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Saturated Models and Ultrafilter Extension for Weakly Aggregative Modal Logics*

Yifeng Ding Jixin Liu

Abstract. Weakly aggregative modal logics (WAML) are a series of natural weakenings of the minimal modal logic K. The natural semantics for them are based on Kripke frames with an $N+1$-ary relation, where $\Box\varphi$ is true at a world iff all of its successor N-tuples has at least one world making φ true. We study the notion of saturated models and ultrafilter extension for this relational semantics of WAML. The Goldblatt-Thomason theorem for WAML is proved as an application.

1 Introduction

Let us call a modality $\Box$ N-weakly aggregative, with N being a positive natural number, if $\Box$ is monotonic and validates $\Box\top$ and the following principle:

$$(\Box p_0 \wedge \Box p_2 \wedge \cdots \wedge \Box p_N) \to \Box((p_0 \wedge p_1) \vee (p_0 \wedge p_2) \vee \cdots \vee (p_{N-1} \wedge p_N)).$$

Modalities of this kind arise naturally whenever we quantify over tuples or sets of size at most N in a $\forall\exists$ manner. For example, let $\Box\varphi$ mean that: "in every pair of twin primes, one of them satisfies property φ". Then, assuming $\Box p_0$ and $\Box p_1$ and $\Box p_2$, for every pair of twin primes (a, b), there is a function $f : \{p_0, p_1, p_2\} \to \{a, b\}$ such that $f(p_i)$ satisfies p_i. By the pigeonhole principle, f is not injective, and either a or b satisfies two of the properties in $\{p_0, p_1, p_2\}$. This means $\Box((p_0 \wedge p_1) \vee (p_0 \wedge p_2) \vee (p_1 \wedge p_2))$ is true.

Over Kripke models with an $N+1$-ary relation, this $\forall\exists$-style quantification is a natural way to talk about them using a unary modality. More formally, $\Box\varphi$ is true at w iff for any $v_1, \ldots, v_N$ such that $R(w, v_1, \ldots, v_N)$, there is $v \in \{v_1, \ldots, v_N\}$ such that φ is true at v. It is easy to see using the pigeonhole principle that $\Box$ thus defined is an N-weakly aggregative modality.

This relational semantics based on standard Kripke models with an $N+1$-ary relation is first defined in [13], and the authors there also introduce the name "Weakly

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Yifeng Ding Department of Philosophy, Peking University
yf.ding@pku.edu.cn
Jixin Liu Department of Philosophy, Sichuan University
xuetian997@scu.edu.cn

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aggregative modal logic (WAML)”. One can quickly observe that $\Box\varphi$ as understood above is equivalent to $\nabla(\phi, \ldots, \phi)$ where ∇ is the standard polyadic modal operator. In a later paper ([2]), it is proved that the modal logic on all $N+1$-ary frames is exactly the smallest logic in which $\Box$ is N-weakly aggregative. WAML have strong connections to different areas, like paraconsistent reasoning ([13]), epistemic logic of knowing value ([10]), neighborhood semantics ([1]) and group knowledge ([4]). This paper will focus on the model theory on relational semantics of WAML; one may see [11] for further literature on other aspects of WAML.

The aim of this paper is to find the right notion of saturation and ultrafilter extension (canonical extension) for WAML. This continues the study of model theory for WAML in [12] and the canonical model construction used in [8]. There are some related works on these topics, such as [3] and [6, 7]. But the first one mainly considers the special case of $N = 1$ and the other two concern neighborhood models. Though they all work with the same language with a single unary modality as in this paper, our work concerns the general $N+1$-ary relational semantics.

The remaining parts of this paper are structured as follows. In Section 2 we recall the model theoretical set up of WAML and define the complex algebra of frames. In Section 3, we study the saturation condition for models. In Section 4, we study the ultrafilter (canonical) extension for any Boolean algebra expanded with a weakly aggregative modality. When applied to the complex algebra of the frame of a model while keeping the valuation information, we obtain the ultrafilter extension of the said model and show that it is saturated. In Section 5, to further justify the definition of ultrafilter extension, we show the Goldblatt-Thomason theorem for weakly aggregative modal logic. In Section 6, we conclude with some unaddressed questions.

2 Relational Semantics and the Algebraic Perspective

Throughout the paper, let N be a fixed positive natural number denoting the arity parameter of the weakly aggregative modal logic we consider.

