omega\bigcup)$.
3. $\mathcal{H}$ is a *complete lattice of sets/Alexandroff topological space*, if μ satisfies $(Bound)$, $(\bigcap)$ and $(\bigcup)$.
4. $\mathcal{H}$ is a *field of sets*, if μ satisfies $(Bound)$, $(2\bigcap)$, $(2\bigcup)$ and (Com).
5. $\mathcal{H}$ is a *σ-field of sets*, if μ satisfies $(Bound)$, $(\omega\bigcap)$, $(\omega\bigcup)$ and (Com).
6. $\mathcal{H}$ is a *complete field of sets*, if μ satisfies $(Bound)$, $(\bigcap)$, $(\bigcup)$ and (Com).
7. $\mathcal{H}$ is a *power set algebra*, if $\mu = \wp(W)$.

3 Subset Space Bisimulation

Similar to the satisfaction relation, the notion of bisimulation in topological semantics of modal logic also makes sense even if the mathematical structures involved are not topological spaces. Hence we have the following definition of (subset space) bisimulation:

Definition 14 ((Subset Space) Bisimulation)**.** Let $\mathcal{M} = (W, \mu, V)$ and $\mathcal{M}' = (W', \mu', V')$ be two models. A *(subset space) bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ is a non-empty relation $Z \subseteq W \times W'$ such that all of the following are true:

(Atom) for any $i \in \omega$, $w \in W$ and $w' \in W'$ such that wZw', $w \in V(p_i)$ if and only if $w' \in V(p_i)$;

(Forth) for any $w \in W$ and $w' \in W'$ such that wZw', for each $U \in \mu$, if $w \in U$, then there is a $U' \in \mu'$ such that $w' \in U'$ and, for each $u' \in U'$, there is a $u \in U$ such that uZu';

(Back) for any $w \in W$ and $w' \in W'$ such that wZw', for each $U' \in \mu'$, if $w' \in U'$, then there is a $U \in \mu$ such that $w \in U$ and, for each $u \in U$, there is a $u' \in U'$ such that uZu'.

Two models are *bisimular*, if there is a bisimulation between them.

A significance of bisimulation is that it preserves the satisfaction relation.

Theorem 15. *Let $\mathcal{M} = (W, \mu, V)$ and $\mathcal{M}' = (W', \mu', V')$ be two models and $Z \subseteq W \times W'$ a bisimulation. For any $w \in W$ and $w' \in W'$ such that wZw', for each $\varphi \in Form$, $\mathcal{M}, w \Vdash \varphi \Leftrightarrow \mathcal{M}', w' \Vdash \varphi$.*

Proof. We use induction on the structure of formulas. In the base step, we consider propositional letters, and the result in this case follows directly from (Atom) in the definition. In the induction step, we consider three cases. The cases for negation and conjunction follow from the induction hypothesis easily. Hence we only consider the case when φ is $\Box\psi$ and the induction hypothesis is true for ψ. Let $w \in W$ and $w' \in W'$ satisfy wZw'.

First assume that $\mathcal{M}, w \Vdash \Box\psi$. Then there is a $U \in \mu$ such that $w \in U$ and $\mathcal{M}, u \Vdash \psi$ is true for each $u \in U$. By (Forth) there is a $U' \in \mu'$ such that $w' \in U'$ and, for each $u' \in U'$, there is a $u \in U$ such that uZu'. Hence, for each $u' \in U'$, there is a $u \in U$ such that uZu', then $\mathcal{M}, u \Vdash \psi$ and thus by the induction hypothesis $\mathcal{M}', u' \Vdash \psi$. Whence U' is such that $w' \in U'$ and $\mathcal{M}', u' \Vdash \psi$ is true for each $u' \in U'$. Therefore, $\mathcal{M}', u' \Vdash \Box\psi$.

Second assume that $\mathcal{M}', w' \Vdash \Box\psi$. Symmetrical to the above reasoning, using (Back) instead of (Forth), we can show that $\mathcal{M}, w \Vdash \Box\psi$. □

The following is a useful construction on subset spaces which results in a model bisimular to the original one.

Definition 16. Let $\mathcal{H} = (W, \mu)$ be a subset space and $\mathcal{M} = (\mathcal{H}, V)$ a model.

1. $\mu^* \stackrel{\text{def}}{=} \{\bigcup \mathcal{U} \mid \mathcal{U} \subseteq \mu\}$ is called the *union closure* of μ.
2. $\mathcal{H}^* \stackrel{\text{def}}{=} (H, \mu^*)$ is called the *union closure* of $\mathcal{H}$.
3. $\mathcal{M}^* \stackrel{\text{def}}{=} (\mathcal{H}^*, V) = (H, \mu^*, V)$ is called the *union closure* of $\mathcal{M}$.

Remark 17. For each subset space $\mathcal{H} = (W, \mu)$, $\mathcal{H}^*$ satisfies $(\bigcup)$ and $\mu \subseteq \mu^*$.

Proposition 18. *Let $\mathcal{M} = (W, \mu, V)$ be a model and $\mathcal{M}^* = (W, \mu^*, V)$ its union closure.* id_W *is a bisimulation between $\mathcal{M}$ and $\mathcal{M}^*$.*

Proof. For (Atom), it is obviously true.

For (Forth), assume that $w \in W$ and $U \in \mu$ such that $w \in U$. Consider U itself. First, obviously $w \in U$. Second, since $U \in \mu$, $U = \bigcup\{U\} \in \mu^*$. Third, for each $u \in U$, $u \in U$ and $(u, u) \in \mathrm{id}_W$.

For (Back), assume that $w \in W$ and $U' \in \mu^*$ such that $w \in U'$. By definition there is a $\rho \subseteq \mu$ such that $U' = \bigcup \rho$. Since $w \in U'$, there is a $U \in \rho$ such that $w \in U$. Consider this U. First, obviously $w \in U$. Second, since $\rho \subseteq \mu$, $U \in \mu$. Third, for each $u \in U$, $u \in U \subseteq \bigcup \rho = U'$ and $(u, u) \in \mathrm{id}_W$. □

4 Four Correspondence Results

In this section, we prove four correspondence results to exemplify the expressive power of our semantics and to facilitate the proofs of soundness theorems afterwards.

Proposition 19. *Let $\mathcal{H} = (W, \mu)$ be a subset space. $\mathcal{H} \Vdash \Box p \wedge \Box q \to \Box(p \wedge q)$, if and only if the following is true:*

$$(Cap) \qquad \textit{for any } U, V \in \mu\textit{, } U \cap V = \bigcup\{T \in \mu \mid T \subseteq U \cap V\}$$

Proof. The "if" part: Assume that (Cap) is true in $\mathcal{H}$. Let $V : \mathbf{P} \to \wp(W)$ be a function and $w \in W$ be such that $(\mathcal{H}, V), w \Vdash \Box p \wedge \Box q$. Then there are $U_p, U_q \in \mu$ such that $w \in U_p$, $U_p \subseteq V(p)$, $w \in U_q$ and $U_q \subseteq V(q)$. Since $w \in U_p \cap U_q$, By (Cap) $w \in \bigcup\{T \in \mu \mid T \subseteq U_p \cap U_q\}$, so there is a $V \in \mu$ such that $w \in V$ and $V \subseteq U_p \cap U_q \subseteq V(p) \cap V(q) = \|p \wedge q\|$. Therefore, $(\mathcal{H}, V), w \Vdash \Box(p \wedge q)$.

