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Discrete and Topological Correspondence Theory for Modal Meet-Implication Logic and Modal Meet-Semilattice Logic in Filter Semantics*

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Abstract. In the present paper, we give a systematic study of the discrete correspondence theory and topological correspondence theory of modal meet-implication logic and modal meet-semilattice logic, in the semantics provided in [21]. The special features of the present paper include the following three points: the first one is that the semantic structure used is based on a semilattice rather than an ordinary partial order; the second one is that the propositional variables are interpreted as filters rather than upsets, and the nominals, which are the "first-order" counterparts of propositional variables, are interpreted as principal filters rather than principal upsets; the third one is that in topological correspondence theory, the collection of admissible valuations is not closed under taking disjunction, which makes the proof of the topological Ackermann lemma different from existing settings.

1 Introduction

In the studies of propositional logics, one part is the study of the different syntactic fragments of well-known propositional logics. The $[\wedge, \rightarrow]$-fragment of intuitionistic propositional logic as well as implicative meet-semilattices are well studied ([4, 11, 16, 18, 19, 20]).

Duality theory plays an impoartant role in the study of propositional logics. The duality theory for distributive and implicative meet-semilattices are also well-studied ([1, 2, 3]). For meet-semilattices, in [12], a duality for meet-semilattices was given, where the topological spaces have a partial order which is also a meet-semilattice (as a partial order).

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In [21], de Groot and Pattinson study the $[\Box, \wedge, \rightarrow]$-fragment of intuitionistic modal logics, which turns out to be a common modal fragment of many different formalizations of intuitionistic modal logics. The special feature in [21] is that the relational semantics used there is based on meet-semilattices rather than just posets, and propositional variables are interpreted as filters rather than just upsets. We call this kind of semantics "filter semantics" in our paper.

In [17], Fornasiere and Moraschini studied the correspondence and canonicity theory for different fragments of intuitionistic propositional logic. Their result is essentially based on generalized Esakia duality ([14, 15]).

The present paper aims at obtaining correspondence theory results in the language of modal meet-implication logic (resp. modal meet-semilattice logic), characterizing formulas in the language of modal implicative semilattices (resp. modal meet-semilattices) in terms of first-order conditions on the dual modal I-spaces (resp. descriptive modal M-frames), which can be regarded as topological correspondence theory, as well as modal I-frames (resp. modal M-frames), which can be regarded as discrete correspondence theory.

Our methodology follows algorithmic correspondence theory ([9, 10]), which uses an algorithm ALBA[1] to transform an input formula/inequality in the language of certain algebras into its first-order correspondent on the dual topological spaces.

The special feature in the present paper includes the following three points: the first one is that the semantic structure used is based on a semilattice rather than an ordinary partial order; the second one is that the propositional variables are interpreted as filters rather than upsets, and the nominals, which are the "first-order" counterparts of propositional variables, are interpreted as principal filters rather than principal upsets; the third one is that in topological correspondence theory, the collection of admissible valuations is not closed under taking disjunction, which makes the proof of the topological Ackermann lemma different from existing settings. The present work can also be taken as another step towards correspondence theory for logics which are not based on bounded lattices, after [24].

The paper is organized as follows: Section 2 gives the preliminaries on the algebraic, topological and relational structures that are related to modal meet-implication logic and modal meet-semilattice logic, as well as the object-level duality and equivalence involved. Section 3 gives the languages to describe modal meet-implication logic and modal meet-semilattice logic, as well as their semantics. Section 4 gives the expanded modal languages that the algorithms manipulate as well as the first-order correspondence languages. Section 5 defines the inductive inequalities and inductive formulas for the language of modal meet-implication logic, and Sahlqvist inequalities and inductive quasi-inequalities for the language of modal meet-semilattice logic. Section 6 describes the algorithms that compute the first-order correspon-

[1]Here ALBA means "Ackermann Lemma Based Algorithm".

dents of the modal formulas/inequalities/quasi-inequalities. Section 7 shows that the algorithms succeed on inductive formulas/Sahlqvist inequalities/inductive quasi-inequalities. Section 8 proves that the algorithms are sound for inductive formulas/inductive quasi-inequalities with respect to arbitrary valuations. Section 9 does the same with respect to admissible valuations and gives the proof of the right-handed topological Ackermann lemma. Section 10 gives conclusions of the paper.

2 Preliminaries

In the present section, we give the preliminaries on the algebraic, topological and relational structures that are related to modal meet-implication logic and modal meet-semilattice logic, as well as the object-level duality and equivalence involved. For more details, see [21].

2.1 Basic algebraic, topological and relational structures

Given a poset $(X, \leq)$, we say a subset $Y \subseteq X$ is an *upset* (resp. a *downset*) if for any $x, y \in X$, if $x \in Y$ and $x \leq y$ (resp. $y \leq x$) then $y \in Y$. We use $\uparrow U$ (resp. $\downarrow U$) to denote the upward (resp. downward) closure of U under the partial order $\leq$. We use $\uparrow x$ (resp. $\downarrow x$) to denote the *principal* upset (resp. downset) $\uparrow\{x\}$ (resp. $\downarrow\{x\}$). Given a full set X, we use Y^c to denote the complement of Y relative to X, i.e. $X - Y$.

2.1.1 Semilattices

Definition 1 (Semilattice, see page 3 in [21]). A *meet-semilattice* is a poset $(A, \leq)$ where any two element a, b has a greatest lower bound (which we call the *meet* of a and b), denoted as $a \wedge b$, and there is a greatest element $\top$ with respect to the partial order $\leq$. We use interchangeably $(A, \wedge, \top)$ and $(A, \leq)$ for a semilattice, where $a \leq b$ iff $a \wedge b = a$.

A *filter* in a meet-semilattice A is a non-empty upset $F \subseteq A$ which is closed under taking meets. It is easy to see that principal upsets of the form $\uparrow x$ are filters.

We write $\mathcal{F}(A)$ as the collection of filters of a meet-semilattice A. With conjunction $\cap$ and top element A, the set $\mathcal{F}(A)$ forms a meet-semilattice. For every $a \in A$ the collection $\tilde{a} = \{F \in \mathcal{F}(A) \mid a \in F\}$ is a filter in $\mathcal{F}(A)$.

2.1.2 M-spaces

On the object-level, the duals of meet-semilattices are M-spaces.

Definition 2 (M-Space, see page 4 in [21]). An *M-space* is a tuple $(X, \sqcap, \overline{\top}, \tau)$ such that $(X, \sqcap, \overline{\top})$ is a meet-semilattice, and (X, τ) is a Stone space generated by a subbase of clopen filters and their complements.

Let SL be the collection of semilattices, MSpace be the collection of M-spaces.

Define $\mathcal{F}^t : \mathrm{SL} \to \mathrm{MSpace}$ sending a semilattice A to the M-space $(\mathcal{F}(A), \cap, A, \tau_A)$ where $\mathcal{F}(A)$ is the semilattice of filters of A, τ_A is a topology generated by $\tilde{a} = \{F \in \mathcal{F}(A) \mid a \in F\}$ and $\tilde{a}^c = \{F \in \mathcal{F}(A) \mid a \notin F\}$ with a ranging over A.

Define $Clp^f : \mathrm{MSpace} \to \mathrm{SL}$ sending an M-space $(X, \sqcap, \overline{\top}, \tau)$ to $(Clp^f(X), \cap, X)$ where $Clp^f(X)$ is the collection of clopen filters of X, with meet $\cap$ and top X.

Then we have that $Clp^f(\mathcal{F}^t(A))$ is a semilattice isomorphic to A, and $\mathcal{F}^t(Clp^f(X))$ is homeomorphic to X which is also order-isomorphic (see Theorem 2.3 in [21]).

2.1.3 Distributive semilattices and implicative semilattices

Definition 3 (Distributive Semilattice, see page 5 in [21]). A meet-semilattice A is called *distributive* if for all $a, b, c \in A$, if $a \wedge b \leq c$ then there are $a', b' \in A$ such that $a \leq a'$, $b \leq b'$ and $c = a' \wedge b'$.

In a distributive meet-semilattice A we can define the smallest filter containing two given filters F and F' by $\langle F, F' \rangle = \{a \wedge b \mid a \in F, b \in F'\}$. It is easy to see that it is a filter.

Proposition 1 (Proposition 2.4 in [21]). *If A is a distributive semilattice, then so is $\mathcal{F}(A)$.*

We now define the algebraic structure of our study, i.e. implicative semilattices:

Definition 4 (Implicative Semilattice, see page 5 in [21]). A semilattice A is *implicative* if we can define a binary operation $\to$ such that $x \leq y \to z$ iff $x \wedge y \leq z$ for all $x, y, z \in A$.

Proposition 2 (Proposition 2.6 and Corollary 2.7 in [21]).

- *The semilattice underlying an implicative semilattice is a distributive semilattice.*
- *If A is a distributive semilattice, then so is $\mathcal{F}(A)$.*
- *If A is an implicative semilattice, then so is $\mathcal{F}(A)$.*

It is clear that an implicative semilattice satisfies all the intuitionistic propositional logic laws that involve $\wedge, \to, \top$ only.

2.1.4 I-spaces

We now define the "Stone spaces" of our setting, i.e. I-spaces, which are dual to implicative semilattices.

Definition 5 (I-Space, Definition 3.6 in [21]). An *I-space* is an M-space whose underlying partial order $\sqsubseteq$ is an implicative semilattice and whose collection of clopen filters is closed under the operation $\rightarrow$ given by

$$a \rightarrow b = \{x \in X \mid \text{ for all } y \sqsupseteq x, \text{ if } y \in a \text{ then } y \in b\}.$$

2.1.5 relational structures: I-frames and descriptive I-frames

We now define the "Kripke frames" of our setting, i.e. I-frames.

Definition 6 (I-Frame, Definition 3.2 in [21]). An *I-frame* is an implicative semilattice regarded as a Kripke frame $(X, \sqsubseteq)$ with $\sqsubseteq$ as the partial order of the implicative semilattice. That is, an I-frame is a poset $(X, \sqsubseteq)$ with a meet $\sqcap$ and an implication $\rightharpoonup$.[2]

In the definition of the semantics of formulas, rather than using upsets as denotations of formulas, we use filters (i.e. upsets closed under finite meets), since the collection of filters is closed under intersections, but not under unions.

We now define the "descriptive general frames" of our setting, i.e. *descriptive I-frames* which are also dual to implicative semilattices:

Definition 7 (Descriptive I-Frame, Definition 3.5 in [21]). A *descriptive I-frame* is a tuple $(X, \sqsubseteq, A)$ consisting of an I-frame $(X, \sqsubseteq)$ and a collection $A \subseteq \mathcal{F}(X, \sqsubseteq)$ of filters such that

- A is closed under $\cap$ and $\rightarrow$, given by

$$a \rightarrow b = \{x \in X \mid \text{ for all } y \sqsupseteq x, \text{ if } y \in a \text{ then } y \in b\}.$$

- A is differentiated: if $x \not\sqsubseteq y$ in $(X, \leq)$ then there exists $F \in A$ such that $x \in F$ and $y \notin F$;
- $(X, \sqsubseteq)$ is compact: every cover of X consisting of elements in A and complements of elements in A has a finite subcover.

Descriptive I-frames are equivalent to I-spaces (see Proposition 3.7 in [21]):

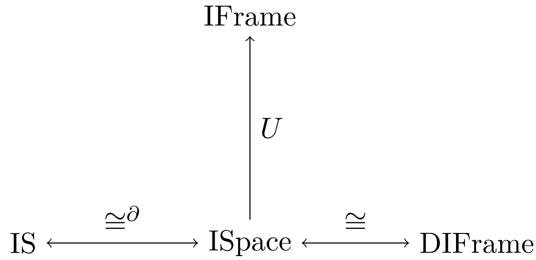
- If $(X, \sqcap, \rightharpoonup, \overline{\top}, \tau)$ is an I-space, we can define an order $\sqsubseteq$ on X by $x \sqsubseteq y$ iff $x \sqcap y = x$. Then the tuple $(X, \sqsubseteq, Clp^f(X))$ is a descriptive I-frame.
- Given a descriptive I-frame $(X, \sqsubseteq, A)$, we have an implicative semilattice $(X, \sqcap, \rightharpoonup, \overline{\top})$. Generate a topology τ_A from the subbase $A \cup \{a^c \mid a \in A\}$. Then $(X, \sqcap, \rightharpoonup, \overline{\top}, \tau_A)$ is an I-space.

[2] In the present paper, we use $\wedge, \rightarrow, \top, \leq$ to denote the meet, implication, top element and order in an algebraic structure, and $\sqcap, \rightharpoonup, \overline{\top}, \sqsubseteq$ to denote the meet, implication, top element (defined from the partial order) and the partial order in a relational structure and a topological structure.

In the remainder of the paper, we associate a descriptive I-frame with the topology of its equivalent I-space.

By the constructions $\mathcal{F}^t$ and Clp^f, it is easy to see the object-level duality between I-spaces and implicative semilattices.

We can use the following picture to describe relations between the structures that we are using (again here we only consider the object-level): we use IS to denote the class of implicative semilattices, ISpace to denote the class of I-spaces, IFrame to denote the class of I-frames, $\mathrm{DIFrame}$ to denote the class of descriptive I-frames. Here $\cong$ means equivalence, $\cong^\partial$ means dual equivalence, U is the operation of forgetting the topology (in categorical settings it is called the forgetful functor).



2.2 Adding modalities

2.2.1 Adding modalities to implicative semilattices

In this subsection, we add modalities for implicative semilattices, and relations for I-frames and descriptive I-frames.

Definition 8 (Modal Implicative Semilattice, see page 11 in [21]). [3] A *modal implicative semilattice* is a tuple $A = (A, \wedge, \to, \top, \Box)$ such that $(A, \wedge, \to, \top)$ is a implicative semilattice, $\Box\top = \top$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in A$.

Definition 9 (Modal I-Frame, Definition 4.1 in [21]). [4] A *modal I-frame* is a tuple $\mathbb{F} = (X, \sqsubseteq, R)$ where $(X, \sqsubseteq)$ is an I-frame and R is a binary relation on X satisfying:

- $\overline{\top} Rx$ iff $x = \overline{\top}$, and $xR\overline{\top}$ for all $x \in X$;
- If $xRy \sqsubseteq z$ then xRz;
- If xRy and $x'Ry'$ then $(x \sqcap x')R(y \sqcap y')$;
- If $(x \sqcap x')Rz$ then there are $y, y' \in X$ such that xRy and $x'Ry'$ and $y \sqcap y' = z$.

Definition 10 (General Modal I-Frame and Descriptive Modal I-Frame, Definition 5.1 in [21]). [5] A *general modal I-frame* is a tuple $\mathbb{F} = (X, \sqsubseteq, R, A)$ where $(X, \sqsubseteq$

[3]In [21] it is called "implicative semilattices with operators".

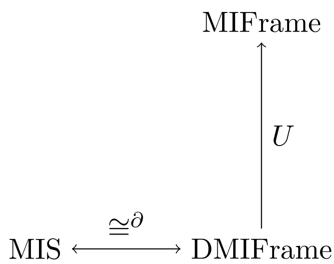
[4]In [21] it is called "□-frame".

[5]In [21] they are called "general □-frame" and "descriptive □-frame".

$, R)$ is a modal I-frame and $(X, \sqsubseteq, A)$ is a descriptive I-frame such that A is closed under $\Box(a) := \{x \in X \mid R[x] \subseteq a\}$ (here $R[x] := \{y \in X \mid xRy\}$). It is called a *descriptive modal I-frame* if $R[x] = \bigcap\{a \in A \mid R[x] \subseteq a\}$ for all $x \in X$.

The duality between implicative semilattices and descriptive I-frames can be naturally extended to modal implicative semilattices and descriptive modal I-frames.