Definition 1. Let $\mathcal{L}$ be the language of basic modal logic defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

where p is in a countably infinite set Prop of propositional atoms. We employ the usual abbreviations, including $\Diamond\varphi := \neg\Box\neg\varphi$.

Definition 2. An N-Kripke frame is a pair (W, R) where W is a non-empty set and R is an $N+1$-ary relation. An N-Kripke model is a triple (W, R, V) where (W, R) is an N-Kripke frame and $V : \mathrm{Prop} \to \wp(W)$. Given any N-Kripke model

$\mathcal{M} = (W, R, V)$, we define the weakly aggregative semantics for $\mathcal{L}$ as follows:

$$\begin{aligned}
\mathcal{M}, w \models p &\iff w \in V(p)\\
\mathcal{M}, w \models \neg\varphi &\iff \mathcal{M}, w \not\models \varphi\\
\mathcal{M}, w \models \varphi \wedge \psi &\iff \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi\\
\mathcal{M}, w \models \Box\varphi &\iff \forall u_1, u_2, \ldots, u_N \in W : R(w, u_1, \ldots, u_N) \implies\\
&\qquad \mathcal{M}, u_1 \models \varphi \text{ or } \mathcal{M}, u_2 \models \varphi \text{ or } \cdots \text{ or } \mathcal{M}, u_N \models \varphi.
\end{aligned}$$

From the above clause for $\Box\phi$, one can directly verify the semantics for $\Diamond\phi$:

$$\begin{aligned}
\mathcal{M}, w \models \Diamond\varphi \iff &\exists u_1, u_2, \ldots, u_N \in W : R(w, u_1, \ldots, u_N) \text{ and}\\
&\mathcal{M}, u_1 \models \varphi \text{ and } \mathcal{M}, u_2 \models \varphi \text{ and } \cdots \text{ and } \mathcal{M}, u_N \models \varphi.
\end{aligned}$$

We introduce some notations to help shorten the above semantic clause for $\Box$. Let $[1, N]$ be the set of natural numbers from 1 to N inclusive, and let W^N be the set of all N-tuples using elements in W. Then for any $\vec{u} \in W^N$ and any $i \in [1, N]$, $\vec{u}[i]$ is the ith element of $\vec{u}$ ($\vec{u} = (\vec{u}[1], \ldots, \vec{u}[n])$). Finally, when $\vec{b} \in W^N$, we write $R(a, \vec{b})$ for $R(a, \vec{b}[1], \ldots, \vec{b}[N])$. Thus, the above semantic clause for $\Box$ can be written as $\forall \vec{u} \in W^N (R(w, \vec{u}) \Rightarrow \exists i \in [1, N] \mathcal{M}, \vec{u}[i] \models \varphi)$.

To better understand the above semantics for $\Box$, note that for the interpretation of weakly aggregative modal logic, N-Kripke frames contain redundant information. For example, say $N = 2$ and consider $(\{1, 2, 3\}, \{(1, 2, 3)\})$ and $(\{1, 2, 3\}, \{(1, 3, 2)\})$. Our language can never pick up the difference between these two N-Kripke frames since they only differ in the ordering of the second and the third argument of their respective ternary relations. For any tuple $\vec{u}$, let set$(\vec{u})$ be the set of elements used in $\vec{u}$. Then it is clear from the semantic clause for $\Box$ that when $R(w, \vec{u})$, only set$(\vec{u})$ matters in a monotonic way. If we want to remove the redundant information, we can consider the following condition on N-Kripke frames.

Definition 3. An N-weakly aggregative frame (N-WA frame for short) is a pair (W, R) such that

- W is a non-empty set, R is an $N + 1$-ary relation on W;
- for any $w \in W$ and any $\vec{u}, \vec{v} \in W^N$, if $R(w, \vec{u})$ and set$(\vec{u}) \subseteq$ set$(\vec{v})$, then $R(w, \vec{v})$.

Note that the second condition is a first-order condition expressible by the formula:

$$\forall x \forall y_1 \ldots \forall y_N \forall z_1 \ldots \forall z_N ((R(x, y_1, \ldots, y_N) \wedge \bigwedge_{i \in [1,N]} \bigvee_{j \in [i,N]} y_i = z_j) \\ \to R(x, z_1, \ldots, z_N)).$$

But to exactly capture the semantic content of an N-Kripke frame, it is best to consider the algebra they generate.