The "only if" part: Assume that $\mathcal{H} \Vdash \Box p \wedge \Box q \to \Box(p \wedge q)$. Let $S, T \in \mu$ be arbitrary. The aim is to show that $S \cap T = \bigcup\{U \in \mu \mid U \subseteq S \cap T\}$. The "$\supseteq$" part is obvious. For the "$\subseteq$" part, let $w \in S \cap T$ be arbitrary. Let $V : \mathbf{P} \to Form$ be a function such that $V(p) = S$ and $V(q) = T$. Then $(\mathcal{H}, V), w \Vdash \Box p \wedge \Box q$. By the assumption $(\mathcal{H}, V), w \Vdash \Box(p \wedge q)$. Hence there is a $U \in \mu$ such that $w \in U$ and $U \subseteq \|p \wedge q\| = V(p) \cap V(q) = S \cap T$, so $w \in \bigcup\{U \in \mu \mid U \subseteq S \cap T\}$. Therefore, $S \cap T \subseteq \bigcup\{U \in \mu \mid U \subseteq S \cap T\}$. □

Proposition 20. *Let $\mathcal{H} = (W, \mu)$ be a subset space. $\mathcal{H} \Vdash p \to \Box p$, if and only if the following is true:*

$$(TR) \qquad \textit{for each } w \in W\textit{, } \{w\} \in \mu$$

Proof. The "if" part: Assume that (TR) is true in $\mathcal{H}$. Let $V : \mathbf{P} \to \wp(W)$ be a function and $w \in W$ be such that $(\mathcal{H}, V), w \Vdash p$. Then by (TR) $\{w\}$ is such that $w \in \{w\}$, $\{w\} \in \mu$ and $\{w\} \subseteq V(p)$. Therefore, $(\mathcal{H}, V), w \Vdash \Box p$.

The "only if" part: Assume that $\mathcal{H} \Vdash p \to \Box p$. Let $w \in W$ be arbitrary. Also let $V : \mathbf{P} \to Form$ be a function such that $V(p) = \{w\}$. Then $(\mathcal{H}, V), w \Vdash p$. By the assumption $(\mathcal{H}, V), w \Vdash \Box p$. Hence there is a $U \in \mu$ such that $w \in U$ and $U \subseteq V(p) = \{w\}$. It follows that $\{w\} = U \in \mu$. □

Proposition 21. *Let $\mathcal{H} = (W, \mu)$ be a subset space. $\mathcal{H} \Vdash \Diamond p \to \Box\Diamond p$, if and only if the following is true:*

$$(UCom) \qquad \textit{for each } \pi \subseteq \mu\textit{, } W \setminus \bigcup\pi = \bigcup\{U \in \mu \mid U \subseteq W \setminus \bigcup\pi\}$$

Proof. The "if" part: Assume that $(UCom)$ is true in $\mathcal{H}$. Let $V : \mathbf{P} \to \wp(W)$ be a function and $w \in W$ be such that $(\mathcal{H}, V), w \Vdash \Diamond p$. By Remark 10, for each $U \in \mu$, $w \in U$ implies that $U \cap V(p) \neq \varnothing$. Since $w \in \bigcap\{W \setminus U \mid U \in \mu \text{ and } w \notin U\} = W \setminus$

$\bigcup\{U \in \mu \mid w \notin U\}$, by $(UCom)$ $w \in \bigcup\{T \in \mu \mid T \subseteq W \setminus \bigcup\{U \in \mu \mid w \notin U\}\}$. Hence there is a $T \in \mu$ such that $w \in T$ and $T \subseteq W \setminus \bigcup\{U \in \mu \mid w \notin U\} = \bigcap\{W \setminus U \mid U \in \mu \text{ and } w \notin U\}$.

We finish the proof by showing that, for each $t \in T$, $(\mathcal{H}, V), t \Vdash \Diamond p$. Let $t \in T$ and $S \in \mu$ such that $t \in S$. Then $t \in \bigcap\{W \setminus U \mid U \in \mu \text{ and } w \notin U\}$. Hence $w \in S$; otherwise, we would have $t \in W \setminus S$, contradicting that $t \in S$. Since $(\mathcal{H}, V), w \Vdash \Diamond p$, by Remark 10 $S \cap V(p) \neq \varnothing$. Therefore, for each $t \in T$, $(\mathcal{H}, V), t \Vdash \Diamond p$.

The "only if" part: Let $\pi \subseteq \mu$ be arbitrary. Assume that $W \setminus \bigcup\pi \neq \bigcup\{U \in \mu \mid U \subseteq W \setminus \bigcup\pi\}$. Then $W \setminus \bigcup\pi \nsubseteq \bigcup\{U \in \mu \mid U \subseteq W \setminus \bigcup\pi\}$. Hence there is a $w \in W \setminus \bigcup\pi$ such that $w \notin \bigcup\{U \in \mu \mid U \subseteq W \setminus \bigcup\pi\}$. Let $V : \mathbf{P} \to \wp(W)$ be a function such that $V(p) = W \setminus \bigcup\pi$. On the one hand, for each $U \in \mu$, if $w \in U$, $w \in U \cap V(p)$, so $U \cap V(p) \neq \varnothing$. By Remark 10 $(\mathcal{H}, V), w \Vdash \Diamond p$.

On the other hand, let $S \in \mu$ be arbitrary. Assume that $w \in S$. Since $w \notin \bigcup\{U \in \mu \mid U \subseteq W \setminus \bigcup\pi\}$, $S \nsubseteq W \setminus \bigcup\pi$, so $S \cap \bigcup\pi \neq \varnothing$. Hence there is a $T \in \pi$ and $u \in W$ such that $u \in S \cap T$. By the definition of V $T \cap V(p) = \varnothing$. Since $u \in T$ and $T \in \mu$, by Remark 10 $(\mathcal{H}, V), u \nVdash \Diamond p$. Since $u \in S$, $S \nsubseteq \|\Diamond p\|$. Since S is arbitrary, $(\mathcal{H}, V), w \nVdash \Box\Diamond p$.

Therefore, $\mathcal{H} \nVdash \Diamond p \to \Box\Diamond p$. □

Proposition 22. *Let $\mathcal{H} = (W, \mu)$ be a subset space. The following are equivalent:*

(i) for each $\varphi \in Form$, $\mathcal{H} \Vdash \varphi$ implies $\mathcal{H} \Vdash \Box\varphi$;
(ii) the following is true:

$$(Full) \qquad W = \bigcup\mu$$

Proof. **From (i) to (ii)**: By definition $\mathcal{H} \Vdash p \vee \neg p$. By (i) $\mathcal{H} \Vdash \Box(p \vee \neg p)$. Then, for any $w \in W$ and function $V : \mathbf{P} \to \wp(W)$, $(\mathcal{H}, V), w \Vdash \Box(p \vee \neg p)$, so there is a $U \in \mu$ such that $w \in U$ and $U \subseteq \|p \vee \neg p\|$ and thus $w \in \bigcup\mu$. Hence $W \subseteq \bigcup\mu$. $\bigcup\mu \subseteq W$ is obvious, so $(Full)$ is true.