We can use the following picture to describe relations between the structures here: we use MIS to denote the class of modal implicative semilattices, $\mathrm{MIFrame}$ to denote the class of modal I-frames, $\mathrm{DMIFrame}$ to denote the class of descriptive modal I-frames.



2.2.2 Adding modalities to meet-semilattices

The following definitions for meet-semilattices are similar to their counterparts in the setting of implicative semilattices:

Definition 11 (Modal Meet-Semilattice). A *modal meet semilattice* is a tuple $A = (A, \wedge, \top, \Box)$ such that $(A, \wedge, \top)$ is a meet-semilattice, $\Box\top = \top$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in A$.

Definition 12 (M-Frame, Definition 3.2 in [21]). An *M-frame* is a meet-semilattice regarded as a Kripke frame $(X, \sqsubseteq)$ with $\sqsubseteq$ as the partial order of the meet-semilattice. That is, an M-frame is a poset $(X, \sqsubseteq)$ with a meet $\sqcap$.

Definition 13 (Modal M-Frame). A *modal M-frame* is a tuple $\mathbb{F} = (X, \sqsubseteq, R)$ where $(X, \sqsubseteq)$ is an M-frame and R is a binary relation on X satisfying:

- $\overline{\top}Rx$ iff $x = \overline{\top}$, and $xR\overline{\top}$ for all $x \in X$;
- If $xRy \sqsubseteq z$ then xRz;
- If xRy and $x'Ry'$ then $(x \sqcap x')R(y \sqcap y')$;
- If $(x \sqcap x')Rz$ then there are $y, y' \in X$ such that xRy and $x'Ry'$ and $y \sqcap y' \sqsubseteq z$.

Notice that due to the lack of distributivity, the fourth condition is different from the implicative semilattice setting.

The following definition is equivalent to an M-space, like the equivalence between descriptive I-frames and I-spaces. Therefore, we associate a descriptive M-frame with the topology of its equivalent M-space.

Definition 14 (Descriptive M-Frame)**.** A *descriptive M-frame* is a tuple $(X, \sqsubseteq, A)$ consisting of an M-frame $(X, \sqsubseteq)$ and a collection $A \subseteq \mathcal{F}(X, \sqsubseteq)$ of filters such that

- A is closed under $\cap$;
- A is differentiated: if $x \not\sqsubseteq y$ in $(X, \leq)$ then there exists $F \in A$ such that $x \in F$ and $y \notin F$;
- $(X, \sqsubseteq)$ is compact: every cover of X consisting of elements in A and complements of elements in A has a finite subcover.

Definition 15 (General Modal M-Frame and Descriptive Modal M-Frame)**.** A *general modal M-frame* is a tuple $\mathbb{F} = (X, \sqsubseteq, R, A)$ where $(X, \sqsubseteq, R)$ is a modal M-frame and $(X, \sqsubseteq, A)$ is a descriptive M-frame such that A is closed under $\Box(a) := \{x \in X \mid R[x] \subseteq a\}$ (here $R[x] := \{y \in X \mid xRy\}$). It is called a *descriptive modal M-frame* if $R[x] = \bigcap\{a \in A \mid R[x] \subseteq a\}$ for all $x \in X$.

3 Syntax and Semantics

In the present section, we give the syntax and semantics of the logical formulas for modal meet-implication logic and modal meet-semilattice logic. We follow the presentation of [21, 23, 24].

3.1 Modal meet-implication logic

3.1.1 Language and syntax

Definition 16. Let us fix a countable set Prop of propositional variables.

- The *$\mathcal{L}$-formulas* of the modal language $\mathcal{L}$ is defined as follows:

$$\varphi ::= p \mid \top \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \Box\varphi,$$

  where $p \in \mathrm{Prop}$.
- The *$\mathcal{L}$-inequalities* of the modal language $\mathcal{L}$ is of the form $\varphi \leq \psi$ where φ and ψ are $\mathcal{L}$-formulas. Intuitively, $\varphi \leq \psi$ expresses the model-level truth of the implicative formula $\varphi \to \psi$.
- The *$\mathcal{L}$-quasi-inequalities* of the modal language $\mathcal{L}$ is of the form $\varphi_1 \leq \psi_1$ & ... & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ where $\varphi_1 \leq \psi_1$, ..., $\varphi_n \leq \psi_n$, $\varphi \leq \psi$ are $\mathcal{L}$-inequalities.
- We use the notation $\overline{p}$ to denote a list of propositional variables and use $\varphi(\overline{p})$ to indicate that the propositional variables occur in φ are all in $\vec{p}$.

- We call a formula *pure* if it does not contain propositional variables.
- We use the notation $\overline{\theta}$ to indicate a finite list of formulas.
- We use the notation $\theta(\eta/p)$ to indicate uniformly substituting p by η.

3.1.2 Semantics

We interpret formulas on the modal I-frames $(X, \sqsubseteq, R)$ and descriptive modal I-frames $(X, \sqsubseteq, R, A)$, with two different kinds of valuations, namely *arbitrary valuations* which interpret propositional variables as arbitrary filters, and *admissible valuations* which interpret propositional variables as elements in A (i.e. clopen filters in the associated topology of the descriptive modal I-frames).

In a modal I-frame $(X, \sqsubseteq, R)$ or a descriptive modal I-frame $(X, \sqsubseteq, R, A)$, we abuse notation to use X to denote the frame. We call X the *domain* of the frame.

Definition 17. An *arbitrary modal I-model* is a tuple $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a modal I-frame and $V : \mathrm{Prop} \to \mathcal{F}(X)$ is an *arbitrary valuation* sends a propositional variable to a filter of X.

An *admissible modal I-model* is a tuple $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a descriptive modal I-frame and $V : \mathrm{Prop} \to A$ is an *admissible valuation* sends a propositional variable to an element of A, i.e. a clopen (with respect to the associated topology of $\mathbb{F}$) filter of the space.

The interpretation of formulas in a modal I-model $\mathbb{M} = (X, \sqsubseteq, V)$ is defined as follows:

$\mathbb{M}, w \Vdash p$	iff	$w \in V(p)$
$\mathbb{M}, w \Vdash \top$	:	always
$\mathbb{M}, w \Vdash \varphi \wedge \psi$	iff	$\mathbb{M}, w \Vdash \varphi$ and $\mathbb{M}, w \Vdash \psi$
$\mathbb{M}, w \Vdash \varphi \to \psi$	iff	for all $v \sqsupseteq w$, if $\mathbb{M}, v \Vdash \varphi$ then $\mathbb{M}, v \Vdash \psi$
$\mathbb{M}, w \Vdash \Box\varphi$	iff	for all v such that wRv, we have $\mathbb{M}, v \Vdash \varphi$

Given a valuation V, a propositional variable $p \in \mathrm{Prop}$, an filter $F \subseteq X$, we can define V^p_F, the *p-variant of V* as follows: $V^p_F(q) = V(q)$ for all $q \neq p$ and $V^p_F(p) = F$.

For any formula φ, we let $V(\varphi) = \{w \in X \mid \mathbb{M}, w \Vdash \varphi\}$ denote the *truth set* of φ in $\mathbb{M}$.

Lemma 1. *1. In a modal I-frame $\mathbb{F} = (X, \sqsubseteq, R)$, we have that if $x \sqsubseteq yRz$, then xRz.*

2. If $F \subseteq X$ is a filter of X, then $\Box F := \{w \in X \mid R[w] \subseteq F\}$ is again a filter of X.

3. For any formula φ, any arbitrary modal I-model $(\mathbb{F}, V)$, $V(\varphi)$ is a filter of X.

4. *For any formula φ, any admissible modal I-model $(\mathbb{F}, V)$, $V(\varphi) \in A$, i.e. $V(\varphi)$ is a clopen filter of X.*

Proof. 1. If $x \sqsubseteq yRz$, then $xR\overline{\top}$ and yRz, therefore by Definition 9, we have that $(x \sqcap y)R(\overline{\top} \sqcap z)$, i.e. xRz.

2. To show that $\Box F$ is a filter of X, it suffices to show that $\overline{\top} \in \Box F$, $\Box F$ is a $\sqsubseteq$-upset and $\Box F$ is closed under taking $\sqcap$.
For any R-successor x of $\overline{\top}$, by Definition 9, $x = \overline{\top}$, since F is a filter, we have $x = \overline{\top} \in F$, therefore $\overline{\top} \in \Box F$.
For any $x \in \Box F$ and $y \in X$ such that $x \sqsubseteq y$, it suffices to show that $y \in \Box F$. Consider any $z \in X$ such that yRz, we have that $x \sqsubseteq yRz$, by item 1 we have xRz. Since $x \in \Box F$, we get $z \in F$. Therefore $y \in \Box F$, so $\Box F$ is a $\sqsubseteq$-upset.
For any $x, y \in \Box F$, to show that $x \sqcap y \in \Box F$, consider any $z \in X$ such that $x \sqcap yRz$, then there are $x', y' \in X$ such that xRx', yRy' and $x' \sqcap y' = z$. Therefore $x', y' \in F$, and since F is a filter, we have $z = x' \sqcap y' \in F$. Therefore $x \sqcap y \in \Box F$, $\Box F$ is closed under taking $\sqcap$.

3. We prove by induction on the formula structure.

- For p, it follows from the definition of valuation.
- For $\top$, it follows from that X is a filter of X.
- For $\wedge$, since $\mathcal{F}(X)$ is closed under taking intersection, we have that if $V(\varphi)$ and $V(\psi)$ are filters of X, then $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$ is again a filter of X.
- For $\to$, assume that $V(\varphi)$ and $V(\psi)$ are filters of X. It is easy to see that $V(\varphi \to \psi)$ is a $\sqsubseteq$-upset and $\overline{\top} \in V(\varphi \to \psi)$. To show that $V(\varphi \to \psi)$ is closed under taking $\sqcap$, assume that $x, y \in V(\varphi \to \psi)$, it suffices to show that $x \sqcap y \in V(\varphi \to \psi)$.
  Consider any $z \sqsupseteq x \sqcap y$, then if $z \in V(\varphi)$, then since X is an implicative semilattice (hence distributive), there are $x', y' \in X$ such that $x \sqsubseteq x'$, $y \sqsubseteq y'$ and $x' \sqcap y' = z$. Since $z \in V(\varphi)$, we have that $x' \sqcap y' \in V(\varphi)$. Since $V(\varphi) \in \mathcal{F}(X)$, we have $x', y' \in V(\varphi)$. Since $x, y \in V(\varphi \to \psi)$, by $x \sqsubseteq x'$, $y \sqsubseteq y'$, $x', y' \in V(\varphi)$ we have $x', y' \in V(\psi)$. Since $V(\psi) \in \mathcal{F}(X)$, we have $z = x' \sqcap y' \in V(\psi)$. Therefore $x \sqcap y \in V(\varphi \to \psi)$, $V(\varphi \to \psi)$ is a filter of X.
- For $\Box$, it follows from item 2.

4. We prove by induction on the formula structure.

- For p, it follows from the definition of admissible valuation.
- For $\top$, it follows from that X is a clopen filter of X.
- For $\wedge, \to, \Box$, it follows from the definition of descriptive modal I-frame.

$\Box$

Definition 18. We say that

- φ is *globally true* on $\mathbb{M}$ (notation: $\mathbb{M} \Vdash \varphi$) if $\mathbb{M}, w \Vdash \varphi$ for every $w \in W$.
- φ is *admissibly valid* on a descriptive modal I-frame $\mathbb{F}$ (notation: $\mathbb{F} \Vdash_A \varphi$) if φ is globally true on $(\mathbb{F}, V)$ for every admissible valuation V.
- φ is *valid* on a modal I-frame $\mathbb{F}$ (notation: $\mathbb{F} \Vdash \varphi$) if φ is globally true on $(\mathbb{F}, V)$ for every arbitrary valuation V.

For the semantics of inequalities and quasi-inequalities,

- $\mathbb{M} \Vdash \varphi \leq \psi$ iff $V(\varphi) \subseteq V(\psi)$;
- $\mathbb{M} \Vdash \varphi_1 \leq \psi_1$ & $\ldots$ & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ iff $\mathbb{M} \Vdash \varphi \leq \psi$ holds whenever $\mathbb{M} \Vdash \varphi_i \leq_i \psi_i$ for all $i = 1, \ldots, n$.

The definitions of validity are similar to formulas.

3.2 Syntax and semantics for modal meet-semilattice logic

The language of modal meet-semilattice logic is defined similar to its modal meet-implication counterpart, the only difference is that there is no implication $\rightarrow$ here. The semantic interpretation is on modal M-frames and descriptive modal M-frames, and the semantic clauses for the formulas are almost the same as modal meet-implication logic, except that there is no clause for $\rightarrow$. Other semantic definitions are similar to the modal meet-implication setting (we just replace the letter I by the letter M).

The proof of the semilattice counterpart of Lemma 1 is a bit different, which we give below:

Lemma 2. *1. In a modal M-frame $\mathbb{F} = (X, \sqsubseteq, R)$, we have that if $x \sqsubseteq yRz$, then xRz.*

2. *If $F \subseteq X$ is a filter of X, then $\Box F := \{w \in X \mid R[w] \subseteq F\}$ is again a filter of X.*
3. *For any formula φ, any arbitrary modal M-model $(\mathbb{F}, V)$, $V(\varphi)$ is a filter of X.*
4. *For any formula φ, any admissible modal M-model $(\mathbb{F}, V)$, $V(\varphi) \in A$, i.e. $V(\varphi)$ is a clopen filter of X.*

Proof. 1. This proof is the same as Lemma 1.1.

2. This proof is a bit different from Lemma 1.2. To show that $\Box F$ is a filter of X, again it suffices to show that $\overline{\top} \in \Box F$, $\Box F$ is a $\sqsubseteq$-upset and $\Box F$ is closed under taking $\sqcap$.

   To show that $\overline{\top} \in \Box F$ and $\Box F$ is a $\sqsubseteq$-upset, the proofs are the same as in Lemma 1.2. The proof that $\Box F$ is closed under taking $\sqcap$ is a bit different:

For any $x, y \in \Box F$, to show that $x \sqcap y \in \Box F$, consider any $z \in X$ such that $x \sqcap yRz$, then there are $x', y' \in X$ such that xRx', yRy' and $x' \sqcap y' \sqsubseteq z$. Therefore $x', y' \in F$, and since F is a filter, we have $x' \sqcap y' \in F$, since a filter is also an $\sqsubseteq$-upset, we have that $x' \sqcap y' \sqsubseteq z \in F$. Therefore $x \sqcap y \in \Box F$, $\Box F$ is closed under taking $\sqcap$.

3. The proof is almost the same as Lemma 1.3, except that we do not need the clause for $\to$.
4. The proof is almost the same as Lemma 1.4, except that we do not need the clause for $\to$.

□

4 Preliminaries on Algorithmic Correspondence

In this section, we give preliminaries on the correspondence algorithms in the style of [10, 22, 23]. The algorithm for modal meet-implication logic transforms the input formula $\varphi \to \psi$ into an equivalent pure quasi-inequality which contains no propositional variable, and therefore can be translated into the first-order correspondence language via the standard translation of the expanded language (see page 40). The algorithm for modal meet-semilattice logic does similar things, but the input is either an inequality or a quasi-inequality.

4.1 Semantic environment of modal meet-implication logic

In the present subsection, we will provide the semantic environment for the correspondence algorithm ALBA in the settings of modal meet-implication logic, in the style of [22, Section 3]. We will show the semantic properties which will be used for the interpretation of the expanded modal language of the algorithm ALBA in Section 4.1.3.