From (ii) to (i): Assume that $\mathcal{H} \Vdash \varphi$. Let $V : \mathbf{P} \to Form$ be a function and $w \in W$. By (ii) there is a $U \in \mu$ such that $w \in U$. For each $u \in U$, by the assumption $(\mathcal{H}, V), u \Vdash \varphi$. By definition $(\mathcal{H}, V), w \Vdash \Box\varphi$. Since both V and w are arbitrary, by definition $\mathcal{H} \Vdash \Box\varphi$. □

5 Soundness Theorems

In this section, we prove soundness theorems for some modal logics.

Definition 23 (Soundness)**.** Let Λ be a modal logic and $\mathbb{H}$ a class of subset spaces.

1. Λ is *weakly sound* with respect to $\mathbb{H}$, if $\mathbb{H} \Vdash \varphi$ is true for each $\varphi \in \Lambda$.

2. Λ is *strongly sound* with respect to $\mathbb{H}$, if, for any $\Gamma \cup \{\varphi\} \subseteq Form$, $\Gamma \vdash_\Lambda \varphi$ implies that $\Gamma \Vdash_{\mathbb{H}} \varphi$.

Remark 24.

1. According to the definition of syntactic consequence, weak soundness and strong soundness are equivalent. In the following, we only discuss weak soundness, whose definition is simpler.
2. By the definition of the satisfaction relation each propositional tautology is valid on all subset spaces, and both (MP) and (US) can be proved using the usual method to preserve validity on subset spaces. Therefore, to prove soundness, we only need to show that the modal axioms are valid and the modal rules preserve validity on the subset spaces under concern.
3. Let Λ be a modal logic and $\mathbb{H}$ and $\mathbb{H}'$ two classes of subset spaces. If $\mathbb{H} \subseteq \mathbb{H}'$ and Λ is sound with respect to $\mathbb{H}'$, then Λ is sound with respect to $\mathbb{H}$.

Theorem 25. **S** *is sound with respect to the class of subset spaces.*

Proof. By Remark 24 it suffices to show that (M), (T) and (4) are valid and that (RM) preserves validity. Arbitrarily we take a subset space $\mathcal{H} = (W, \mu)$, a function $V : \mathbf{P} \to Form$ and $w \in W$.

- For (M), assume that $(\mathcal{H}, V), w \Vdash \Box(p \wedge q)$. Then there is a $U \in \mu$ such that $w \in U$ and $U \subseteq \|p \wedge q\| = V(p) \cap V(q)$. Hence $U \in \mu$ is such that $w \in U$, $U \subseteq V(p)$ and $U \subseteq V(q)$. It follows that $(\mathcal{H}, V), w \Vdash \Box p$ and $(\mathcal{H}, V), w \Vdash \Box q$, so $(\mathcal{H}, V), w \Vdash \Box p \wedge \Box q$.
- For (T), assume that $(\mathcal{H}, V), w \Vdash \Box p$. Then there is a $U \in \mu$ such that $w \in U$ and $U \subseteq V(p)$. Hence $w \in V(p)$, so $(\mathcal{H}, V), w \Vdash p$.
- For (4), assume that $(\mathcal{H}, V), w \Vdash \Box p$. Then there is a $U \in \mu$ such that $w \in U$ and $U \subseteq V(p)$. Note that $U \subseteq \|\Box p\|$: for each $u \in U$, $U \in \mu$ is such that $u \in U$ and $U \subseteq V(p)$, so $(\mathcal{H}, V), u \Vdash \Box p$. Hence $(\mathcal{H}, V), w \Vdash \Box\Box p$.
- For (RM), assume that $\mathcal{H} \Vdash \varphi \to \psi$ and $(\mathcal{H}, V), w \Vdash \Box\varphi$. Then there is a $U \in \mu$ such that $w \in U$ and $U \subseteq \|\varphi\|$. Since $\mathcal{H} \Vdash \varphi \to \psi$, $\|\varphi\| \subseteq \|\psi\|$. Hence $U \in \mu$ is such that $w \in U$ and $U \subseteq \|\psi\|$. Therefore, $(\mathcal{H}, V), w \Vdash \Box\psi$. Since V and w are arbitrary, $\mathcal{H} \Vdash \Box\varphi \to \Box\psi$.

$\Box$

Theorem 26.

1. $\mathbf{S}^+$ *is sound with respect to the class of subset spaces satisfying* (Cap).
2. $\mathbf{S}^+$ *is sound with respect to the class of intersection structures.*

Proof. For Item 1, by Remark 24 and the proof of Theorem 25 it suffices to show that (C) is valid, which follows from Proposition 19.

Item 2 follows from Remark 24 and Item 1, just noting that (Cap) follows from $(\bigcap)$: for any $S, T \in \mu$, $S \cap T \in \mu$ by $(\bigcap)$, so $S \cap T = \bigcup\{U \in \mu \mid U \subseteq S \cap T\}$. □

Theorem 27.

1. **S4** *is sound with respect to the class of subset spaces satisfying* (Cap) *and* $(Full)$.
2. **S4** *is sound with respect to the class of topped intersection structures.*
3. **S4** *is sound with respect to the class of lattices of sets.*
4. **S4** *is sound with respect to the class of* σ*-lattices of sets.*
5. **S4** *is sound with respect to the class of topological spaces.*
6. **S4** *is sound with respect to the class of complete lattices of sets.*

Proof. For Item 1, by Remark 24 and the proof of Theorem 26 it suffices to show that (RN) preserves validity in subset spaces satisfying $(Full)$, which follows from Proposition 22.

Both Item 2 and Item 3 follows from Remark 24 and Item 1 just noting that each topped intersection structure and each lattice of sets satisfy (Cap) and $(Full)$. The other items follow from Remark 24, Item 2 and Item 3. □

Theorem 28.

1. **S5** *is sound with respect to the class of subset spaces satisfying* (Cap)*,* $(Full)$ *and* $(UCom)$.
2. **S5** *is sound with respect to the class of complete fields of sets.*

Proof. For Item 1, by Remark 24 and the proof of Theorem 27 it suffices to show that (5) is validity in subset spaces satisfying $(UCom)$, which follows from Proposition 21.

Item 2 follows from Item 1, just noting that $(UCom)$ holds in complete fields of sets: Let (W, μ) be a complete field of sets. For each $\pi \subseteq \mu$, by completeness $\bigcup \pi \in \mu$, then by (Com) $W \backslash \bigcup \pi \in \mu$ and thus $W \backslash \bigcup \pi = \bigcup\{U \in \mu \mid U \subseteq W \backslash \bigcup \pi\}$. □

Theorem 29.

1. **Tr** *is sound with respect to the class of subset spaces satisfying* (TR).
2. **Tr** *is sound with respect to the class of discrete topological spaces/power set algebras.*

Proof. For Item 1, by Remark 24 and the proof of Theorem 28 it suffices to show that (Tr) is validity in subset spaces satisfying (TR), which follows from Proposition 20.

Item 2 follows from Remark 24 and Item 1, just noting that every power set algebra satisfies (TR). □

We end this section with a negative result about **S5**. While by Theorem 27 **S4** is sound with respect to the class of lattices of sets, that of σ-lattices of sets and that of complete lattices of sets, **S5** is no longer sound when the class of complete fields of sets is extended a bit to that of σ-fields of sets. The problem is that $\Diamond p \to \Box\Diamond p$ is no longer valid. By Proposition 21 this formula corresponds to (UCom). Roughly speaking, (UCom) requires that complements of arbitrary unions of elements of μ, not just complements of elements of μ, are in μ; and this fails when μ is not closed under arbitrary union. The proof of the following proposition gives a concrete counter-model.

Proposition 30. (5)*, i.e.* $\Diamond p \to \Box\Diamond p$*, is* not *valid in a σ-field of sets. Thus,* **S5** *is* not *sound with respect to the class of σ-fields of sets.*

Proof. Consider $\mathcal{H} = (\mathbb{R}, \mu)$, where

1. $\mathbb{R}$ is the set of real numbers;
2. $\mu = \{U \subseteq \mathbb{R} \mid U \in \mathit{CPR} \text{ or } \mathbb{R} \setminus U \in \mathit{CPR}\}$, where $\mathit{CPR} = \{U \subseteq \mathbb{R}^+ \mid |U| \leq \aleph_0\}$ and $|U|$ denotes the cardinality of U.

First, we verify that $\mathcal{H}$ is a σ-algebra. Since $\{1\} \in \mathit{CPR}$, $\{1\} \in \mu$ and thus $\mu \neq \varnothing$. By definition μ is closed under complement. It remains to show that μ is closed under countable intersection.

Let $\{U_i \mid i \in \omega\} \subseteq \mu$. We consider two cases.

- *Case 1*: There is an $i^* \in \omega$ such that $U_{i^*} \in \mathit{CPR}$.
  Then $\bigcap_{i\in\omega} U_i \subseteq U_{i^*}$ and thus $\bigcap_{i\in\omega} U_i \in \mathit{CPR}$. Hence $\bigcap_{i\in\omega} U_i \in \mu$.
- *Case 2*: For each $i \in \omega$, $\mathbb{R} \setminus U_i \in \mathit{CPR}$.
  Then $\bigcap_{i\in\omega} U_i = \mathbb{R} \setminus (\mathbb{R} \setminus \bigcap_{i\in\omega} U_i) = \mathbb{R} \setminus \bigcup_{i\in\omega}(\mathbb{R} \setminus U_i)$. Since $\mathbb{R} \setminus U_i \in \mathit{CPR}$ is true for each $i \in \omega$, $\bigcup_{i\in\omega}(\mathbb{R} \setminus U_i) \in \mathit{CPR}$, so $\bigcap_{i\in\omega} U_i \in \mu$.

In both cases, $\bigcap_{i\in\omega} U_i \in \mu$.

Second we show that (5), i.e. $\Diamond p \to \Box\Diamond p$, is not valid in $\mathcal{H}$. Let $V : \mathbf{P} \to \mathit{Form}$ be a function such that $V(p) = \{r \in \mathbb{R} \mid r \leq 0\}$. Consider 0. Note that $(\mathcal{H}, V), 0 \Vdash p$ by definition.

Observe that $(\mathcal{H}, V), 0 \Vdash \Diamond p$: Let $U \in \mu$ be such that $0 \in U$. Then $0 \in U \cap V(p)$ and thus $U \cap V(p) \neq \varnothing$. Hence $(\mathcal{H}, V), 0 \Vdash \Diamond p$.

Observe that $(\mathcal{H}, V), 0 \nVdash \Box\Diamond p$: Let $U \in \mu$ be such that $0 \in U$. Then $U \notin \mathit{CPR}$, so $\mathbb{R} \setminus U \in \mathit{CPR}$ by the definition of μ. Since $|\mathbb{R}^+| > \aleph_0$, there is an $r \in \mathbb{R}^+$ such that $r \in U$. Since $r \in \mathbb{R}^+$, $(\mathcal{H}, V), r \nVdash p$. By definition $\{r\} \in \mathit{CPR} \subseteq \mu$, $r \in \{r\}$ and $\{r\} \cap V(p) = \varnothing$. Hence $(\mathcal{H}, V), r \nVdash \Diamond p$. Since $r \in U$ and U is arbitrary, $(\mathcal{H}, V), 0 \nVdash \Box\Diamond p$.

Therefore, $\mathcal{H} \nVdash \Diamond p \to \Box\Diamond p$. □

6 Completeness via Canonical Models

In this section, we prove completeness theorems for some modal logics.

Definition 31 (Completeness)**.** Let Λ be a modal logic and $\mathbb{H}$ a class of subset spaces. Λ is *strongly complete* with respect to $\mathbb{H}$, if, for any $\Gamma \cup \{\varphi\} \subseteq Form$, $\Gamma \Vdash_{\mathbb{H}} \varphi$ implies that $\Gamma \vdash_{\Lambda} \varphi$.

Remark 32. Let Λ be a modal logic and $\mathbb{H}$ and $\mathbb{H}'$ two classes of subset spaces. If $\mathbb{H} \subseteq \mathbb{H}'$ and Λ is strongly complete with respect to $\mathbb{H}$, then Λ is strongly complete with respect to $\mathbb{H}'$.

By the usual reasoning, to show that Λ is strongly complete with respect to $\mathbb{H}$, it suffices to show that every Λ-**MCS** is satisfied at a point in a model based on an element of $\mathbb{H}$. The most important technique is the so-called *canonical model*, which is adapted from topological semantics ([1]) and introduced as follows:

Definition 33. Let Λ be a modal logic. Its *canonical subset space* $\mathcal{H}^{\Lambda}$ is an ordered pair $(W^{\Lambda}, \mu^{\Lambda})$, where

1. W^{Λ} is the set of Λ-**MCS**s;
2. $\mu^{\Lambda} = \{\widehat{\Box\varphi} \mid \varphi \in Form\}$, where $\widehat{\varphi} = \{\Gamma \in W^{\Lambda} \mid \varphi \in \Gamma\}$ for each $\varphi \in Form$.

Its *canonical model* $\mathcal{M}^{\Lambda}$ is an ordered pair $(\mathcal{H}^{\Lambda}, V^{\Lambda})$, where

1. $\mathcal{H}^{\Lambda}$ is the canonical subset space of Λ;
2. $V^{\Lambda}(p_i) = \widehat{p_i}$, for each $i \in \omega$.

One key feature of the canonical model is manifested as the Truth Lemma. To prove the Truth Lemma, we need the Existence Lemma.

Lemma 34 (Existence Lemma)**.** *Let Λ be a modal logic such that* **S** $\sqsubseteq \Lambda$*. For any Λ-**MCS** Γ and $\varphi \in Form$, the following are equivalent:*

(i) $\Box\varphi \in \Gamma$*;*

*(ii) there is a $\theta \in Form$ such that $\Box\theta \in \Gamma$ and, for any Λ-**MCS** Δ, $\Box\theta \in \Delta$ implies $\varphi \in \Delta$.*

In our notation, (ii) can also be written as: there is a $\theta \in Form$ such that $\Gamma \in \widehat{\Box\theta}$ and, for each $\Delta \in \widehat{\Box\theta}$, $\varphi \in \Delta$.

Proof. **From (i) to (ii)**: Assume that $\Box\varphi \in \Gamma$. Then $\varphi \in Form$ is such that $\Box\varphi \in \Gamma$ and, for any Λ-**MCS** Δ, if $\Box\varphi \in \Delta$, then $\varphi \in \Delta$ by the properties of Λ-**MCS**s and the fact that $\Box\varphi \to \varphi \in$ **S** $\subseteq \Lambda \subseteq \Delta$.

From (ii) to (i): Assume that (ii) is true.