The first notable feature of this language is that it includes special variables (besides the propositional variables), the so-called *nominals*. In existing settings (see the table below), they are interpreted as atoms (in complete atomic Boolean algebras), completely join-prime elements (in perfect distributive lattices), completely join-irreducible elements (in perfect (non-distributive) lattices), regular open closures of singletons (in the setting of possibility semantics). The common feature of these settings are that the selected class of elements join-generates the relevant complete lattices. Therefore, since in our setting, given a modal I-frame $(X, \sqsubseteq, R)$, the relevant algebraic structure is $\mathcal{F}(X)$, the implicative semilattice of filters of X (which is the "perfect" counterpart in the setting of implicative semilattices). Since arbitrary intersection of filters of a semilattice is still a filter, $\mathcal{F}(X)$ is a complete semilattice, hence is a complete lattice. This is the algebraic structure we are focusing on. We will show that the principal filters are a suitable class of interpretants for nominal variables.

Propositional base	Nominals/join-generators
perfect Boolean algebras	atoms
perfect distributive lattices	complete join-primes
perfect general lattices	complete join-irreducibles
constructive canonical extensions	closed elements
possibility semantics	regular open closures of singletons
“perfect” implicative semilattices	principal filters

The second notable feature of the expanded language of ALBA is that it includes additional modal operators interpreted as the adjoints of the modal operators of the original language. In what follows, we will show that also these connectives have a natural interpretation in $\mathcal{F}(X)$.

The third feature that we will focus on is the interpretation of the disjunction in the expanded language. Although we do not have disjunction in our original language, we still have a natural interpretation of disjunction in $\mathcal{F}(X)$.

4.1.1 A class of interpretants for nominals, as well as the interpretation of $\vee$

As mentioned early on, the key requirement for a suitable class of interpretants for nominals is that it is join-dense in $\mathcal{F}(X)$. We have the implicative semilattice structure of $\mathcal{F}(X)$ as $(\mathcal{F}(X), \cap, \to, X)$. Since arbitrary intersections of filters of X is again a filter, $(\mathcal{F}(X), \bigcap, X)$ is a complete meet-semilattice. Therefore, $\mathcal{F}(X)$ is a complete lattice where the arbitrary join operation is defined as follows:

$$\bigvee_{F_i \in Y} F_i := \bigcap\{F \in \mathcal{F}(X) \mid F_i \subseteq F \text{ for all } F_i \in Y\}.$$

Indeed, we can show the following proposition:

Proposition 3 (cf. Lemma 2.14 in [5])**.**

$$\bigvee_{F_i \in Y} F_i = \{x_1 \sqcap \ldots \sqcap x_n \mid x_j \in F_j \textit{ for a finite collection of filters } F_j \textit{ in } Y\}.$$

Proof. We denote $\{x_1 \sqcap \ldots \sqcap x_n \mid x_j \in F_j$ for a finite collection of filters F_j in $Y\}$ as Z. First of all, Z contains all elements in $\{x_i \mid x_i \in F_i \in Y\}$, i.e. F_i. Therefore,

$$Z \supseteq \bigcup_{F_i \in Y} F_i.$$

Next we prove that Z is a filter of X. It is easy to see that Z is closed under taking $\sqcap$ and contains $\overline{\top}$, and therefore is non-empty. Now take any $x_1 \sqcap \ldots \sqcap x_n \in Z$. Then for any $y_1 \in X$ such that $x_1 \sqcap \ldots \sqcap x_n \sqsubseteq y_1$, since X is an implicative semilattice, it is a distributive semilattice, therefore there are $z_1, y_2 \in X$ such that $x_1 \sqsubseteq z_1$ and $x_2 \sqcap \ldots \sqcap x_n \sqsubseteq y_2$ and $z_1 \sqcap y_2 = y_1$. By repeating this procedure, there are $z_2, \ldots, z_n \in X$ such that $x_2 \sqsubseteq z_2$, …, $x_n \sqsubseteq z_n$, and $z_1 \sqcap \ldots \sqcap z_n = y_1$. Now since $x_i \in F_i$ for $i = 1, \ldots, n$, we have $z_i \in F_i$ for $i = 1, \ldots, n$, therefore $y_1 \in Z$. So Z us closed under taking $\sqsubseteq$-upsets. Therefore $Z \in \mathcal{F}(X)$.

From the above we have that $Z \in \{F \in \mathcal{F}(X) \mid F_i \subseteq F \text{ for all } F_i \in Y\}$. It suffices to prove that $Z \subseteq \bigcap\{F \in \mathcal{F}(X) \mid F_i \subseteq F \text{ for all } F_i \in Y\}$.

For any $F \in \mathcal{F}(X)$ such that $F_i \subseteq F$ for all $F_i \in Y$, it suffices to prove that $Z \subseteq F$. For any $y \in Z$, y is of the form $x_1 \sqcap \ldots \sqcap x_n$ where $x_i \in F_i$ and $F_i \in Y$ for $i = 1, \ldots, n$. Therefore $x_1, \ldots, x_n \in F$, and thus $y = x_1 \sqcap \ldots \sqcap x_n \in F$. So $Z \subseteq F$. □

Therefore, we have the following proposition:

Proposition 4. *For any filter $F \in \mathcal{F}(X)$, $F = \bigvee\{\uparrow x \mid x \in F\}$.*

Proof. It is easy to see that for any $x \in F$, $\uparrow x \subseteq F$, therefore $\bigvee\{\uparrow x \mid x \in F\} \subseteq F$. For any $y \in F$, we have $y \in \uparrow y \subseteq \bigvee\{\uparrow x \mid x \in F\}$, therefore $F \subseteq \bigvee\{\uparrow x \mid x \in F\}$. □

We will interpret nominals as principal upsets (i.e. principal filters) in the next subsection.

Although in the signature of the modal language, we do not have $\vee$, in the expanded modal language, we will have $\vee$ which is interpreted as the join of the complete lattice $\mathcal{F}(X)$, i.e.

$$F_1 \vee F_2 = \{x_1 \sqcap x_2 \mid x_1 \in F_1 \text{ and } x_2 \in F_2\}.$$

For similar interpretations of $\vee$, see [5] and [13, Section 3.3].

4.1.2 Interpreting $\blacklozenge$: the left adjoint of $\Box$

As we know, $\mathcal{F}(X)$ is a complete lattice, and the operation $\Box : \mathcal{F}(X) \to \mathcal{F}(X)$ where $\Box F := \{w \in X \mid R[w] \subseteq F\}$ is clearly completely meet-preserving, therefore $\Box$ has a left adjoint, which are denoted by $\blacklozenge$. It will be used in the semantic interpretation of the additional connectives in the expanded modal language in the next subsection.

In what follows we will prove that $\blacklozenge : \mathcal{F}(X) \to \mathcal{F}(X)$ is exactly $\blacklozenge F := R[F] = \{w \in X \mid (\exists v \in F)(Rvw)\}$.

Proof. It suffices to prove the following two items:

1. For any $F \in \mathcal{F}(X)$, $R[F] \in \mathcal{F}(X)$.
   We prove that $\overline{\top} \in R[F]$, $R[F]$ is a $\sqsubseteq$-upset and $R[F]$ is closed under taking $\sqcap$.
   By the definition of a filter, we have that $F \neq \varnothing$, therefore $\overline{\top} \in F$. By Definition 9, we have that $\overline{\top} R \overline{\top}$, so $\overline{\top} \in R[F]$.
   For any $x \in R[F]$ and $y \sqsupseteq x$, there is a $z \in F$ such that $zRx \sqsubseteq y$, by Definition 9 we have zRy, therefore $y \in R[F]$. So $R[F]$ is a $\sqsubseteq$-upset.
   For any $x, y \in R[F]$, we have $x', y' \in F$ such that $x'Rx$ and $y'Ry$. By Definition 9 we have $x' \sqcap y' R x \sqcap y$. Since F is a filter, it is closed under taking $\sqcap$, so $x' \sqcap y' \in F$, therefore $x \sqcap y \in R[F]$. So $R[F]$ is closed under taking $\sqcap$.
2. For any $F, G \in \mathcal{F}(X)$, $R[F] \subseteq G$ iff $F \subseteq \Box(G)$.
   $\Rightarrow$: Assume that $R[F] \subseteq G$ and $x \in F$. Then for any $y \in X$ such that xRy, $y \in R[F] \subseteq G$, so $x \in \Box(G)$. Therefore $F \subseteq \Box(G)$.
   $\Leftarrow$: Assume that $F \subseteq \Box(G)$. For any $x \in R[F]$, there is a $y \in F$ such that yRx. So $y \in F \subseteq \Box(G)$. So by yRx we have $x \in G$. Therefore $R[F] \subseteq G$.

□

4.1.3 The expanded modal language

In the present subsection, we give the definition of the expanded modal language $\mathcal{L}^+$, which will be used in ALBA:

$$\varphi ::= p \mid \mathbf{i} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \blacklozenge\varphi$$

where $\mathbf{i} \in \mathrm{Nom}$ is called a *nominal*. $\mathcal{L}^+$-inequalities and $\mathcal{L}^+$-quasi-inequalities are defined in the expected ways.

- For **i**, it is interpreted as a principal upset (i.e. principal filter).
- For $\blacklozenge$, it is interpreted as the diamond modality on the inverse relation R^{-1}.
- For $\bot$, it is interpreted as the bottom element of $\mathcal{F}(X)$, i.e. the smallest filter of X, i.e. $\{\overline{\top}\}$.
- For $\vee$, it is interpreted as the join of $\mathcal{F}(X)$.

Also notice that although $\bot, \vee$ are not in the language of the modal implicative semilattices, they are in the expanded language for the sake of the algorithm.

For the semantics of the expanded language, the valuation V is extended to $\mathrm{Prop} \cup \mathrm{Nom}$ such that $V(\mathbf{i}) = {\uparrow} i$ for some $i \in X$ for each $\mathbf{i} \in \mathrm{Nom}$.[6] The additional semantic clauses can be given as follows:

[6]Notice that we allow admissible valuations to interpret nominals as ${\uparrow} w$, even if it might not be in A (i.e. it might not be a clopen filter). The admissibility restrictions are only for the propositional variables.

$\mathbb{M}, w \Vdash \bot$	iff	$w = \overline{\top}$
$\mathbb{M}, w \Vdash \mathbf{i}$	iff	$i \sqsubseteq w$
$\mathbb{M}, w \Vdash \varphi \vee \psi$	iff	there exist $u, v \in X$ such that $w = u \sqcap v$ and $\mathbb{M}, u \Vdash \varphi$ and $\mathbb{M}, v \Vdash \psi$
$\mathbb{M}, w \Vdash \blacklozenge\varphi$	iff	$\exists v(Rvw$ and $\mathbb{M}, v \Vdash \varphi)$

4.1.4 The first-order correspondence language and the standard translation

In the first-order correspondence language, we have:

- Binary predicate symbols $R, \sqsubseteq$ corresponding to the binary relation and the partial order in the (descriptive) modal I-frames;
- Binary function symbol $\sqcap$ corresponding to the meet of the partial order in the (descriptive) modal I-frames;
- Constant symbol $\overline{\top}$ corresponding to the top element of the partial order in the (descriptive) modal I-frames;
- Unary predicate symbols P corresponding to propositional variables p;
- Constant symbols i corresponding to nominals $\mathbf{i}$.

Definition 19. The standard translation of the expanded language is defined as follows:

- $ST_x(p) := Px$
- $ST_x(\mathbf{i}) := i \sqsubseteq x$
- $ST_x(\bot) := x = \overline{\top}$
- $ST_x(\top) := x = x$
- $ST_x(\varphi \wedge \psi) := ST_x(\varphi) \wedge ST_x(\psi)$
- $ST_x(\varphi \vee \psi) := \exists y \exists z((x = y \sqcap z) \wedge ST_y(\varphi) \wedge ST_z(\psi))$
- $ST_x(\varphi \to \psi) := \forall y(x \sqsubseteq y \wedge ST_y(\varphi) \to ST_y(\psi))$
- $ST_x(\Box\varphi) := \forall y(Rxy \to ST_y(\varphi))$
- $ST_x(\blacklozenge\varphi) := \exists y(Ryx \wedge ST_y(\varphi))$
- $ST(\varphi \leq \psi) := \forall x(ST_x(\varphi) \to ST_x(\psi))$
- $ST(\varphi_1 \leq \psi_1 \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n \ \Rightarrow \ \varphi \leq \psi) := ST(\varphi_1 \leq \psi_1) \wedge \ldots \wedge ST(\varphi_n \leq \psi_n) \to ST(\varphi \leq \psi)$

It is easy to see that this translation is correct:

Proposition 5. *For any (descriptive) modal I-model* $\mathbb{M}$*, any* $w \in X$ *and any* $\mathcal{L}^+$*-formula* φ*,*

$$\mathbb{M}, w \Vdash \varphi \textit{ iff } \mathbb{M} \vDash ST_x(\varphi)[w].$$

Proposition 6. *For any (descriptive) modal I-model* $\mathbb{M}$*, any* $w \in X$ *and any* $\mathcal{L}^+$*-inequality* Ineq*,* $\mathcal{L}^+$*-quasi-inequality* Quasi*,*

$$\mathbb{M} \Vdash \text{Ineq } \textit{iff } \mathbb{M} \vDash ST(\text{Ineq});$$
$$\mathbb{M} \Vdash \text{Quasi } \textit{iff } \mathbb{M} \vDash ST(\text{Quasi}).$$

4.2 The Case of Modal Meet-Semilattice Logic

In this subsection, we will discuss the semantic environment of the modal meet-semilattice logic, define the expanded modal language and the standard translation.

Since in finding the potential interpretations of nominals and disjunction, in Proposition 3, the distributivity of implicative semilattice is essentially used, so here we need to find new interpretations for nominals and disjunction.

4.2.1 A class of interpretants for nominals, as well as the interpretation of $\vee$

Again, the key requirement for a suitable class of interpretants for nominals is that it is join-dense in $\mathcal{F}(X)$. We have the semilattice structure of $\mathcal{F}(X)$ as $(\mathcal{F}(X), \cap, X)$. Since arbitrary intersections of filters of X is again a filter, $(\mathcal{F}(X), \bigcap, X)$ is a complete meet-semilattice. Therefore, $\mathcal{F}(X)$ is a complete lattice where the arbitrary join operation is defined as follows:

$$\bigvee_{F_i \in Y} F_i := \bigcap\{F \in \mathcal{F}(X) \mid F_i \subseteq F \text{ for all } F_i \in Y\}.$$

For the concrete definition of arbitrary joins, it is different from Proposition 3, which is given below:

Proposition 7. *$\bigvee_{F_i \in Y} F_i = \bigcup\{\uparrow(x_1 \sqcap \ldots \sqcap x_n) \mid x_j \in F_j$ for a finite collection of filters F_j in $Y\}$.*

Proof. We denote $\bigcup\{\uparrow(x_1 \sqcap \ldots \sqcap x_n) \mid x_j \in F_j$ for a finite collection of filters F_j in $Y\}$ as Z. First of all, Z contains all elements in $\{x_i \mid x_i \in F_i \in Y\}$, i.e. F_i. Therefore,

$$Z \supseteq \bigcup_{F_i \in Y} F_i.$$

Next we prove that Z is a filter of X. It is easy to see that Z is closed under taking $\sqsubseteq$-upsets and contains $\top$ (therefore is non-empty). To show that Z is closed under taking $\sqcap$, suppose that $y_1, y_2 \in Z$, then there are $x_1 \in F_1 \in Y, \ldots, x_n \in F_n \in Y$, $x'_1 \in F'_1 \in Y, \ldots, x'_m \in F'_m \in Y$ such that $x_1 \sqcap \ldots \sqcap x_n \sqsubseteq y_1$ and $x'_1 \sqcap \ldots \sqcap x'_m \sqsubseteq y_2$, so $y_1 \sqcap y_2 \in \uparrow(x_1 \sqcap \ldots \sqcap x_n \sqcap x'_1 \sqcap \ldots \sqcap x'_m) \subseteq Z$.