Note that $\vdash_\Lambda \neg(\Box\theta \wedge \neg\varphi)$: Suppose (towards a contradiction) that $\nvdash_\Lambda \neg(\Box\theta \wedge \neg\varphi)$. Then $\Box\theta \wedge \neg\varphi$ is Λ-consistent. By Lindenbaum's Lemma (Lemma 4.17 in [3]) there is a Λ-**MCS** Δ such that $\Box\theta \wedge \neg\varphi \in \Delta$. Hence $\Box\theta \in \Delta$ and $\varphi \notin \Delta$, contradicting (ii).

Then $\vdash_\Lambda \Box\theta \to \varphi$. Since $\mathbf{S} \sqsubseteq \Lambda$, by (RM) $\vdash_\Lambda \Box\Box\theta \to \Box\varphi$. Moreover, $\vdash_\Lambda \Box\theta \to \Box\Box\theta$, so $\vdash_\Lambda \Box\theta \to \Box\varphi$. Since $\Box\theta \in \Gamma$, $\Box\varphi \in \Gamma$. □

Now we prove the Truth Lemma.

Lemma 35 (Truth Lemma)**.** *Let Λ be a modal logic such that $\mathbf{S} \sqsubseteq \Lambda$. For any $\Gamma \in W^\Lambda$ and $\varphi \in Form$, $\mathcal{M}^\Lambda, \Gamma \Vdash \varphi \Leftrightarrow \varphi \in \Gamma$.*

Proof. We use induction on the structure of formulas. In the base step, we consider propositional letters, and the result in this case follows directly from the definition. In the induction step, we consider three cases. The cases for negation and conjunction follow from the induction hypothesis easily. Hence we only consider the case when φ is $\Box\psi$ and the induction hypothesis is true for ψ.

First assume that $\mathcal{M}^\Lambda, \Gamma \Vdash \Box\psi$. Then there is a $U \in \mu^\Lambda$ such that $\Gamma \in U$ and, for each $\Delta \in U$, $\mathcal{M}^\Lambda, \Delta \Vdash \psi$. By definition there is a $\theta \in Form$ such that $U = \widehat{\Box\theta}$. Then by the induction hypothesis, for each $\Delta \in \widehat{\Box\theta}$, $\psi \in \Delta$. By Existence Lemma $\Box\psi \in \Gamma$.

Second assume that $\Box\psi \in \Gamma$. By Existence Lemma there is a $\theta \in Form$ such that $\Gamma \in \widehat{\Box\theta}$ and, for each $\Delta \in \widehat{\Box\theta}$, $\psi \in \Delta$. By definition $\widehat{\Box\theta} \in \mu^\Lambda$. Then by the induction hypothesis, for each $\Delta \in \widehat{\Box\theta}$, $\mathcal{M}^\Lambda, \Delta \Vdash \psi$. Hence $\mathcal{M}^\Lambda, \Gamma \Vdash \Box\psi$. □

Now we are ready to prove some completeness results.

Theorem 36. **S** *is strongly complete with respect to the class of subset spaces.*

Proof. Let Γ be an **S**-consistent set. By Lindenbaum's Lemma there is an **S-MCS** Γ^+ such that $\Gamma \subseteq \Gamma^+$. By the Truth Lemma $\mathcal{M}^{\mathbf{S}}, \Gamma^+ \Vdash \Gamma$. By definition $\mathcal{H}^{\mathbf{S}}$ is a subset space. □

Theorem 37. $\mathbf{S}^+$ *is strongly complete with respect to the class of subset spaces satisfying* $(2\bigcap)$.

Proof. For Item 1, given the proof of Theorem 36 the crucial step is to show that $\mathcal{H}^{\mathbf{S}^+}$ satisfies $(2\bigcap)$. Let $S, T \in \mu^{\mathbf{S}^+}$ be arbitrary. By definition there are $\theta, \eta \in Form$ such that $S = \widehat{\Box\theta}$ and $T = \widehat{\Box\eta}$. We show that $\widehat{\Box(\theta \wedge \eta)} = \widehat{\Box\theta} \cap \widehat{\Box\eta} = S \cap T$, and thus $S \cap T \in \mu^{\mathbf{S}^+}$. Let $\Gamma \in W^{\mathbf{S}^+}$ be arbitrary.

First assume that $\Gamma \in \widehat{\Box(\theta \wedge \eta)}$. Then $\Box(\theta \wedge \eta) \in \Gamma$. Since $\vdash_{\mathbf{S}+} \Box(\theta \wedge \eta) \to \Box\theta \wedge \Box\eta$, $\Box\theta \wedge \Box\eta \in \Gamma$. Hence $\Box\theta \in \Gamma$ and $\Box\eta \in \Gamma$, and thus $\Gamma \in \widehat{\Box\theta}$ and $\Gamma \in \widehat{\Box\eta}$. Therefore, $\Gamma \in \widehat{\Box\theta} \cap \widehat{\Box\eta}$.

Second assume that $\Gamma \in \widehat{\Box\theta} \cap \widehat{\Box\eta}$. Then $\Gamma \in \widehat{\Box\theta}$ and $\Gamma \in \widehat{\Box\eta}$, and thus $\Box\theta \in \Gamma$ and $\Box\eta \in \Gamma$. Hence $\Box\theta \wedge \Box\eta \in \Gamma$. Since $\vdash_{\mathbf{S}+} \Box\theta \wedge \Box\eta \to \Box(\theta \wedge \eta)$, $\Box(\theta \wedge \eta) \in \Gamma$, and thus $\Gamma \in \widehat{\Box(\theta \wedge \eta)}$. □

Theorem 38. **S4** *is strongly complete with respect to the class of subset spaces satisfying* $(Bound)$ *and* $(2\bigcap)$.

Proof. For Item 1, given the proof of Theorem 37 the crucial step is to show that $\mathcal{H}^{\mathbf{S4}}$ satisfies $(Bound)$.

We show that $\widehat{\Box(p \vee \neg p)} = W^{\mathbf{S4}}$, and thus $W^{\mathbf{S4}} \in \mu^{\mathbf{S4}}$. The "$\subseteq$" part is obvious. For the "$\supseteq$" part, for each $\Gamma \in W^{\mathbf{S4}}$, since $\vdash_{\mathbf{S4}} \Box(p \vee \neg p)$ and Γ is an **S4-MCS**, $\Box(p \vee \neg p) \in \Gamma$ and thus $\Gamma \in \widehat{\Box(p \vee \neg p)}$.

We show that $\widehat{\Box(p \wedge \neg p)} = \varnothing$, and thus $\varnothing \in \mu^{\mathbf{S4}}$. The "$\supseteq$" part is obvious. For the "$\subseteq$" part, since $\vdash_{\mathbf{S4}} \neg\Box(p \wedge \neg p)$, no **S4-MCS** includes $\Box(p \wedge \neg p)$. □

Finally we use the technique of union closure to prove an important result in topological semantics of modal logic. This proof is essentially the one in [1].

Theorem 39. **S4** *is strongly complete with respect to the class of topological spaces.*

Proof. Let Γ be an **S4**-consistent set. By Lindenbaum's Lemma there is an **S4-MCS** Γ^+ such that $\Gamma \subseteq \Gamma^+$. By the Truth Lemma $\mathcal{M}^{\mathbf{S4}}, \Gamma^+ \Vdash \Gamma$, i.e. $(\mathcal{H}^{\mathbf{S4}}, V^{\mathbf{S4}}), \Gamma^+ \Vdash \Gamma$. By Proposition 18 $(\mathcal{H}^{\mathbf{S4}*}, V^{\mathbf{S4}}), \Gamma^+ \Vdash \Gamma$.

We show that $\mathcal{H}^{\mathbf{S4}*}$ is a topological space. By the proof of Theorem 38 and Remark 17 $\varnothing, W^{\mathbf{S4}} \in \mu^{\mathbf{S4}} \subseteq \mu^{\mathbf{S4}*}$ and $\mathcal{H}^{\mathbf{S4}*}$ satisfies $(\bigcup)$. It remains to show that $\mathcal{H}^{\mathbf{S4}*}$ satisfies $(2\bigcap)$. Let $S, T \in \mu^{\mathbf{S4}*}$ be arbitrary. Then there are $\rho, \pi \in \wp(\mu^{\mathbf{S4}})$ such that $S = \bigcup\rho$ and $T = \bigcup\pi$. A proof similar to the one for Theorem 37 shows that $\mathcal{H}^{\mathbf{S4}}$ satisfies $(2\bigcap)$, so $\{U \cap V \mid U \in \rho \text{ and } V \in \pi\} \subseteq \mu^{\mathbf{S4}}$ and thus $S \cap T = \bigcup\rho \cap \bigcup\pi = \bigcup\{U \cap V \mid U \in \rho \text{ and } V \in \pi\} \in \mu^{\mathbf{S4}*}$. □

7 Completeness via Intersection-closed Canonical Models

Some extensions of $\mathbf{S}^+$ are strongly complete with respect to interesting classes of subset spaces. However, the canonical model does not work, and we need the notion of intersection-closed canonical model.

Definition 40 (Intersection-closed Canonical Model)**.** Let Λ be a modal logic. Its *intersection-closed canonical subset space* $\overline{\mathcal{H}^{\Lambda}}$ is an ordered pair $(W^{\Lambda}, \overline{\mu^{\Lambda}})$, where

1. W^Λ is the set of Λ-**MCS**s;
2. $\overline{\mu^\Lambda} = \{\widehat{\Box\Gamma} \mid \Gamma \subseteq Form \text{ and } \Gamma \neq \varnothing\}$, where $\Box\Gamma = \{\Box\varphi \mid \varphi \in \Gamma\}$ and $\widehat{\Box\Gamma} = \{w \in W^\Lambda \mid \Box\Gamma \subseteq w\}$.

Its *intersection-closed canonical model* $\overline{\mathcal{M}^\Lambda}$ is the ordered pair $(\overline{\mathcal{H}^\Lambda}, V^\Lambda)$.

Basic properties of intersection-closed canonical models are collected in the following lemma.

Lemma 41. *Let* Λ *be a modal logic.*

1. $\mu^\Lambda \subseteq \overline{\mu^\Lambda}$.
2. $\overline{\mathcal{H}^\Lambda}$ *satisfies* $(\bigcap)$.

Proof. For Item 1, for each $U \in \mu^\Lambda$, there is a $\varphi \in Form$ such that $U = \widehat{\Box\varphi}$, so $U = \widehat{\Box\{\varphi\}} \in \overline{\mu^\Lambda}$.

For Item 2, let I be a non-empty set and $\{U_i \mid i \in I\} \subseteq \overline{\mu^\Lambda}$. For each $i \in I$, there is a non-empty set $\Gamma_i \subseteq Form$ such that $U_i = \widehat{\Box\Gamma_i}$. Then $\bigcup_{i\in I} \Gamma_i$ is a non-empty subset of $Form$ and

$$\bigcap_{i\in I} U_i = \bigcap_{i\in I} \widehat{\Box\Gamma_i} = \widehat{\Box\bigcup_{i\in I}\Gamma_i} \in \overline{\mu^\Lambda}$$

□

For intersection-closed canonical models, Truth Lemma is also true, but only for extensions of $\mathbf{S}^+$. To prove this, we first need to prove a new version of Existence Lemma.

Lemma 42 (Existence Lemma for Extensions of $\mathbf{S}^+$)**.** *Let* Λ *be a modal logic such that* $\mathbf{S}^+ \sqsubseteq \Lambda$. *For any* Λ-***MCS*** Γ *and* $\varphi \in Form$, *the following are equivalent:*

(i) $\Box\varphi \in \Gamma$;
(ii) there is a non-empty set $\Delta \in Form$ *such that* $\Box\Delta \subseteq \Gamma$ *and, for any* Λ-***MCS*** Γ', $\Box\Delta \subseteq \Gamma'$ *implies* $\varphi \in \Gamma'$.

Proof. **From (i) to (ii)**: Assume that $\Box\varphi \in \Gamma$. Then $\{\varphi\} \subseteq Form$ is such that $\Box\{\varphi\} \subseteq \Gamma$ and, for any Λ-**MCS** Γ', if $\Box\{\varphi\} \subseteq \Gamma'$, then $\Box\varphi \in \Gamma'$ and thus $\varphi \in \Gamma'$ by the properties of Λ-**MCS**s and the fact that $\Box\varphi \to \varphi \in \mathbf{S}^+ \subseteq \Lambda \subseteq \Gamma'$.

From (ii) to (i): Assume that (ii) is true. Suppose (towards a contradiction) that $\Box\varphi \notin \Gamma$.

Note that $\Box\Delta \cup \{\neg\varphi\}$ is Λ-consistent: Suppose (towards a contradiction) that it is not Λ-consistent. It follows that there is an $n \in \omega$ and $\psi_1, \ldots, \psi_n \in \Box\Delta$[3] such that $\vdash_\Lambda$

[3] We adopt the convention that, when $n = 0$, $\psi_1, \ldots, \psi_n$ is the empty sequence.

$\neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \neg\varphi)$ and thus $\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_n \to \varphi$. By (RM) $\vdash_\Lambda \Box(\psi_1 \wedge \cdots \wedge \psi_n) \to \Box\varphi$. Since $\mathbf{S}^+ \sqsubseteq \Lambda$, it can be proved that $\vdash_\Lambda \Box\psi_1 \wedge \cdots \wedge \Box\psi_n \to \Box(\psi_1 \wedge \cdots \wedge \psi_n)$. Hence $\vdash_\Lambda \Box\psi_1 \wedge \cdots \wedge \Box\psi_n \to \Box\varphi$. Since $\psi_1, \ldots, \psi_n \in \Box\Delta$, there are $\theta_1, \ldots, \theta_n \in \Delta$ such that $\psi_i = \Box\theta_i$ for each $i \in \{1, \ldots, n\}$. Hence $\vdash_\Lambda \Box\Box\theta_1 \wedge \cdots \wedge \Box\Box\theta_n \to \Box\varphi$. Since $\mathbf{S}^+ \sqsubseteq \Lambda$, it can be proved that $\vdash_\Lambda \Box\theta_1 \wedge \cdots \wedge \Box\theta_n \to \Box\Box\theta_1 \wedge \cdots \wedge \Box\Box\theta_n$. Hence $\vdash_\Lambda \Box\theta_1 \wedge \cdots \wedge \Box\theta_n \to \Box\varphi$, i.e. $\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_n \to \Box\varphi$. Since $\psi_1, \ldots, \psi_n \in \Box\Delta$, $\Box\Delta \subseteq \Gamma$ and Γ is an Λ-**MCS**, $\Box\varphi \in \Gamma$, contradicting the supposition at the beginning.