From the above we have that $Z \in \bigcap\{F \in \mathcal{F}(X) \mid F_i \subseteq F \text{ for all } F_i \in Y\}$. It suffices to prove that $Z \subseteq \bigcap\{F \in \mathcal{F}(X) \mid F_i \subseteq F \text{ for all } F_i \in Y\}$.

For any $F \in \mathcal{F}(X)$ such that $F_i \subseteq F$ for all $F_i \in Y$, it suffices to prove that $Z \subseteq F$. For any $y \in Z$, $y \sqsupseteq x_1 \sqcap \ldots \sqcap x_n$ where $x_i \in F_i$ and $F_i \in Y$ for $i = 1, \ldots, n$. Therefore $x_1, \ldots, x_n \in F$, and thus $x_1 \sqcap \ldots \sqcap x_n \sqsubseteq y \in F$. So $Z \subseteq F$. $\square$

Therefore, we have the following proposition, the proof of which is the same as Proposition 4:

Proposition 8. *For any filter $F \in \mathcal{F}(X)$, $F = \bigvee\{\uparrow x \mid x \in F\}$.*

We will interpret nominals as principal upsets (i.e. principal filters) in the next subsection.

The interpretation of $\vee$ in the expanded modal language is different from the implicative setting, which is given as follows:

$$F_1 \vee F_2 = \bigcup\{\uparrow(x_1 \sqcap x_2) \mid x_1 \in F_1 \text{ and } x_2 \in F_2\},$$

i.e. $x \in F_1 \vee F_2$ iff there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $x_1 \sqcap x_2 \sqsubseteq x$.

4.2.2 The interpretation of ⧫

The interpretation of ⧫ is exactly the same as in the modal meet-implication logic setting.

4.2.3 The expanded modal language

The expanded modal language $\mathcal{L}^+$ is also similar to the modal meet-implication logic setting, except that we do not have $\rightarrow$.

$$\varphi ::= p \mid \mathbf{i} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \blacklozenge\varphi.$$

For the semantics of the expanded language, the valuation V is extended to $\mathrm{Prop} \cup \mathrm{Nom}$ such that $V(\mathbf{i}) = \uparrow i$ for some $i \in X$ for each $\mathbf{i} \in \mathrm{Nom}$. The additional semantic clauses can be given as follows:

$\mathbb{M}, w \Vdash \bot$ iff $w = \overline{\top}$
$\mathbb{M}, w \Vdash \mathbf{i}$ iff $i \sqsubseteq w$
$\mathbb{M}, w \Vdash \varphi \vee \psi$ iff there exist $u, v \in X$ such that $u \sqcap v \sqsubseteq w$ and $\mathbb{M}, u \Vdash \varphi$ and $\mathbb{M}, v \Vdash \psi$
$\mathbb{M}, w \Vdash \blacklozenge\varphi$ iff $\exists v(Rvw \text{ and } \mathbb{M}, v \Vdash \varphi)$

4.2.4 The first-order correspondence language and the standard translation

The first-order correspondence language is defined the same as in the modal meet-implication logic setting. The standard translation is almost the same as in the modal meet-implication logic setting except that we do not have $\rightarrow$ here and $\vee$ is interpreted differently:

$$ST_x(\varphi \vee \psi) := \exists y \exists z((y \sqcap z \sqsubseteq x) \wedge ST_y(\varphi) \wedge ST_z(\psi))$$

It is easy to see that this translation is correct, similar to Proposition 5 and 6.

5 Inductive Formulas/Quasi-Inequalities in the Two Settings

5.1 The modal meet-implication logic case

In this subsection, we define inductive inequalities and inductive formulas for modal meet-implication logic. The definition is similar to [24, Section 5].

We first define positive formulas with propositional variables in $A \subseteq \mathrm{Prop}$ as follows:

$$\mathrm{POS}_A ::= p \mid \top \mid \Box\mathrm{POS} \mid \mathrm{POS} \wedge \mathrm{POS}$$

where $p \in A \subseteq \mathrm{Prop}$.

Then we define the dependence order on propositional variables as any irreflexive and transitive binary relation $<_\Omega$ on them. Then we define the PIA_p formulas[7] with main variable p (where p is a fixed variable) as follows:

$$\mathrm{PIA}_p ::= p \mid \top \mid \Box\mathrm{PIA}_p \mid \mathrm{POS}_{A_p} \to \mathrm{PIA}_p$$

where $A_p = \{q \in \mathrm{Prop} \mid q <_\Omega p\}$ in PIA_p. Then we define the inductive antecedent as follows:

$$\mathrm{Ant} ::= \mathrm{PIA}_p \mid \mathrm{Ant} \wedge \mathrm{Ant}$$

where p ranges over Prop. Then we define the inductive succedent as follows:

$$\mathrm{Suc} ::= p \mid \top \mid \mathrm{PIA}_q \to \mathrm{Suc} \mid \Box\mathrm{Suc} \mid \mathrm{Suc} \wedge \mathrm{Suc}$$

where p, q range over Prop.

Finally, an Ω-*inductive formula* is a formula of the form $\mathrm{Ant} \to \mathrm{Suc}$. An inductive formula is an Ω-inductive formula for some Ω.

5.2 The modal meet-semilattice setting

In the modal meet-semilattice setting, a notable phenomenon is that *all inequalities* of the form $\varphi \leq \psi$ are Sahlqvist inequalities. We will also consider quasi-inequalities.

Definition 20 (Inductive Quasi-inequality)**.** Given an irreflexive and transitive binary relation $<_\Omega$ on propositional variables (i.e. a dependence order), we say that a quasi-inequality $\varphi_1 \leq \psi_1$ & $\ldots$ & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ is Ω-*inductive*, if for each $\varphi_i \leq \psi_i$, each propositional variable p occurring in φ_i and each propositional variable q occurring in ψ_i, we have $p <_\Omega q$. A quasi-inequality is *inductive* if it is Ω-inductive for some $<_\Omega$.

[7]Here PIA means "Positive Implies Atomic".

6 Algorithms

In this section, we give the first-order correspondence algorithms for both modal meet-implication logic and modal meet-semilattice logic.

6.1 The modal meet-implication logic case

In the present subsection, we define the algorithm ALBA which computes the first-order correspondence of the input formula in the style of [10, 24]. The algorithm ALBA proceeds in three stages. Firstly, ALBA receives a formula $\mathrm{Ant} \to \mathrm{Suc}$ as input and transforms it into the inequality $\mathrm{Ant} \leq \mathrm{Suc}$.

1. **Preprocessing and first approximation**:
   (a) We apply the following distribution rules exhaustively: In Suc, rewrite every subformula of the former form into the latter form:
   - $\alpha \to \beta \wedge \gamma$, $(\alpha \to \beta) \wedge (\alpha \to \gamma)$
   - $\Box(\alpha \wedge \beta)$, $\Box\alpha \wedge \Box\beta$

   (b) Apply the splitting rule:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma}$$

   Now for each obtained inequality $\varphi_i \leq \psi_i$, We apply the following first-approximation rule:

$$\frac{\varphi_i \leq \psi_i}{\mathbf{i}_0 \leq \varphi_i \;\Rightarrow\; \mathbf{i}_0 \leq \psi_i}$$

   Now we focus on each quasi-inequality $\mathbf{i}_0 \leq \varphi_i \;\Rightarrow\; \mathbf{i}_0 \leq \psi_i$, which we call a *system*, and use S to denote a meta-conjunction of inequalities. When S is empty, we denote it as $\varnothing$. We use parentheses around the quasi-inequality to separate it from other quasi-inequalities when necessary.
2. **The reduction-elimination cycle**:
   In this stage, for each $\mathbf{i}_0 \leq \varphi_i \;\Rightarrow\; \mathbf{i}_0 \leq \psi_i$, we apply the following rules to eliminate all the propositional variables:

   (a) Splitting rules:

$$\frac{\mathrm{S} \;\Rightarrow\; \alpha \leq \beta \wedge \gamma}{(\mathrm{S} \;\Rightarrow\; \alpha \leq \beta) \quad (\mathrm{S} \;\Rightarrow\; \alpha \leq \gamma)}$$

$$\frac{\mathrm{S} \;\&\; \alpha \leq \beta \wedge \gamma \;\Rightarrow\; \varphi \leq \psi}{\mathrm{S} \;\&\; \alpha \leq \beta \;\&\; \alpha \leq \gamma \;\Rightarrow\; \varphi \leq \psi}$$

   (b) Residuation rules:

$$\frac{\mathrm{S} \;\Rightarrow\; \alpha \leq \Box\beta}{\mathrm{S} \;\Rightarrow\; \blacklozenge\alpha \leq \beta}$$

$$\frac{\mathrm{S}\ \&\ \alpha \leq \Box\beta \ \Rightarrow\ \varphi \leq \psi}{\mathrm{S}\ \&\ \blacklozenge\alpha \leq \beta \ \Rightarrow\ \varphi \leq \psi}$$

$$\frac{\mathrm{S} \ \Rightarrow\ \alpha \leq \beta \rightarrow \gamma}{\mathrm{S} \ \Rightarrow\ \alpha \wedge \beta \leq \gamma}$$

$$\frac{\mathrm{S}\ \&\ \alpha \leq \beta \rightarrow \gamma \ \Rightarrow\ \varphi \leq \psi}{\mathrm{S}\ \&\ \alpha \wedge \beta \leq \gamma \ \Rightarrow\ \varphi \leq \psi}$$

(c) Approximation rule:

$$\frac{\mathrm{S} \ \Rightarrow\ \varphi \leq \psi}{\mathrm{S}\ \&\ \mathbf{i} \leq \varphi \ \Rightarrow\ \mathbf{i} \leq \psi}$$

The nominal introduced by the approximation rule must not occur in the system before applying the rule.

(d) Deleting rules:

$$\frac{\mathrm{S}\ \&\ \alpha \leq \top \ \Rightarrow\ \varphi \leq \psi}{\mathrm{S} \ \Rightarrow\ \varphi \leq \psi}$$

$$\frac{\mathrm{S} \ \Rightarrow\ \alpha \leq \top}{\varnothing \ \Rightarrow\ \alpha \leq \top}$$

(e) The right-handed Ackermann rule. This rule eliminates propositional variables and is the core of the algorithm, the other rules are aimed at reaching a shape in which the rule can be applied.

$$\frac{\&_{i=1}^{n} \theta_i \leq p\ \&\ \&_{j=1}^{m} \eta_j \leq \iota_j \ \Rightarrow\ \varphi \leq \psi}{\&_{j=1}^{m} \eta_j(\theta/p) \leq \iota_j(\theta/p) \ \Rightarrow\ \varphi(\theta/p) \leq \psi(\theta/p)}$$

where:

i. p does not occur in $\theta_1, \ldots, \theta_n$;
ii. Each η_i, ψ is positive, and each ι_i, φ negative in p, for $1 \leq i \leq m$;
iii. $\theta := \theta_1 \vee \ldots \vee \theta_n$.

3. **Output**: If in the previous stage, for some systems, the algorithm gets stuck, i.e. some propositional variables cannot be eliminated, then the algorithm halts and output "failure". Otherwise, each initial system after the first approximation has been reduced to a set of pure quasi-inequalities Reduce($\mathbf{i}_0 \leq \varphi_i \ \Rightarrow\ \mathbf{i}_0 \leq \psi_i$), and then the output is a set of pure quasi-inequalities $\bigcup_{i \in I}$Reduce($\mathbf{i}_0 \leq \varphi_i \ \Rightarrow\ \mathbf{i}_0 \leq \psi_i$). Then we can use the conjunction of the standard translations of the quasi-inequalities to obtain the first-order correspondence (notice that in the standard translation of each quasi-inequality, we need to universally quantify over all the individual variables).

Example 1. We give an example of how ALBA is executed. Here for the sake of clarity we add propositional quantifiers and nominal quantifiers before the quasi-inequality.

$$\begin{aligned}
&\forall p(\Box p \to p)\\
&\forall p(\Box p \leq p)\\
&\forall p \forall \mathbf{i}(\mathbf{i} \leq \Box p \ \Rightarrow \mathbf{i} \leq p)\\
&\forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \ \Rightarrow \mathbf{i} \leq p)\\
&\forall \mathbf{i}(\mathbf{i} \leq \blacklozenge \mathbf{i})\\
&\forall i \forall x(ST_x(\mathbf{i}) \to ST_x(\blacklozenge \mathbf{i}))\\
&\forall i \forall x(i \sqsubseteq x \to \exists y(Ryx \wedge ST_y(\mathbf{i})))\\
&\forall i \forall x(i \sqsubseteq x \to \exists y(Ryx \wedge i \sqsubseteq y)).
\end{aligned}$$

Example 2. We give another example showing how ALBA is executed, up to the end of Stage 2.

$$\begin{aligned}
&\forall p \forall q((\Box(p \to q) \wedge \Box\Box p) \to \Box(p \wedge q))\\
&\forall p \forall q(\Box(p \to q) \wedge \Box\Box p \leq \Box(p \wedge q))\\
&\forall p \forall q \forall \mathbf{i}(\mathbf{i} \leq \Box(p \to q) \wedge \Box\Box p \ \Rightarrow \ \mathbf{i} \leq \Box(p \wedge q))\\
&\forall p \forall q \forall \mathbf{i}(\mathbf{i} \leq \Box(p \to q) \ \&\ \mathbf{i} \leq \Box\Box p \ \Rightarrow \ \mathbf{i} \leq \Box(p \wedge q))\\
&\forall p \forall q \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \to q \ \&\ \blacklozenge\blacklozenge \mathbf{i} \leq p \ \Rightarrow \ \mathbf{i} \leq \Box(p \wedge q))\\
&\forall p \forall q \forall \mathbf{i}(\blacklozenge \mathbf{i} \wedge p \leq q \ \&\ \blacklozenge\blacklozenge \mathbf{i} \leq p \ \Rightarrow \ \mathbf{i} \leq \Box(p \wedge q))\\
&\forall p \forall q \forall \mathbf{i}(\blacklozenge \mathbf{i} \wedge \blacklozenge\blacklozenge \mathbf{i} \leq q \ \Rightarrow \ \mathbf{i} \leq \Box(\blacklozenge\blacklozenge \mathbf{i} \wedge q))\\
&\forall p \forall q \forall \mathbf{i}(\&\varnothing \ \Rightarrow \ \mathbf{i} \leq \Box(\blacklozenge\blacklozenge \mathbf{i} \wedge (\blacklozenge \mathbf{i} \wedge \blacklozenge\blacklozenge \mathbf{i}))).
\end{aligned}$$

Here $\&\varnothing$ means that there is no inequality in the antecedent part of the quasi-inequality.

6.2 The modal meet-semilattice logic case

In the present subsection, we define the algorithm ALBA which computes the first-order correspondence of the input inequality and quasi-inequality in the style of [10, 24]. The algorithm ALBA proceeds in three stages. Firstly, ALBA receives an inequality of the form $\varphi \leq \psi$ or a quasi-inequality of the form

$$\varphi_1 \leq \psi_1 \ \&\ \ldots \ \&\ \varphi_n \leq \psi_n \ \Rightarrow \ \varphi \leq \psi,$$

and treat it as a quasi-inequality (in case of an inequality, we take n to be 0).

1. **Preprocessing and first approximation**:
   (a) Apply the following *distribution rule* exhaustively: In $\varphi_1 \leq \psi_1 \ \&\ \ldots \ \&\ \varphi_n \leq \psi_n \ \Rightarrow \ \varphi \leq \psi$, in $\psi_1, \ldots, \psi_n, \varphi$, rewrite every subformula of the form $\Box(\alpha \wedge \beta)$ into $\Box\alpha \wedge \Box\beta$.