By Lindenbaum's Lemma there is a Λ-**MCS** Γ' such that $\Box\Delta \cup \{\neg\varphi\} \subseteq \Gamma'$, so $\Box\Delta \subseteq \Gamma'$ but $\varphi \notin \Gamma'$, contradicting (ii). □

Now we can prove the new Truth Lemma.

Lemma 43 (Truth Lemma for Intersection-closed Canonical Model)**.** *Let Λ be a modal logic such that $\mathbf{S}^+ \sqsubseteq \Lambda$. For any $w \in W^\Lambda$ and $\varphi \in Form$, $\overline{\mathcal{M}^\Lambda}, w \Vdash \varphi \Leftrightarrow \varphi \in w$.*

Proof. We use induction on the structure of formulas. In the base step, we consider propositional letters, and the result in this case follows directly from the definition. In the induction step, we consider three cases. The cases for negation and conjunction follow from the induction hypothesis easily. Hence we only consider the case when φ is $\Box\psi$ and the induction hypothesis is true for ψ.

First assume that $\overline{\mathcal{M}^\Lambda}, w \Vdash \Box\psi$. Then there is a $U \in \overline{\mu^\Lambda}$ such that $w \in U$ and, for each $u \in U$, $\mathcal{M}^\Lambda, u \Vdash \psi$. By definition there is a non-empty set $\Gamma \subseteq Form$ such that $U = \widehat{\Box\Gamma}$. Then by the induction hypothesis, for each $u \in \widehat{\Box\Gamma}$, $\psi \in u$. By Lemma 42 $\Box\psi \in w$.

Second assume that $\Box\psi \in w$. By Lemma 42 there is a non-empty $\Gamma \subseteq Form$ such that $w \in \widehat{\Box\Gamma}$ and, for each $u \in \widehat{\Box\Gamma}$, $\psi \in u$. By definition $\widehat{\Box\Gamma} \in \overline{\mu^\Lambda}$. By the induction hypothesis, for each $u \in \widehat{\Box\Gamma}$, $\overline{\mathcal{M}^\Lambda}, u \Vdash \psi$. Hence $\overline{\mathcal{M}^\Lambda}, w \Vdash \Box\psi$. □

Now we use intersection-closed canonical models to prove some completeness results.

Theorem 44. $\mathbf{S}^+$ *is strongly complete with respect to the class of intersection structures, i.e. subset spaces satisfying* $(\bigcap)$.

Proof. The proof of Item 1 is the same as that of Theorem 37, except that we use $\overline{\mathcal{H}^{\mathbf{S}^+}}$ which satisfies $(\bigcap)$ and Lemma 43. □

Theorem 45.

1. **S4** *is strongly complete with respect to the class of complete lattices of sets.*
2. **S4** *is strongly complete with respect to the class of σ-lattices of sets.*
3. **S4** *is strongly complete with respect to the class of lattices of sets.*

4. **S4** *is strongly complete with respect to the class of topped intersection structures.*

Proof. Let Γ be an **S4**-consistent set. By Lindenbaum's Lemma there is an **S4-MCS** Γ^+ such that $\Gamma \subseteq \Gamma^+$. By Lemma 43 $\overline{\mathcal{M}^{\mathbf{S4}}}, \Gamma^+ \Vdash \Gamma$, i.e. $(\overline{\mathcal{H}^{\mathbf{S4}}}, \overline{V^{\mathbf{S4}}}), \Gamma^+ \Vdash \Gamma$. By Proposition 18 $(\overline{\mathcal{H}^{\mathbf{S4}}}^*, \overline{V^{\mathbf{S4}}}), \Gamma^+ \Vdash \Gamma$. It remains to show that $\overline{\mathcal{H}^{\mathbf{S4}}}^*$ is a complete lattice of sets.

First note that $W^{\mathbf{S4}} \in \overline{\mu^{\mathbf{S4}}}^*$: $W^{\mathbf{S4}} = \widehat{\Box\{p \vee \neg p\}} \in \overline{\mu^{\mathbf{S4}}} \subseteq \overline{\mu^{\mathbf{S4}}}^*$.

Second by Remark 17 $\overline{\mathcal{H}^{\mathbf{S4}}}^*$ satisfies $(\bigcup)$ and $\varnothing \in \overline{\mu^{\mathbf{S4}}}^*$.

Finally we show that $\overline{\mathcal{H}^{\mathbf{S4}}}^*$ satisfies $(\bigcap)$. Let I be a non-empty set and $\{U_i \mid i \in I\} \subseteq \overline{\mathcal{H}^{\mathbf{S4}}}^*$. Since we have proved that $\varnothing \in \overline{\mathcal{H}^{\mathbf{S4}}}^*$, without loss of generality we assume that $U_i \neq \varnothing$ for each $i \in I$. By definition, for each $i \in I$, there is a $\rho_i \subseteq \overline{\mathcal{H}^{\mathbf{S4}}}$ such that $U_i = \bigcup \rho_i$, then there is a non-empty set J_i and a set of non-empty sets $\{\Gamma_{j_i} \mid j_i \in J_i\} \subseteq \wp(Form)$ such that $\rho_i = \{\widehat{\Box\Gamma_{j_i}} \mid j_i \in J_i\}$. Then

$$\begin{aligned}\bigcap_{i\in I} U_i &= \bigcap_{i\in I}\bigcup \rho_i \\ &= \bigcap_{i\in I}\bigcup_{j_i\in J_i} \widehat{\Box\Gamma_{j_i}} \\ &= \bigcup\left\{\bigcap_{i\in I}\widehat{\Box\Gamma_{k(i)}} \mid k \in \prod_{i\in I} J_i\right\} \\ &= \bigcup\left\{\widehat{\Box\bigcup_{i\in I}\Gamma_{k(i)}} \mid k \in \prod_{i\in I} J_i\right\},\end{aligned}$$

where $\prod_{i\in I} J_i$ denotes the Cartesian product of $\{J_i \mid i \in I\}$. Since for each $i \in I$ and $k \in \prod_{i\in I} J_i$, $\Gamma_{k(i)}$ is a non-empty set of formulas, so, for each $k \in \prod_{i\in I} J_i$, $\widehat{\Box\bigcup_{i\in I}\Gamma_{k(i)}} \in \overline{\mu^{\mathbf{S4}}}$. Hence $\bigcap_{i\in I} U_i = \bigcup\left\{\widehat{\Box\bigcup_{i\in I}\Gamma_{k(i)}} \mid k \in \prod_{i\in I} J_i\right\}$ is in $\overline{\mu^{\mathbf{S4}}}^*$.