After this step, we obtain a quasi-inequality of the form

$$\varphi_1 \leq \psi'_1 \; \& \; \ldots \; \& \; \varphi_n \leq \psi'_n \; \Rightarrow \; \varphi' \leq \psi.$$

(b) Apply the *splitting rule* exhaustively to each ψ'_i in $\varphi_1 \leq \psi'_1 \; \& \; \ldots \; \& \; \varphi_n \leq \psi'_n \; \Rightarrow \; \varphi' \leq \psi$ (notice that this rule operates on an inequality rather than the whole quasi-inequality):

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma}$$

After this step, we obtain a quasi-inequality of the form

$$\mathrm{S} \; \Rightarrow \; \varphi' \leq \psi,$$

where S is a (possibly empty) meta-conjunction of inequalities.

(c) Apply the *first-approximation rule* to the quasi-inequality $\mathrm{S} \; \Rightarrow \varphi' \leq \psi$ obtained in the previous stage (notice that this rule operates on the whole quasi-inequality):

$$\frac{\mathrm{S} \; \Rightarrow \varphi' \leq \psi}{\mathrm{S} \; \& \; \mathbf{i} \leq \varphi' \; \Rightarrow \mathbf{i} \leq \psi}$$

We call a quasi-inequality also a *system*, and use S to denote a meta-conjunction of inequalities.

2. **The reduction-elimination cycle**:
In this section, for the system $\mathrm{S} \; \Rightarrow \mathbf{i} \leq \psi$, we use the splitting rule and the residuation rule to the inequalities in S and finally apply the right-handed Ackermann rule to the whole system. In this stage, the splitting rule and the residuation rule operate on a single inequality and the right-handed Ackermann rule operates on the whole quasi-inequality.

(a) Splitting rule:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \; \& \; \alpha \leq \gamma}$$

(b) Residuation rule:

$$\frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta}$$

(c) The right-handed Ackermann rule. This rule eliminates propositional variables and is the core of the algorithm, the other rules are aimed at reaching a shape in which the rule can be applied.

$$\frac{\&_{i=1}^{n} \theta_i \leq p \; \& \; \&_{j=1}^{m} \eta_j \leq \iota_j \; \Rightarrow \; \mathbf{i} \leq \alpha}{\&_{j=1}^{m} \eta_j(\theta/p) \leq \iota_j \; \Rightarrow \; \mathbf{i} \leq \alpha(\theta/p)}$$

where:

i. p does not occur in $\theta_1, \ldots, \theta_n$;
ii. p does not occur in $\iota_1, \ldots, \iota_m$;
iii. $\theta := \theta_1 \vee \ldots \vee \theta_n$. When $n = 0$, $\theta := \bot$.

3. **Output**: If in the previous stage, the algorithm gets stuck, i.e. some propositional variables cannot be eliminated, then the algorithm halts and output "failure". Otherwise, the initial system after the first approximation has been reduced to a pure quasi-inequality Reduce($\varphi_1 \leq \psi_1$ & ... & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$), and then the output is the pure quasi-inequality and its standard translation (notice that in the standard translation of the quasi-inequality, we need to universally quantify over all the individual variables).

7 Success of the Algorithm

In the present section, we show the success of the algorithm on any inductive formula for the modal meet-implication logic case and the success of the algorithm on any Sahlqvist inequality and any inductive quasi-inequality for the modal meet-semilattice case.

7.1 The modal meet-implication logic case

In the present subsection, we show the success of ALBA on any inductive formula.

Theorem 9. ALBA *succeeds on any inductive formula $\varphi \to \psi$ and outputs a set of pure quasi-inequalities and a first-order formula.*

Proof. The proof is similar to [24, Section 7]. We check the shape of the inequality or system in each stage, for the input formula $\mathrm{Ant} \to \mathrm{Suc}$:

Stage 1. After applying the distribution rules, it is easy to see that Ant is of the form $\bigwedge \mathrm{PIA}_p$, and Suc becomes the form $\bigwedge \mathrm{Suc}'$, where

$$\mathrm{Suc}' ::= p \mid \top \mid \mathrm{PIA}_q \to \mathrm{Suc}' \mid \Box \mathrm{Suc}'.$$

Then by applying the splitting rule, we get a set of inequalities of the form $\bigwedge \mathrm{PIA}_p \leq \mathrm{Suc}'$.

After the first approximation rule, each system is of the form $\mathbf{i}_0 \leq \bigwedge \mathrm{PIA}_p \Rightarrow \mathbf{i}_0 \leq \mathrm{Suc}'$.

Stage 2. In this stage, we deal with each system $\mathbf{i}_0 \leq \bigwedge \mathrm{PIA}_p \Rightarrow \mathbf{i}_0 \leq \mathrm{Suc}'$.

- For the inequality $\mathbf{i}_0 \leq \bigwedge \mathrm{PIA}_p$, by first applying the splitting rule for $\wedge$ and then exhaustively applying the residuation rules for $\Box$ and $\to$, we get inequalities of the form $\mathrm{MinVal}_p \leq p$ or $\mathrm{MinVal}_p \leq \top$, where

$$\mathrm{MinVal}_p ::= \mathbf{i} \mid \blacklozenge \mathrm{MinVal}_p \mid \mathrm{MinVal}_p \wedge \mathrm{POS}_{A_p},$$

where $A_p = \{q \in \mathrm{Prop} \mid q <_\Omega p\}$.
Now the system is of the form $\&(\mathrm{MinVal}_p \leq p)$ & $\&(\mathrm{MinVal}_p \leq \top) \Rightarrow \mathbf{i}_0 \leq \mathrm{Suc}'$. By applying the deleting rule, we get a system of the form $\&(\mathrm{MinVal}_p \leq p) \Rightarrow \mathbf{i}_0 \leq \mathrm{Suc}'$.

- Now we deal with the $\mathbf{i}_0 \leq \mathrm{Suc}'$ part.
  - If the system is of the form
$$\&(\mathrm{MinVal}_p \leq p) \Rightarrow \mathbf{i}_0 \leq \mathrm{PIA}_q \to \mathrm{Suc}',$$
then we apply the residuation rule for $\to$ and get
$$\&(\mathrm{MinVal}_p \leq p) \Rightarrow \mathbf{i}_0 \wedge \mathrm{PIA}_q \leq \mathrm{Suc}',$$
then we apply the approximation rule for the succedent and get
$$\&(\mathrm{MinVal}_p \leq p) \,\&\, \mathbf{j} \leq \mathbf{i}_0 \wedge \mathrm{PIA}_q \Rightarrow \mathbf{j} \leq \mathrm{Suc}',$$
then we apply the splitting rule for the antecedent and get
$$\&(\mathrm{MinVal}_p \leq p) \,\&\, \mathbf{j} \leq \mathbf{i}_0 \,\&\, \mathbf{j} \leq \mathrm{PIA}_q \Rightarrow \mathbf{j} \leq \mathrm{Suc}',$$
then by the reduction for $\mathbf{i}_0 \leq \mathrm{PIA}_p$, we get a system of the form
$$\&(\mathrm{MinVal}_p \leq p) \,\&\, \mathbf{j} \leq \mathbf{i}_0 \Rightarrow \mathbf{j} \leq \mathrm{Suc}'.$$
  - If the system is of the form
$$\&(\mathrm{MinVal}_p \leq p) \Rightarrow \mathbf{i}_0 \leq \Box\mathrm{Suc}',$$
then we apply the residuation rule for $\Box$ and get
$$\&(\mathrm{MinVal}_p \leq p) \Rightarrow \blacklozenge\mathbf{i}_0 \leq \mathrm{Suc}',$$
then we apply the approximation rule for the succedent and get
$$\&(\mathrm{MinVal}_p \leq p) \,\&\, \mathbf{j} \leq \blacklozenge\mathbf{i}_0 \Rightarrow \mathbf{j} \leq \mathrm{Suc}'.$$

Therefore, by the reduction strategies above, we get a quasi-inequality of the form
$$\&(\mathrm{MinVal}_p \leq p) \,\&\, \mathrm{Pure} \Rightarrow \mathbf{j} \leq q$$
or
$$\&(\mathrm{MinVal}_p \leq p) \,\&\, \mathrm{Pure} \Rightarrow \mathbf{j} \leq \top.$$
The second case can be further reduced by the deleting rule for the succedent to
$$\varnothing \Rightarrow \mathbf{j} \leq \top,$$
and all propositional variables are eliminated.

- Now we are ready to apply the right-handed Ackermann rule to an Ω-miminal variable q to eliminate it. Then since there are only finitely many propositional variables, we can always find another Ω-miminal variable to eliminate. Finally we eliminate all propositional variables and get a pure quasi-inequality and its standard translation.

$\square$

7.2 The modal meet-semilattice logic case

In the present subsection, we prove that the algorithm succeeds on any inequality $\varphi \leq \psi$ and any inductive quasi-inequality $\varphi_1 \leq \psi_1 \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n \ \Rightarrow \ \varphi \leq \psi$, i.e. there exists an execution such that the algorithm output a quasi-inequality and the standard translation.

Theorem 10. ALBA *succeeds on any inequality* $\varphi \leq \psi$.

Proof. In the preprocessing stage, by the distribution rule, $\varphi \leq \psi$ is transformed into $\bigwedge_i \alpha_i \leq \psi$, where each α_i is of the form $\Box^{k_i} p$ or $\Box^{k_i} \top$ or $\Box^{k_i} \bot$ for some propositional variable p and $k_i \geq 0$ (we call this kind of α_i a *boxed atom*).

By the first-approximation rule, we get $\mathbf{i} \leq \bigwedge_i \alpha_i \ \Rightarrow \ \mathbf{i} \leq \psi$.

In the reduction-elimination cycle, by the splitting rule, we have $\&_i \, \mathbf{i} \leq \alpha_i \ \Rightarrow \ \mathbf{i} \leq \psi$.

By the residuation rule, we have $\&_i \, \blacklozenge^{k_i} \mathbf{i} \leq p \ \& \ \mathrm{Pure} \ \Rightarrow \ \mathbf{i} \leq \psi$, where Pure is a meta-conjunction (possibly empty) of pure inequalities.

Then by the right-handed Ackermann rule we can eliminate all propositional variables. $\square$

Given a dependence order $<_\Omega$, we say that a quasi-inequality $\alpha_1 \leq \beta_1 \ \& \ \ldots \ \& \ \alpha_n \leq \beta_n \ \Rightarrow \ \alpha \leq \beta$ satisfies the *variable occurrence restriction* of $<_\Omega$, if for each $\alpha_i \leq \beta_i$, each propositional variable p occurring in α_i and each propositional variable q occurring in β_i, we have $p <_\Omega q$.

Theorem 11. ALBA *succeeds on any inductive quasi-inequality* $\varphi_1 \leq \psi_1 \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n \ \Rightarrow \ \varphi \leq \psi$.

Proof. We assume that the dependence order is $<_\Omega$.

In the preprocessing stage, by applying the distribution rule exhaustively, we get a quasi-inequality of the form $\varphi_1 \leq \psi_1' \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n' \ \Rightarrow \ \varphi' \leq \psi$, where each

ψ'_i and φ' are conjunctions of boxed atoms, where the variable occurrence restriction of $<_\Omega$ is still satisfied.

By applying the splitting rule exhaustively, we get a quasi-inequality of the form $\mathrm{S} \ \Rightarrow \ \varphi' \leq \psi$, where φ' is a conjunction of boxed atoms, and for each inequality $\alpha \leq \beta$ in S, we have that β is a boxed atom, and the variable occurrence restriction of $<_\Omega$ is still satisfied.

By applying the first-approximation rule, we get a quasi-inequality of the form $\mathrm{S}\ \&\ \mathbf{i} \leq \varphi' \ \Rightarrow \ \mathbf{i} \leq \psi$, where S and φ' are described as above, and the variable occurrence restriction of $<_\Omega$ is still satisfied.

In the reduction-elimination cycle, by the splitting rule, we have $\mathrm{S}\ \&\ \mathrm{S}' \ \Rightarrow\ \mathbf{i} \leq \psi$, where for each inequality $\alpha \leq \beta$ in S, we have that β is a boxed atom, and each inequality in S' is of the form $\mathbf{i} \leq \beta$ where β is a boxed atom, and the variable occurrence restriction of $<_\Omega$ is still satisfied.

By applying the residuation rule, we have $\mathrm{T}\ \&\ \mathrm{T}' \ \Rightarrow \ \mathbf{i} \leq \psi$, where in each inequality $\gamma \leq \delta$ in T, δ is a propositional variable or $\bot$ or $\top$, and in each inequality $\gamma \leq \delta$ in T', γ is pure and δ is a propositional variable or $\bot$ or $\top$. The variable occurrence restriction of $<_\Omega$ is still satisfied.

Now we can choose a propositional variable p which is $<_\Omega$-minimal. By the $<_\Omega$-minimality of p and the variable occurrence restriction of $<_\Omega$, for each inequality $\gamma \leq p$ in T and T', we have that γ is pure.

Therefore, we can apply the right-handed Ackermann rule. After applying this rule, we can check that the quasi-inequality is of the form $\mathrm{U}\ \&\ \mathrm{U}' \ \Rightarrow\ \mathbf{i} \leq \psi'$, where

- in each inequality $\gamma \leq \delta$ in U, δ is a propositional variable or $\bot$ or $\top$;
- in each inequality $\gamma \leq \delta$ in U', γ is pure and δ is a propositional variable or $\bot$ or $\top$;
- the variable occurrence restriction of $<_\Omega$ is still satisfied.

Therefore we can choose a $<_\Omega$-minimal propositional variable from the remaining variables, and apply the right-handed Ackermann rule on it, and after applying this rule, the three conditions above are still satisfied. By repeating this procedure, we can eliminate all the propositional variables. $\square$

8 Soundness of the Algorithms: The Discrete Case

8.1 The modal meet-implication logic case

In this subsection we show the soundness of the algorithm for inductive formulas with respect to modal I-frames. The soundness proof follows the style of [10].

Theorem 12 (Soundness)**.** *If* ALBA *runs according to the success proof on an input inductive formula* $\varphi \to \psi$ *and outputs a first-order formula* $\mathrm{FO}(\varphi \to \psi)$*, then for any modal I-frame* $\mathbb{F} = (X, \sqsubseteq, R)$*,*

$$\mathbb{F} \Vdash \varphi \to \psi \textit{ iff } \mathbb{F} \vDash \mathrm{FO}(\varphi \to \psi).$$

Proof. The proof goes similarly to [10, Theorem 8.1]. Let $\varphi \to \psi$ denote the input formula, $\{\mathbf{i}_0 \leq \varphi_i \ \Rightarrow \ \mathbf{i}_0 \leq \psi_i\}_{i\in I}$ denote the set of quasi-inequalities after the first-approximation rule, let $\{\mathrm{Reduce}(\mathbf{i}_0 \leq \varphi_i \ \Rightarrow \ \mathbf{i}_0 \leq \psi_i)\}_{i\in I}$ denote the sets of quasi-inequalities after Stage 2, let $\mathrm{FO}(\varphi \to \psi)$ denote the standard translation of the quasi-inequalities in Stage 3 into first-order formulas, then it suffices to show the equivalence from (1) to (4) given below:

$$\mathbb{F} \Vdash \varphi \to \psi \tag{1}$$

$$\mathbb{F} \Vdash \mathbf{i}_0 \leq \varphi_i \ \Rightarrow \ \mathbf{i}_0 \leq \psi_i, \text{ for all } i \in I \tag{2}$$

$$\mathbb{F} \Vdash \mathrm{Reduce}(\mathbf{i}_0 \leq \varphi_i \ \Rightarrow \ \mathbf{i}_0 \leq \psi_i), \text{ for all } i \in I \tag{3}$$

$$\mathbb{F} \vDash \mathrm{FO}(\varphi \to \psi) \tag{4}$$

The equivalence between (1) and (2) follows from Proposition 13;
The equivalence between (2) and (3) follows from Propositions 14, 17;
The equivalence between (3) and (4) follows from Proposition 6. □

In the remainder of this section, we prove the soundness of the rules in each stage.

Proposition 13. *The distribution rules, the splitting rule are sound in* $\mathbb{F}$*, and the first-approximation rule is sound in* $\mathbb{F}$*, i.e.* (1) *and* (2) *are equivalent.*

Proof. For the distribution rules and the splitting rule, the proof is similar to [10].

For the distribution rules, it follows from the following equivalences: for any modal I-frame $\mathbb{F}$, any valuation V on $\mathbb{F}$, any world $w \in X$,

- $\mathbb{F}, V, w \Vdash \alpha \to \beta \wedge \gamma$ iff $\mathbb{F}, V, w \Vdash (\alpha \to \beta) \wedge (\alpha \to \gamma)$;
- $\mathbb{F}, V, w \Vdash \Box(\alpha \wedge \beta)$ iff $\mathbb{F}, V, w \Vdash \Box\alpha \wedge \Box\beta$.

For the splitting rule, it follows from the following equivalence: for any modal I-frame $\mathbb{F}$, any valuation V on $\mathbb{F}$,

$$\mathbb{F}, V \Vdash \alpha \leq \beta \wedge \gamma \text{ iff } \mathbb{F}, V \Vdash \alpha \leq \beta \text{ and } \mathbb{F}, V \Vdash \alpha \leq \gamma.$$

For the first-approximation rule, we have the following lemma:

Lemma 3. *For any modal I-frame $\mathbb{F}$, any valuation V on $\mathbb{F}$, if $V(\mathbf{i}) = {\uparrow} i$, then*

$$\mathbb{F}, V \Vdash \mathbf{i} \leq \alpha \textit{ iff } \mathbb{F}, V, i \Vdash \alpha.$$

Proof of Lemma 3

$$\begin{array}{ll} & \mathbb{F}, V \Vdash \mathbf{i} \leq \alpha \\ \text{iff} & V(\mathbf{i}) \subseteq V(\alpha) \\ \text{iff} & {\uparrow} i \subseteq V(\alpha) \\ \text{iff} & i \in V(\alpha) \\ \text{iff} & \mathbb{F}, V, i \Vdash \alpha. \end{array}$$

□

Therefore, for any modal I-frame $\mathbb{F}$,

- Assume $\mathbb{F} \Vdash \varphi_i \leq \psi_i$. For any valuation V on $\mathbb{F}$, if $\mathbb{F}, V \Vdash \mathbf{i} \leq \varphi_i$, then $V(\mathbf{i}) = {\uparrow} i$ for some $i \in X$. Thus $\mathbb{F}, V, i \Vdash \varphi_i$, so $i \in V(\varphi_i)$. Since $\mathbb{F} \Vdash \varphi_i \leq \psi_i$, we have $V(\varphi_i) \subseteq V(\psi_i)$, so $i \in V(\psi_i)$, therefore $\mathbb{F}, V, i \Vdash \psi_i$, i.e. $\mathbb{F}, V \Vdash \mathbf{i} \leq \psi_i$. So we get $\mathbb{F}, V \Vdash \mathbf{i} \leq \varphi_i \ \Rightarrow \ \mathbf{i} \leq \psi_i$ for any V on $\mathbb{F}$.
- For the other direction, assume that $\mathbb{F} \Vdash \mathbf{i} \leq \varphi_i \ \Rightarrow \ \mathbf{i} \leq \psi_i$. Then for any valuation V on $\mathbb{F}$, it suffices to prove that $V(\varphi_i) \subseteq V(\psi_i)$. Take any $i \in V(\varphi_i)$, consider the valuation $V' = V^{\mathbf{i}}_{\uparrow\{i\}}$, then since $\mathbf{i}$ does not occur in φ_i, we have that $V'(\mathbf{i}) = {\uparrow} i$ and $V'(\varphi_i) = V(\varphi_i)$. So we have $\mathbb{F}, V', i \Vdash \varphi_i$, by Lemma 3 we have $\mathbb{F}, V' \Vdash \mathbf{i} \leq \varphi_i$, so from $\mathbb{F} \Vdash \mathbf{i} \leq \varphi_i \ \Rightarrow \ \mathbf{i} \leq \psi_i$ we get $\mathbb{F}, V' \Vdash \mathbf{i} \leq \psi_i$ and $\mathbb{F}, V', i \Vdash \psi_i$. So $i \in V'(\psi_i)$, since $\mathbf{i}$ does not occur in ψ_i, we have that $V'(\psi_i) = V(\psi_i)$. So $i \in V(\psi_i)$, therefore $V(\varphi_i) \subseteq V(\psi_i)$. Since V and i are arbitrary, we have $\mathbb{F} \Vdash \varphi_i \leq \psi_i$.

□

The next step is to show the soundness of each rule of Stage 2. For each rule, before the application of this rule we have a system $\mathrm{S} \ \Rightarrow \mathrm{Ineq}$, after applying the rule we get a system $\mathrm{S}' \ \Rightarrow \mathrm{Ineq}'$, the soundness of Stage 2 is then the equivalence of the following:

- $\mathbb{F} \Vdash \mathrm{S} \ \Rightarrow \mathrm{Ineq}$
- $\mathbb{F} \Vdash \mathrm{S}' \ \Rightarrow \mathrm{Ineq}'$

It suffices to show the following property for the splitting rule, residuation rules and the deleting rules:

$$\text{For any } \mathbb{F}\text{, any arbitrary valuation } V\text{,}$$
$$\mathbb{F}, V \Vdash \mathrm{S} \ \Rightarrow \mathrm{Ineq} \text{ iff } \mathbb{F}, V \Vdash \mathrm{S}' \ \Rightarrow \mathrm{Ineq}'.$$

For the first-approximation rule, we prove it directly.

For the right-handed Ackermann rule, we also prove it directly.

Proposition 14. *The splitting rule, the approximation rules, the residuation rules and the deleting rule in Stage 2 are sound in both directions in* $\mathbb{F}$.

Proof. The soundness proofs for the splitting rules and the residuation rules are similar to the soundness of the same rules in [10].

- For the splitting rules, it follows from the following equivalence: for any modal I-frame $\mathbb{F}$, any arbitrary valuation V on $\mathbb{F}$,

$$\mathbb{F}, V \Vdash \alpha \leq \beta \wedge \gamma \text{ iff } \mathbb{F}, V \Vdash \alpha \leq \beta \text{ and } \mathbb{F}, V \Vdash \alpha \leq \gamma.$$

- For the residuation rules, it follows from the following equivalence: for any modal I-frame $\mathbb{F}$, any arbitrary valuation V on $\mathbb{F}$,
  - $\mathbb{F}, V \Vdash \alpha \leq \Box\beta$ iff $\mathbb{F}, V \Vdash \blacklozenge\alpha \leq \beta$;
  - $\mathbb{F}, V \Vdash \alpha \leq \beta \to \gamma$ iff $\mathbb{F}, V \Vdash \alpha \wedge \beta \leq \gamma$.

  The first equivalence above follows from the fact that $\blacklozenge : \mathcal{F}(X) \to \mathcal{F}(X)$ and $\Box : \mathcal{F}(X) \to \mathcal{F}(X)$ form an adjunction pair, and the second equivalence above follows from the fact that $\cap$ and $\to$ (as defined on page 29) form a residuation pair (since $\mathcal{F}(X)$ is an implicative semilattice).
- For the approximation rule, the soundness proof is similar to the first-approximation rule: for any modal I-frame $\mathbb{F}$,
  - Assume $\mathbb{F} \Vdash \mathrm{S} \ \Rightarrow \ \varphi \leq \psi$. For any valuation V on $\mathbb{F}$, if $\mathbb{F}, V \Vdash \mathrm{S}$ and $\mathbb{F}, V \Vdash \mathbf{i} \leq \varphi$, then $V(\mathbf{i}) = {\uparrow} i$ for some $i \in X$. Thus $\mathbb{F}, V, i \Vdash \varphi$, so $i \in V(\varphi)$. Since $\mathbb{F} \Vdash \varphi \leq \psi$, we have $V(\varphi) \subseteq V(\psi)$, so $i \in V(\psi)$, therefore $\mathbb{F}, V, i \Vdash \psi$, i.e. $\mathbb{F}, V \Vdash \mathbf{i} \leq \psi$. So we get $\mathbb{F}, V \Vdash \mathrm{S}\ \&\ \mathbf{i} \leq \varphi \ \Rightarrow \ \mathbf{i} \leq \psi$ for any V on $\mathbb{F}$.
  - For the other direction, assume that $\mathbb{F} \Vdash \mathrm{S}\ \&\ \mathbf{i} \leq \varphi \ \Rightarrow \ \mathbf{i} \leq \psi$. Then for any valuation V on $\mathbb{F}$, assume that $\mathbb{F}, V \Vdash \mathrm{S}$, it suffices to prove that $V(\varphi) \subseteq V(\psi)$. Take any $i \in V(\varphi)$, consider the valuation $V' = V^{\mathbf{i}}_{\uparrow\{i\}}$, then since $\mathbf{i}$ does not occur in φ and S, we have that $V'(\mathbf{i}) = {\uparrow} i$, $V'(\varphi) = V(\varphi)$ and $\mathbb{F}, V' \Vdash \mathrm{S}$. So we have $\mathbb{F}, V', i \Vdash \varphi$, by Lemma 3 we have $\mathbb{F}, V' \Vdash \mathbf{i} \leq \varphi$, so from $\mathbb{F} \Vdash \mathrm{S}\ \&\ \mathbf{i} \leq \varphi \ \Rightarrow \ \mathbf{i} \leq \psi$ we get $\mathbb{F}, V' \Vdash \mathbf{i} \leq \psi$ and $\mathbb{F}, V', i \Vdash \psi$. So $i \in V'(\psi)$, since $\mathbf{i}$ does not occur in ψ, we have that $V'(\psi) = V(\psi)$. So $i \in V(\psi)$, therefore $V(\varphi) \subseteq V(\psi)$. Since V and i are arbitrary, we have $\mathbb{F} \Vdash \mathrm{S} \ \Rightarrow \ \varphi \leq \psi$.

- The soundness of the deleting rule is trivial, since $\alpha \leq \top$ always holds in any modal I-model.

□

Proposition 15. *The right-handed Ackermann rule applied in the success proof is sound in* $\mathbb{F}$.

Proof. Without loss of generality we assume that $n = 2$ and $m = 1$. Then it suffices to show that the following are equivalent:

- $\mathbb{F} \Vdash \theta_1 \leq p \,\&\, \theta_2 \leq p \,\&\, \eta \leq \iota \;\Rightarrow\; \varphi \leq \psi$;
- $\mathbb{F} \Vdash \eta(\theta_1 \vee \theta_2/p) \leq \iota(\theta_1 \vee \theta_2/p) \;\Rightarrow\; \varphi(\theta_1 \vee \theta_2/p) \leq \psi(\theta_1 \vee \theta_2/p)$.

$\downarrow$: Assume that $\mathbb{F} \Vdash \theta_1 \leq p \,\&\, \theta_2 \leq p \,\&\, \eta \leq \iota \;\Rightarrow\; \varphi \leq \psi$. Then for any arbitrary valuation V on $\mathbb{F}$, if $\mathbb{F}, V \Vdash \eta(\theta_1 \vee \theta_2/p) \leq \iota(\theta_1 \vee \theta_2/p)$, then take $V' = V^p_{V(\theta_1 \vee \theta_2)}$[8], then since p does not occur in θ_1, θ_2, we have that $V'(\theta_1 \vee \theta_2) = V(\theta_1 \vee \theta_2) = V'(p)$, therefore $V(\eta(\theta_1 \vee \theta_2/p)) = V'(\eta(\theta_1 \vee \theta_2/p)) = V'(\eta)$, similarly $V(\iota(\theta_1 \vee \theta_2/p)) = V'(\iota)$, so from $\mathbb{F}, V \Vdash \eta(\theta_1 \vee \theta_2/p) \leq \iota(\theta_1 \vee \theta_2/p)$ we get $V'(\eta) \subseteq V'(\iota)$. Since $V'(\theta_1) \subseteq V'(\theta_1 \vee \theta_2) = V'(p), V'(\theta_2) \subseteq V'(\theta_1 \vee \theta_2) = V'(p)$, we get $\mathbb{F}, V' \Vdash \theta_1 \leq p \,\&\, \theta_2 \leq p \,\&\, \eta \leq \iota$, therefore from $\mathbb{F} \Vdash \theta_1 \leq p \,\&\, \theta_2 \leq p \,\&\, \eta \leq \iota \;\Rightarrow\; \varphi \leq \psi$ we get $\mathbb{F}, V' \Vdash \varphi \leq \psi$. Therefore $V'(\varphi) \subseteq V'(\psi)$. Similar to η and ι we get $V'(\varphi) = V(\varphi(\theta_1 \vee \theta_2/p))$ and $V'(\psi) = V(\psi(\theta_1 \vee \theta_2/p))$, so $\mathbb{F}, V \Vdash \varphi(\theta_1 \vee \theta_2/p) \leq \psi(\theta_1 \vee \theta_2/p)$. By the arbitrariness of V we get $\mathbb{F} \Vdash \eta(\theta_1 \vee \theta_2/p) \leq \iota(\theta_1 \vee \theta_2/p) \;\Rightarrow\; \varphi(\theta_1 \vee \theta_2/p) \leq \psi(\theta_1 \vee \theta_2/p)$.

$\uparrow$: Assume that $\mathbb{F} \Vdash \eta(\theta_1 \vee \theta_2/p) \leq \iota(\theta_1 \vee \theta_2/p) \;\Rightarrow\; \varphi(\theta_1 \vee \theta_2/p) \leq \psi(\theta_1 \vee \theta_2/p)$. Then for any valuation V on $\mathbb{F}$, if $\mathbb{F}, V \Vdash \theta_1 \leq p \,\&\, \theta_2 \leq p \,\&\, \eta \leq \iota$, then $V(\theta_1) \subseteq V(p), V(\theta_2) \subseteq V(p), V(\eta) \subseteq V(\iota)$. Therefore $V(\theta_1 \vee \theta_2) \subseteq V(p)$, so by the polarity of p in $\eta(\theta_1 \vee \theta_2/p)$ and $\iota(\theta_1 \vee \theta_2/p)$ we have that $V(\eta(\theta_1 \vee \theta_2/p)) \subseteq V(\eta) \subseteq V(\iota) \subseteq V(\iota(\theta_1 \vee \theta_2/p))$. So from $\mathbb{F} \Vdash \eta(\theta_1 \vee \theta_2/p) \leq \iota(\theta_1 \vee \theta_2/p) \;\Rightarrow\; \varphi(\theta_1 \vee \theta_2/p) \leq \psi(\theta_1 \vee \theta_2/p)$ we get $V(\varphi) \subseteq V(\varphi(\theta_1 \vee \theta_2/p)) \subseteq V(\psi(\theta_1 \vee \theta_2/p)) \subseteq V(\psi)$, so $\mathbb{F}, V \Vdash \varphi \leq \psi$. Therefore by the arbitrariness of V we get $\mathbb{F} \Vdash \theta_1 \leq p \,\&\, \theta_2 \leq p \,\&\, \eta \leq \iota \;\Rightarrow\; \varphi \leq \psi$. □

8.2 The Modal Meet-Semilattice Logic Case

For the soundness proof of the algorithm for inductive quasi-inequalities with respect to modal M-frames, the proof for the rules here are essentially the same as the same rules in the modal meet-implication logic setting. Notice that the right-handed Ackermann rule here is a special case of the right-handed Ackermann rule there.

[8]This is the point where we will dicuss in the next section.

9 Soundness of the Algorithms: The Descriptive Case

9.1 The modal meet-implication logic case

In this subsection we show the soundness of the algorithm for inductive formulas with respect to descriptive modal I-frames. The proof also follows the style of [10].