Items 2 to 4 follows from Item 1 and Remark 32. $\Box$

Theorem 46. **S5** *is strongly complete with respect to the class of complete fields of sets.*

Proof. Let Γ be an **S5**-consistent set. By Lindenbaum's Lemma there is an **S5-MCS** Γ^+ such that $\Gamma \subseteq \Gamma^+$. By Lemma 43 $\overline{\mathcal{M}^{\mathbf{S5}}}, \Gamma^+ \Vdash \Gamma$, i.e. $(\overline{\mathcal{H}^{\mathbf{S5}}}, \overline{V^{\mathbf{S5}}}), \Gamma^+ \Vdash \Gamma$. By Proposition 18 $(\overline{\mathcal{H}^{\mathbf{S5}}}^*, \overline{V^{\mathbf{S5}}}), \Gamma^+ \Vdash \Gamma$. It remains to show that $\overline{\mathcal{H}^{\mathbf{S5}}}^*$ is a complete algebra of sets. Using reasoning similar to that in the proof of Theorem 45, we can show that $\overline{\mathcal{H}^{\mathbf{S5}}}^*$ satisfies $(Bound)$, $(\bigcup)$ and $(\bigcap)$. It remains to show that $\overline{\mathcal{H}^{\mathbf{S5}}}^*$ satisfies (Com).

Let $U \in \overline{\mu^{\mathbf{S5}}}^*$. Then there is a $\rho \subseteq \overline{\mu^{\mathbf{S5}}}$ such that $U = \bigcup \rho$. If $\rho = \varnothing$, $W^{\mathbf{S5}} \setminus U = W^{\mathbf{S5}} \in \overline{\mu^{\mathbf{S5}}}^*$. In the following, we only need to focus on the case when $\rho \neq \varnothing$. By definition there is a non-empty set I and a set of non-empty sets of formulas $\{\Gamma_i \mid i \in I\}$ such that $\rho = \{\widehat{\Box\Gamma_i} \mid i \in I\}$. Note that, for each $i \in I$, for each $w \in W^{\mathbf{S5}}$,

$$\begin{aligned}
&w \in W^{\mathbf{S5}} \setminus \widehat{\Box\Gamma_i} \\
\Leftrightarrow\ & w \notin \widehat{\Box\Gamma_i} \\
\Leftrightarrow\ & \Box\Gamma_i \nsubseteq w \\
\Leftrightarrow\ & \text{there is a } \varphi \in \Gamma_i \text{ such that } \Box\varphi \notin w \\
\Leftrightarrow\ & \text{there is a } \varphi \in \Gamma_i \text{ such that } \neg\Box\varphi \in w \\
\Leftrightarrow\ & \text{there is a } \varphi \in \Gamma_i \text{ such that } \Box\neg\Box\varphi \in w \qquad (\text{by } \vdash_{\mathbf{S5}} \neg\Box\varphi \leftrightarrow \Box\neg\Box\varphi) \\
\Leftrightarrow\ & \text{there is a } \varphi \in \Gamma_i \text{ such that } w \in \overbrace{\Box\{\neg\Box\varphi\}} \\
\Leftrightarrow\ & w \in \bigcup_{\varphi \in \Gamma_i} \overbrace{\Box\{\neg\Box\varphi\}}
\end{aligned}$$

Hence, for each $i \in I$, $W^{\mathbf{S5}} \setminus \widehat{\Box\Gamma_i} = \bigcup_{\varphi \in \Gamma_i} \overbrace{\Box\{\neg\Box\varphi\}}$. Then

$$\begin{aligned}
W^{\mathbf{S5}} \setminus U &= W^{\mathbf{S5}} \setminus \bigcup \rho \\
&= W^{\mathbf{S5}} \setminus \bigcup_{i \in I} \widehat{\Box\Gamma_i} \\
&= \bigcap_{i \in I} W^{\mathbf{S5}} \setminus \widehat{\Box\Gamma_i} \\
&= \bigcap_{i \in I} \bigcup_{\varphi \in \Gamma_i} \overbrace{\Box\{\neg\Box\varphi\}}
\end{aligned}$$

For any $i \in I$ and $\varphi \in \Gamma_i$, $\overbrace{\Box\{\neg\Box\varphi\}} \in \overline{\mu^{\mathbf{S5}}}$, so, for each $i \in I$, $\bigcup_{\varphi \in \Gamma_i} \overbrace{\Box\{\neg\Box\varphi\}} \in \overline{\mu^{\mathbf{S5}}}^*$. Since we have known that $\overline{\mu^{\mathbf{S5}}}^*$ satisfies $(\bigcap)$, $W^{\mathbf{S5}} \setminus U \in \overline{\mu^{\mathbf{S5}}}^*$. □

Theorem 47. **Tr** *is strongly complete with respect to the class of discrete topological spaces/power set algebras.*

Proof. Let Γ be a **Tr**-consistent set. By Lindenbaum's Lemma there is a **Tr-MCS** Γ^+ such that $\Gamma \subseteq \Gamma^+$. By Lemma 43 $\overline{\mathcal{M}^{\mathbf{Tr}}}, \Gamma^+ \Vdash \Gamma$, i.e. $(\overline{\mathcal{H}^{\mathbf{Tr}}}, V^{\mathbf{Tr}}), \Gamma^+ \Vdash \Gamma$. By Proposition 18 $(\overline{\mathcal{H}^{\mathbf{Tr}}}^*, V^{\mathbf{Tr}}), \Gamma^+ \Vdash \Gamma$. It remains to show that, for each $U \subseteq W^{\mathbf{Tr}}$, $U \in \overline{\mu^{\mathbf{Tr}}}^*$.

Let $U \in W^{\mathbf{Tr}}$. For each $u \in U$, since $\vdash_{\mathbf{Tr}} \varphi \leftrightarrow \Box\varphi$, $\{u\} = \overbrace{\Box\{\varphi \mid \Box\varphi \in u\}} \in \overline{\mu^{\mathbf{Tr}}}$. Hence $U = \bigcup_{u \in U}\{u\} \in \overline{\mu^{\mathbf{Tr}}}^*$. □

8 Conclusion

In this paper, we generalize topological semantics of modal logic to describe subset spaces. We find the minimal modal logic of subset spaces, and prove soundness and completeness theorems between some important classes of subset spaces (e.g. intersection structures, topological spaces and complete fields of sets) and some famous modal logics (e.g. **S4**, **S5** and **Tr**).

The main results are as follows, among which Items 3 and 6 are known results in topological semantics of modal logic and Items 4 to 6 are first proved directly, in detail and without a devour via neighbourhood semantics or relational semantics:

1. **S** is strongly sound and complete with respect to the class of subset spaces.
2. $\mathbf{S}^+$ is strongly sound and complete with respect to the class of intersection structures.
3. **S4** is strongly sound and complete with respect to the class of topological spaces.
4. **S4** is strongly sound and complete with respect to the class of complete lattices of sets/Alexandroff topological space, the class of σ-lattices of sets, the class of lattices of sets and the class of topped intersection structures.
5. **S5** is strongly sound and complete with respect to the class of complete fields of sets, but no longer sound with respect to the class of σ-fields of sets.
6. **Tr** is strongly sound and complete with respect to the class of discrete topological spaces/power set algebra.

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关于子集空间的模态逻辑

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摘　要

在模态逻辑中，拓扑语义是邻域语义的一个直观而自然的特例。本文基于一个观察：拓扑语义的满足关系适用于比拓扑空间更一般的子集空间。本文给出了相对于子集空间所组成的类强可靠和强完全的最小模态逻辑，还证明了一些著名的模态逻辑（例如 **S4**、**S5** 和 **Tr**）相对于多个重要的子集空间类（例如交结构和完备集域）的可靠性和完全性。其中本文不借助邻域语义或关系语义而是直接使用典范模型方法的一些变种来证明一些已知结果，例如 **Tr** 相对于离散拓扑空间类的可靠性和完全性。

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