Theorem 16 (Soundness). *If* ALBA *runs according to the success proof on an input inductive formula* $\varphi \to \psi$ *and outputs a first-order formula* $\mathrm{FO}(\varphi \to \psi)$*, then for any descriptive modal I-frame* $\mathbb{F} = (X, \sqsubseteq, R, A)$*,*

$$\mathbb{F} \Vdash_A \varphi \to \psi \textit{ iff } \mathbb{F} \vDash \mathrm{FO}(\varphi \to \psi).$$

Similar to the modal I-frame case, it suffices to show the soundness of each rule in the algorithm with respect to the admissible semantics. Indeed, for most of the rules, since in the equivalence involved, either they do not change valuation (so admissible valuation remains admissible), or only the valuation of a new nominal changes (so admissibility of valuation does not change either), the proof is essentially the same as in the soundness proof with respect to arbitrary semantics. The only exception is the right-handed Ackermann rule, which changes the valuation of a propositional variable. In arbitrary semantics it does not matter, since the value of $\theta_1 \vee \theta_2$ and other formulas in the minimal valuation part always exist in $\mathcal{F}(X)$. However, we cannot guarantee that the value of $\theta_1 \vee \theta_2$ and other formulas in the minimal valuation part is a clopen filter, so we need another soundness proof of the right-handed Ackermann rule with respect to descriptive modal I-frames.

9.1.1 Analysis of the right-handed Ackermann rule

Before we show the soundness of the right-handed Ackermann rule with respect to descriptive modal I-frames, we consider the application of the right-handed Ackermann rule in the success proof. The analysis is very similar to the one in [24, Section 9.1].

Before the application of the right-handed Ackermann rule, the system is of the shape $\mathrm{S} \;\Rightarrow\; \mathbf{j} \leq t$, where p is the current Ω-minimal propositional variable, and t might be p or another propositional variable or a pure formula, and S consists of the following inequalities:

- $\mathrm{MinVal}'_{p,1} \leq p, \ldots, \mathrm{MinVal}'_{p,n} \leq p$;
- inequalities of the form $\mathrm{MinVal}'_q \leq q$, where $q \neq p$;
- pure inequalities.

Here $\mathrm{MinVal}'_r \in \mathrm{C}_r$ for $r = p, q$, where C_r is defined as follows:

$$\mathrm{C}_r ::= \mathbf{i} \mid \top \mid \bot \mid s \mid \blacklozenge\mathrm{C}_r \mid \Box\mathrm{C}_r \mid \mathrm{C}_r \wedge \mathrm{C}_r \mid \mathrm{C}_r \vee \mathrm{C}_r$$

where s is a propositional variable of dependence order below r. It is easy to see that each $\mathrm{MinVal}'_{p,i}$ is pure (since all propositional variables below p are already eliminated), and MinVal'_q may or may not contain p.

Now denote $\bigvee_i \mathrm{MinVal}'_{p,i}$ as V_p. After the application of the right-handed Ackermann rule, the system is of the shape $\mathrm{S}' \Rightarrow \mathbf{j} \leq t(\mathrm{V}_p/p)$, and S' consists of the following inequalities:

- inequalities of the form $\mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q$;
- pure inequalities.

It is easy to see that in the system, in each non-pure inequality in S', they are of the form $\mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q$, which still fall in the categories described as before the application of the right-handed Ackermann rule. Also, the first application of the right-handed Ackermann rule is a special case of the situation described above.

Therefore, it suffices to show that for the system $\mathrm{S} \Rightarrow \mathbf{j} \leq t$, t is p or another propositional variable or a pure formula, and S consists of the following inequalities:

- $\mathrm{MinVal}'_{p,1} \leq p, \ldots, \mathrm{MinVal}'_{p,n} \leq p$;
- inequalities of the form $\mathrm{MinVal}'_q(p) \leq q$, where $q \neq p$ and $\mathrm{MinVal}'_q(p)$ contains positive occurrences of p;
- inequalities that do not contain p;

The application of the right-handed Ackermann rule on variable p is sound with respect to admissible valuations.

9.1.2 Proof of topological Ackermann Lemma

In what follows, when we refer to $\mathbb{F}$ and X, if we do not mention specifically, they refer to a descriptive modal I-frame. We use $C(X)$ to denote the set of closed filters of X and A to denote the set of clopen filters of X. We also denote $\blacklozenge Y := R[Y]$ and $\Box Y := (R^{-1}(Y^c))^c$.

Lemma 4 (Lemma 2.2 in [21]). *Let $(X, \sqcap, \overline{\top}, \tau)$ be an M-space and $F \subseteq X$ a filter in $(X, \sqcap, \overline{\top})$. Then F is closed in (X, τ) if and only if it is the intersection of all clopen filters that contain F.*

Corollary 1. *For any $Y \in C(X)$, Y is an intersection of a non-empty downward-directed collection of elements in A, i.e. $\forall Y \in C(X)$, $Y = \bigcap_i X_i$ for some non-empty downward-directed $\{X_i\}_{i \in I} \subseteq A$.*

Lemma 5. *A filter of X is closed iff it is principal, i.e. of the form $\uparrow x$. Therefore, $\uparrow x \in C(X)$ for any $x \in X$. Especially, $\{\overline{\top}\} \in C(X)$.*

Proof. The proof is similar to [5, Lemma 2.8].

To prove that every $\uparrow x$ is closed, it suffices to show that $\uparrow x = \bigcap\{F \in A \mid x \in F\}$. It is easy to see that for any $F \in A$ such that $x \in F$, $\uparrow x \subseteq F$, so $\uparrow x \subseteq \bigcap\{F \in A \mid x \in F\}$. If $y \notin \uparrow x$, then $x \not\sqsubseteq y$, by differentiatedness there is an $F \in A$ such that $x \in F$ and $y \notin F$. so $y \notin \bigcap\{F \in A \mid x \in F\}$. Therefore $\bigcap\{F \in A \mid x \in F\} \subseteq \uparrow x$.

To prove that every closed filter is principal, suppose otherwise, there is a closed filter F which is not principal, then for any $x \in F$ there is a $y \in F$ such that $x \not\sqsubseteq y$. Then by differentiatedness there is an $F_x \in A$ such that $x \in F_x$ and $y \notin F_x$, so $F \not\subseteq F_x$. While $F \subseteq \bigcup_{x \in F} F_x$, so by compactness there are $x_1, \ldots, x_n \in F$ such that $F \subseteq \bigcup_{i=1}^n F_{x_i}$. Now for each $x_i \in F$ there is a $y_i \in F$ such that $x_i \not\sqsubseteq y_i$ and $y_i \notin F_{x_i}$. So $y_1 \sqcap \ldots \sqcap y_n \notin F_{x_i}$ for each $i = 1, \ldots, n$, but F is a filter, so $y_1 \sqcap \ldots \sqcap y_n \in F$, a contradiction to $F \subseteq \bigcup_{i=1}^n F_{x_i}$. □

Lemma 6. *If $Y \in C(X)$ and $Z \in C(X)$, then $Y \vee Z \in C(X)$.*

Proof. By Lemma 5, $Y = \uparrow y$ and $Z = \uparrow z$ for some $y, z \in X$. Then $Y \vee Z = \{y' \sqcap z' \mid y \sqsubseteq y' \text{ and } z \sqsubseteq z'\} = \uparrow\{y \sqcap z\} \in C(X)$. □

Lemma 7 (A Variation of Proposition 4.4 in [6])**.** *Let $(X, \sqsubseteq, R, A)$ be a descriptive modal I-frame. If $Y \in C(X)$, then $R[Y] \in C(X)$, i.e. $\blacklozenge Y \in C(X)$.*

Proof. Let $Y \in C(X)$. Let $x \notin R[Y] = \bigcup\{R(y) : y \in Y\}$. Then $x \notin R[y]$ for all $y \in Y$. By Definition 10, $R[y] = \bigcap\{a \in A \mid R[x] \subseteq a\}$, therefore $R[y] \in C(X)$ and there exists $U_y \in A$ such that $x \notin U_y$ and $R(y) \subseteq U_y$. So $y \in \Box U_y$ for all $y \in Y$. Consider the family $\mathcal{I} = \{\Box U_y : y \in Y\}$, then $Y \subseteq \bigcup \mathcal{I}$. By Corollary 1 and compactness, we have that there are $y_1, \ldots, y_n \in Y$ such that $Y \subseteq \Box U_{y_1} \cup \ldots \cup \Box U_{y_n} \subseteq \Box(U_{y_1} \cup \ldots \cup U_{y_n})$ and $x \notin U_{y_1} \cup \ldots \cup U_{y_n}$.

Now take V_x to be $U_{y_1} \vee \ldots \vee U_{y_n}$, then $Y \subseteq \Box(U_{y_1} \cup \ldots \cup U_{y_n}) \subseteq \Box(U_{y_1} \vee \ldots \vee U_{y_n}) \subseteq \Box(V_x)$, then $R[Y] \subseteq V_x$. Then by Lemma 6, $V_x \in C(X)$.

Then we can show that $R[Y] = \bigcap_{x \notin R[Y]} V_x$: If $z \in R[Y]$, then $z \in V_x$ for all $x \notin R[Y]$, so $z \in \bigcap_{x \notin R[Y]} V_x$. If $z \notin R[Y]$, then $z \notin V_z$, so $z \notin \bigcap_{x \notin R[Y]} V_x$. Since each $V_x \in C(X)$, we have that $R[Y] \in C(X)$. □

Lemma 8. *1. $A \subseteq C(X)$.*
2. $\Box \bigcap_i X_i = \bigcap \Box_i X_i$, for any $X_i \subseteq X$.
3. If $Y \in C(X)$, then $\Box Y \in C(X)$.

Proof. 1. Trivial.
2. By the fact that $\Box$ is completely intersection preserving.
3. An easy corollary of items 1 and 2.

□

Lemma 9. *$\blacklozenge \bigcap_i X_i = \bigcap \blacklozenge_i X_i$ for any non-empty downward-directed $\{X_i\}_{i \in I} \subseteq C(X)$.*

Proof. The proof is essentially the same as [24, Lemma 4]. The direction $\blacklozenge \bigcap_i X_i \subseteq \bigcap \blacklozenge_i X_i$ is easy. For the other direction, suppose there is a $Y \in C(X)$ such that $\blacklozenge \bigcap_i X_i \subseteq Y$. Then there is a collection $\{Z_j\}_{j \in J} \in A$ such that $Y = \bigcap_j Z_j$. Therefore, $\blacklozenge \bigcap_i X_i \subseteq Z_j$ for all j. Thus $\bigcap_i X_i \subseteq \Box Z_j$ for all j. By compactness and the downward-directedness of X_i, we have that for all j and some k_j depending on j, $X_{k_j} \subseteq \Box Z_j$, i.e., $\blacklozenge X_{k_j} \subseteq Z_j$, so $\bigcap_i \blacklozenge X_i \subseteq Z_j$ for all j, so $\bigcap_i \blacklozenge X_i \subseteq \bigcap_j Z_j = Y$. Now take $Y = \blacklozenge \bigcap_i X_i$, we have $\bigcap_i \blacklozenge X_i \subseteq \blacklozenge \bigcap_i X_i$. □

Lemma 10. *For any formula $\alpha(\overline{q}, \overline{\mathbf{i}}, p)$ built up from nominals, $\top$, $\bot$, propositional variables using $\blacklozenge, \Box, \wedge, \vee$, consider the following valuation: $V(\overline{q}) = \overline{Y}$, where $\overline{Y} \in A$, $V(\overline{\mathbf{i}}) = \overline{\uparrow x}$, where $x \in X$, $V(p) = Z$, where $Z \in A$, then $V(\alpha(\overline{q}, \overline{\mathbf{i}}, p)) \in C(X)$.*

Proof. By induction on the complexity of $\alpha(\overline{q}, \overline{\mathbf{i}}, p)$.

- For nominals, from Lemma 5, $\uparrow x \in C(X)$ for all $x \in X$.
- For $\top$, it is easy to see that X is a clopen filter of X.
- For $\bot$, from Lemma 5, $\{\overline{\top}\} \in C(X)$.
- For propositional variables, it follows from Lemma 8.
- For $\blacklozenge$, it follows from Lemma 7.
- For $\Box$, it follows from Lemma 8.
- For $\wedge$, it follows from that $C(X)$ is closed under taking intersection.
- For $\vee$, it follows from Lemma 6.

□

Lemma 11. *For any formula $\alpha(\overline{q}, \overline{\mathbf{i}}, p)$ built up from nominals, $\top$, $\bot$, propositional variables using $\blacklozenge, \Box, \wedge, \vee$, for any $\overline{Y} \in A$ corresponding to $\overline{q}$, any $\overline{\uparrow x}$ (where $x \in X$) corresponding to $\overline{\mathbf{i}}$, any non-empty downward-directed $\{Z_i\}_{i \in I} \subseteq C(X)$ corresponding to p, we have $\bigcap_i \beta(\overline{Y}, \overline{\uparrow x}, Z_i) = \beta(\overline{Y}, \overline{\uparrow x}, \bigcap_i Z_i)$.*[9]

Proof. By induction on the complexity of $\alpha(\overline{q}, \overline{\mathbf{i}}, p)$.

- For nominals, $\top$, $\bot$, propositional variables, trivial.
- For $\blacklozenge$, it follows from Lemma 9.
- For $\Box$, it follows from Lemma 8.
- For $\wedge$, trivial.
- For $\vee$, it suffices to prove that $\bigcap_i (\gamma \vee \delta)(\overline{Y}, \overline{\uparrow x}, Z_i) = (\gamma \vee \delta)(\overline{Y}, \overline{\uparrow x}, \bigcap_i Z_i)$. For the sake of simplicity, we write $\theta(\overline{Y}, \overline{\uparrow x}, Z_i)$ as $\theta(Z_i)$. Then we need to prove that $\bigcap_i (\gamma \vee \delta)(Z_i) = (\gamma \vee \delta)(\bigcap_i Z_i)$.
  It is easy to see that $\bigcap_i (\gamma \vee \delta)(Z_i) \supseteq (\gamma \vee \delta)(\bigcap_i Z_i)$. For the other direction, we first prove the following statement:

[9] Here by $\beta(\overline{Y}, \overline{\{x\}}, Z_i)$ we mean $V(\beta)$ under the valuation where $V(\overline{q}) = \overline{Y}$, $V(\overline{\mathbf{i}}) = \overline{\{x\}}$, $V(p) = Z_i$.

For any $U \in A$, if $(\gamma \vee \delta)(\bigcap_i Z_i) \subseteq U$, then $\bigcap_i (\gamma \vee \delta)(Z_i) \subseteq U$.

Proof of statement. Assume that $(\gamma \vee \delta)(\bigcap_i Z_i) \subseteq U$, then $\gamma(\bigcap_i Z_i) \subseteq U$ and $\delta(\bigcap_i Z_i) \subseteq U$. By induction hypothesis, we have $\bigcap_i \gamma(Z_i) = \gamma(\bigcap_i Z_i)$ and $\bigcap_i \delta(Z_i) = \delta(\bigcap_i Z_i)$, so we have $\bigcap_i \gamma(Z_i) \subseteq U$ and $\bigcap_i \delta(Z_i) \subseteq U$. By compactness and downward-directedness of $\{Z_i\}_{i \in I} \subseteq C(X)$, we have that there are $j, k \in I$ such that $\gamma(Z_j) \subseteq U$ and $\delta(Z_k) \subseteq U$, so there is an $l \in I$ such that $Z_l \subseteq Z_j \cap Z_k$ and $\gamma(Z_l) \subseteq U$ and $\delta(Z_l) \subseteq U$, so $(\gamma \vee \delta)(Z_l) \subseteq U$, therefore $\bigcap_i (\gamma \vee \delta)(Z_i) \subseteq U$. □

From this statement we can prove the second statement:

For any $U \in C(X)$, if $(\gamma \vee \delta)(\bigcap_i Z_i) \subseteq U$, then $\bigcap_i (\gamma \vee \delta)(Z_i) \subseteq U$.

Proof of statement. Assume that $(\gamma \vee \delta)(\bigcap_i Z_i) \subseteq U$ and $U = \bigcap_{j \in J} V_j$ where $V_j \in A$ for all $j \in J$. Then $(\gamma \vee \delta)(\bigcap_i Z_i) \subseteq V_j$ for all $j \in J$. By the first statement, we have that $\bigcap_i (\gamma \vee \delta)(Z_i) \subseteq V_j$ for all $j \in J$. Therefore $\bigcap_i (\gamma \vee \delta)(Z_i) \subseteq U$. □

Now take $U = (\gamma \vee \delta)(\bigcap_i Z_i)$, then we get $\bigcap_i (\gamma \vee \delta)(Z_i) \subseteq (\gamma \vee \delta)(\bigcap_i Z_i)$.

□

Now we come to the soundness proof of the right-handed Ackermann rule for $<_\Omega$-minimal propositional variable p with respect to admissible valuations when we have the system $\mathrm{S} \Rightarrow \mathbf{j} \leq t$, where t is p or another propositional variable or a pure formula, and S consists of the following inequalities:

- $\mathrm{MinVal}'_{p,1} \leq p, \ldots, \mathrm{MinVal}'_{p,n} \leq p$;
- inequalities of the form $\mathrm{MinVal}'_q(p) \leq q$, where $q \neq p$ and $\mathrm{MinVal}'_q(p)$ contains positive occurrences of p;
- inequalities that do not contain p;

and after the right-handed Ackermann rule, we have the system of the shape $\mathrm{S}' \Rightarrow \mathbf{j} \leq t(\mathrm{V}_p/p)$, and S' consists of the following inequalities:

- inequalities of the form $\mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q$;
- inequalities that do not contain p before the application.

Proposition 17. *The right-handed Ackermann rule applied as in the success proof is sound in any descriptive modal I-frame* $\mathbb{F}$.

Proof. The spirit of this proof is essentially the same as [10, Lemma 9.3], but the proof details are quite different.

Without loss of generality we assume that S contains the following inequalities and t is p (since when t is not p then $\mathbf{j} \leq t$ does not contain p and the proof will be easier):

- $\mathrm{MinVal}'_{p,1} \leq p, \mathrm{MinVal}'_{p,2} \leq p$;
- one inequality of the form $\mathrm{MinVal}'_q(p) \leq q$, where $q \neq p$ and $\mathrm{MinVal}'_q(p)$ contains positive occurrences of p;
- one inequality Ineq that do not contain p.

Then after the application of the rule, we have the system of the shape $\mathrm{S}' \Rightarrow \mathbf{j} \leq \mathrm{V}_p$, and S' consists of the following inequalities:

- one inequality of the form $\mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q$;
- one inequality that do not contain p before the application.

- $\mathbb{F} \Vdash_A \mathrm{S} \ \Rightarrow \ \mathbf{j} \leq p$;
- $\mathbb{F} \Vdash_A \mathrm{S}' \ \Rightarrow \ \mathbf{j} \leq \mathrm{V}_p$.

For the $\Uparrow$ direction, the proof is essentially the same as in the arbitrary semantics case (just replace "any valuation" by "any admissible valuation").

For the $\Downarrow$ direction, assume that

$$\mathbb{F} \Vdash_A \mathrm{MinVal}'_{p,1} \leq p \ \& \ \mathrm{MinVal}'_{p,2} \leq p \ \& \ \mathrm{MinVal}'_q(p) \leq q \ \& \ \mathrm{Ineq} \ \Rightarrow \ \mathbf{j} \leq p.$$

Take any admissible valuation V on $\mathbb{F}$. If

$$\mathbb{F}, V \Vdash \mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q \ \& \ \mathrm{Ineq} \text{ and } \mathbb{F}, V \nVdash \mathbf{j} \leq \mathrm{V}_p,$$

then we take the arbitrary valuation

$$V' = V^p_{V(\mathrm{V}_p)}$$

which is almost admissible except at p, since

$$V'(p) = V(\mathrm{V}_p) \in C(X)$$

by Lemma 10. Then since p does not occur in V_p, we have that

$$V'(\mathrm{V}_p) = V(\mathrm{V}_p) = V'(p),$$

therefore

$$V(\mathrm{MinVal}'_q(\mathrm{V}_p/p)) = V'(\mathrm{MinVal}'_q(\mathrm{V}_p/p)) = V'(\mathrm{MinVal}'_q(p)),$$

so from $\mathbb{F}, V \Vdash \mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q$ we get

$$V'(\mathrm{MinVal}'_q(p)) \subseteq V(q) = V'(q).$$

Now we denote $\mathrm{MinVal}'_q(p)$ as $\theta(p)$, then we get

$$\theta(V'(p)) = V'(\theta(p)) \subseteq V'(q).[10]$$

[10]See the notation on page 59.

Since $V'(p) \in C(X)$, we have that

$$V'(p) = \bigcap\{Y \in A \mid V'(p) \subseteq Y\}.$$

Since $\theta(p) \in \mathrm{C}_q$ (see page 56), by Lemma 11, we have that

$$\theta(V'(p)) = \theta(\bigcap\{Y\}) = \bigcap \theta(Y),$$

so $\bigcap \theta(Y) \subseteq V'(q)$. Since $Y \in A$, we have $\theta(Y) \in C(X)$ by Lemma 10. By compactness and downward-directness and non-emptiness of $\{Y \in A \mid V'(p) \subseteq Y\} \subseteq A \subseteq C(X)$, there is a $Y \in A$ such that $\theta(Y) \subseteq V'(q)$.

From $\mathbb{F}, V \not\Vdash \mathbf{j} \leq \mathrm{V}_p$ we have

$$V(\mathbf{j}) \nsubseteq V(\mathrm{V}_p) = V'(p) = \bigcap\{Y \in A \mid V'(p) \subseteq Y\}.$$

Assume that $V(\mathbf{j}) = {\uparrow} j$ for $j \in X$, then the above is equivalent to

$$j \notin \bigcap\{Y \in A \mid V'(p) \subseteq Y\},$$

so there is a $Y' \in A$ such that $V'(p) \subseteq Y$ and $j \notin Y'$.

Now take $Z = Y \cap Y'$, then

$$j \notin Z \text{ and } \theta(Z) \subseteq \theta(Y) \subseteq V'(q),$$

so

$$V(\mathbf{j}) \nsubseteq Z \text{ and } \theta(Z) \subseteq V'(q).$$

Now define $V'' := V^p_Z$, then V'' is an admissible valuation. Since p does not occur in V_p, we have

$$V''(\mathrm{V}_p) = V(\mathrm{V}_p) = V'(p) \subseteq Z = V''(p),$$

so

$$\mathbb{F}, V'' \Vdash \mathrm{MinVal}'_{p,1} \leq p \;\&\; \mathrm{MinVal}'_{p,2} \leq p.$$

By $\theta(Z) \subseteq V'(q)$, since $\theta(Z) = V''(\theta)$ and $V''(q) = V'(q)$, we have

$$\mathbb{F}, V'' \Vdash \theta(p) \leq q, \text{ i.e. } \mathbb{F}, V'' \Vdash \mathrm{MinVal}'_q(p) \leq q.$$

Since Ineq does not contain p, from $\mathbb{F}, V \Vdash \mathrm{Ineq}$ we have

$$\mathbb{F}, V'' \Vdash \mathrm{Ineq}.$$

By

$$\mathbb{F} \Vdash_A \mathrm{MinVal}'_{p,1} \leq p \;\&\; \mathrm{MinVal}'_{p,2} \leq p \;\&\; \mathrm{MinVal}'_q(p) \leq q \;\&\; \mathrm{Ineq} \;\Rightarrow\; \mathbf{j} \leq p$$

we get

$$\mathbb{F}, V'' \Vdash \mathbf{j} \leq p,$$

i.e.

$$V(\mathbf{j}) = V''(\mathbf{j}) \subseteq V''(p) = Z,$$

a contradiction to $V(\mathbf{j}) \nsubseteq Z$. So

$$\mathbb{F}, V \Vdash \mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q \ \&\ \mathrm{Ineq} \text{ and } \mathbb{F}, V \nVdash \mathbf{j} \leq \mathrm{V}_p$$

cannot hold, therefore

$$\mathbb{F}, V \Vdash \mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q \ \&\ \mathrm{Ineq} \ \Rightarrow\ \mathbf{j} \leq \mathrm{V}_p.$$

Since V is any admissible valuation, we have

$$\mathbb{F} \Vdash_A \mathrm{MinVal}'_q(\mathrm{V}_p/p) \leq q \ \&\ \mathrm{Ineq} \ \Rightarrow\ \mathbf{j} \leq \mathrm{V}_p.$$

□

9.2 The modal meet-semilattice logic case

For the soundness proof of the algorithm ALBA for inductive quasi-inequalities with respect to descriptive modal M-frames, the proof for the rules here are essentially the same as the same rules in the modal meet-implication logic setting. Notice that the analysis of the right-handed Ackermann rule for descriptive modal I-frame setting can be adapted here without modification.

10 Conclusion

In this paper, we study the correspondence theory for modal meet-implication logic and modal meet-semilattice logic, in the semantics provided in [21]. We study both the discrete correspondence theory over modal I/M-frames, and the topological correspondence theory over descriptive modal I/M-frames.

We give the following concluding remarks:

- The kind of logics we study are not based on algebras that are bounded lattice expansions, but based on (implicative) semilattices which does not necessarily have join or bottom. Therefore, this paper can be regarded as another step of correspondence theory for logics not based on bounded lattices, after [24].
- By the duality given in [21], the relational semantic structures used here (modal I/M-frames and descriptive modal I/M-frames) are based on a semilattice rather than just an ordinary partial order (like in intuitionistic propositional logic), and the propositional variables are interpreted as filters rather than upsets in arbitrary valuations, and in admissible valuations, propositional variables are interpreted as clopen filters rather than clopen upsets.

- In topological correspondence theory, the collection of admissible valuations (which forms a modal (implicative) semilattice) is not necessarily closed under taking disjunction and does not necesarily contain the interpretation of bottom, which makes the proof of the topological Ackermann lemma different from existing settings.

We point out the following future directions:

- **Meet-dense subset.** In the current setting, we are able to find the interprants for the nominals, namely the set that join-generates $\mathcal{F}(X)$. However, we are not able to find a set that meet-generates $\mathcal{F}(X)$. If we can find such a set, we are able to introduce the so-called conominals (see [10]) and have more algorithm rules that are applicable, and maybe even the left-handed Ackermann rule.
- **Spectral correspondence theory.** For modal (implicative) semilattices, the duality that we use in our setting are Priestley-like dualities where the basis for the topology are clopen sets. It is also interesting to investigate the spectral correspondence theory (see e.g. [24]) based on spectral-like dualities for modal (implicative) semilattices, similar to the dualities developed in [7, 8].
- **Complexity issues.** It is worth investigating the complexity of the problem of checking whether a formula/an inequality/a quasi-inequality is inductive, and the complexity of the algorithm given that the input formula/quasi-inequality is inductive.

References

[1] R. Balbes, 1969, "A representation theory for prime and implicative semilattices", *Transactions of the American Mathematical Society*, **136**, 261–267.

[2] G. Bezhanishvili and R. Jansana, 2011, "Priestley style duality for distributive meet-semilattices", *Stud Logica*, **98(1-2):** 83–122.

[3] G. Bezhanishvili and R. Jansana, 2013, "Esakia style duality for implicative semilattices", *Appl. Categorical Struct.*, **21(2):** 181–208.

[4] N. Bezhanishvili, D. Coumans, S. J. v. Gool and D. d. Jongh, 2015, in M. Aher, D. Hole, E. Jeřábek, t. Kupke Clemens" and U. M. for the Meet-Implication Fragment of IPC (eds.), *Logic, Language, and Computation*, pp. 97–116, Berlin, Heidelberg: Springer Berlin Heidelberg.

[5] N. Bezhanishvili, A. Dmitrieva, J. de Groot and T. Moraschini, 2023, "Positive modal logic beyond distributivity", *Annals of Pure and Applied Logic*, 103374.

[6] S. A. Celani, 2022, in preparation, "Relational representation of quasi-modal Tarski algebras".

[7] S. A. Celani, L. M. Cabrer and D. Montangie, 2009, "Representation and duality for Hilbert algebras", *Central European Journal of Mathematics*, **7**, 463–478.

[8] S. A. Celani and t. d. f. T. a. Cabrer Leonardo M.", 2008, *Algebra universalis*, **58(1):** 73–94.

[9] W. Conradie, S. Ghilardi and A. Palmigiano, 2014, "Unified correspondence", in A. Baltag and S. Smets (eds.), *Johan van Benthem on Logic and Information Dynamics*, **Vol. 5**, pp. 933–975, Cham: Springer International Publishing.

[10] W. Conradie and A. Palmigiano, 2012, "Algorithmic correspondence and canonicity for distributive modal logic", *Annals of Pure and Applied Logic*, **163(3):** 338–376.

[11] H. Curry, 1963, *Foundations of Mathematical Logic*, New York: McGraw-Hill.

[12] B. Davey and H. Werner, "Dualities and equivalences for varieties of algebras", in: North-Holland, Amsterdam, 1983, chap. Colloquia Mathematica Societatis Janos Bolyai.

[13] A. Dmitrieva, 2021, *Positive modal logic beyond distributivity: duality, preservation and completeness*, mathesis, University of Amsterdam.

[14] L. Esakia, 1974, "Topological kripke models", *Soviet Math. Dokl.*, **15**, 147–151.

[15] L. Esakia, 2019, *Heyting Algebras: Duality Theory*, A. Evseev(trans.), Springer Cham.

[16] J. M. Font and T. Moraschini, 2014, "Logics of varieties, logics of semilattices and conjunction", *Logic Journal of the IGPL*, **22(6):** 818–843.

[17] D. Fornasiere and T. Moraschini, 2023, "Intuitionistic Sahlqvist theory for deductive systems", *The Journal of Symbolic Logic*, 1–59.

[18] A. Monteiro, 1955, "Axiomes independents pour les algebres de Brouwer", *Revista de la Union Mathematica Argentina y de la Association Fisira Argentina*, **27**, 149–160.

[19] W. Nemitz and T. Whaley, 1971, "Varieties of impicative semilattices", *Pacific Journal of Mathematics*, **37(3):** 759–769.

[20] W. Nemitz, 1965, "Implicative semi-lattices", *Transactions of the American Mathematical Society*, **117**, 128–142.

[21] D. Pattinson and J. de Groot, 2022, "Modal meet-implication logic", *Logical Methods in Computer Science*, **18**, 1:1–1:35.

[22] Z. Zhao, 2021, "Algorithmic correspondence and canonicity for possibility semantics", *Journal of Logic and Computation*, **31(2):** 523–572.

[23] Z. Zhao, 2021, "Algorithmic correspondence for hybrid logic with binder", *Logic Journal of the IGPL*, **31(1):** 39–67.

[24] Z. Zhao, 2022, "Correspondence theory for generalized modal algebras", in A. Ciabattoni, E. Pimentel and R. J. G. B. de Queiroz (eds.), *Logic, Language, Information, and Computation*, pp. 53–69, Cham: Springer International Publishing.

滤语义下模态合取–蕴涵逻辑
与模态交半格逻辑的离散与拓扑对应理论

梁飞　　赵之光

摘　要

本文对模态合取-蕴涵逻辑和模态交半格逻辑的离散与拓扑对应理论进行系统的研究。本研究的主要特点有三点：第一，本文中的语义结构基于半格而非一般的偏序；第二，命题变元解释成滤而非向上封闭的集合，作为命题变元的一阶版本的专名解释成主滤而非单点生成的向上封闭的集合；第三，在拓扑对应理论中，可容许赋值的集合不对析取封闭，从而使拓扑阿克曼引理的证明与已有情形不同。

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