Definition 4. An N-weakly aggregative algebra (N-WA algebra for short) is a pair $\mathcal{B} = (B, \Box)$ where B is a Boolean algebra and $\Box$ is a monotonic function from B to B such that $\Box\top = \top$ and the axiom K_N for the weakly aggregative modal logic is valid:

- for any $a_0, a_1, \ldots, a_N \in B$, $\bigwedge_{i=0}^{N} \Box a_i \leq \Box(\bigvee_{i<j\leq}(a_i \wedge a_j))$.

Here $\leq$ is the order in the Boolean algebra B.

For any N-Kripke frame $F = (W, R)$, define its complex algebra $\mathrm{cmp}(F) = (\wp(W), \Box_R)$ where we endow $\wp(W)$ with its standard Boolean structure by intersection and complementation and $\Box_R$ is defined according to the semantics of $\Box$: $\Box_R(X) = \{w \in W \mid \forall \vec{u} \in W^N,\text{ if } R(w, \vec{u}) \text{ then } X \cap \text{set}(\vec{u}) \neq \varnothing\}$. In the following, the dual $\Diamond_R$ will be used much more often, so we note its definition explicitly: $\Diamond_R(X) = \{w \in W \mid \exists \vec{u} \in W^N, R(w, \vec{u}) \text{ and set}(\vec{u}) \subseteq X\}$. We often write $\Diamond_R X$ for $\Diamond_R(X)$, and further just write $\Diamond$ for $\Diamond_R$ when the context is clear. Observe that $\mathrm{cmp}(F)$ must be an N-WA algebra.

Given an N-Kripke frame (W, R), consider $R^+ \subseteq W^{N+1}$ defined by $R^+(w, \vec{u})$ iff there is $\vec{v} \in W^N$ such that $R(w, \vec{v})$ and set$(\vec{v}) \subseteq$ set$(\vec{u})$. Then it is easy to observe that (W, R^+) is an N-WA frame and $\mathrm{cmp}(W, R^+) = \mathrm{cmp}(W, R)$. In other words, every N-Kripke frame can be equivalently transformed into an N-WA frame in a canonical way. Conversely, it can be checked that in all our later methods of constructing a N-Kripke frame from an N-WA algebra, the resulting frame will in fact be an N-WA frame.

3 Saturation

Saturation is an important concept in model theory. In the context of modal logic, saturation is the most natural condition under which we can derive structural equivalence, usually a version of bisimulation, from syntactical equivalence, usually defined by satisfying the same formulas.

The right kind of bisimulation is already defined in [12]. We recall it here:

Definition 5. Let $\mathcal{M}_1 = (W_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, V_2)$ be N-Kripke models. A bisimulation E between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a binary relation from W_1 to W_2 such that for any $(w_1, w_2) \in E$:

- for any $p \in \mathrm{Prop}$, $w_1 \in V(p)$ iff $w_2 \in V(p)$;
- for any $\vec{v_1} \in W_1^N$ such that $R_1(w_1, \vec{v_1})$ there is $\vec{v_2} \in W_2^N$ such that $R_2(w_2, \vec{v_2})$ and for any $i \in [1, N]$ there is $j \in [1, N]$ such that $\vec{v_1}[j] E \vec{v_2}[i]$;

- for any $\vec{v_2} \in W_2^N$ such that $R_2(w_2, \vec{v_2})$ there is $\vec{v_1} \in W_1^N$ such that $R_1(w_1, \vec{v_1})$ and for any $i \in [1, N]$ there is $j \in [1, N]$ such that $\vec{v_1}[i]E\vec{v_2}[j]$.

The condition that for any $i \in [1, N]$ there is $j \in [1, N]$ such that $\vec{v_1}[j]E\vec{v_2}[i]$ can also be understood as that E restricted to set$(\vec{v_1}) \times$ set$(\vec{v_2})$ is total on the set$(\vec{v_2})$ side.

Two pointed models $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ are bisimilar, written $\mathcal{M}_1, w_1 \leftrightarroweq \mathcal{M}_2, w_2$ if there is a bisimulation E between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $w_1 E w_2$.

Let us also introduce a notation of syntactic equivalence:

Definition 6. Let $\mathcal{M}_1 = (W_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, V_2)$ be N-Kripke models and $w_1 \in W_1$ and $w_2 \in W_2$. We say that the two pointed models $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ are modally equivalent in language $\mathcal{L}$, written $\mathcal{M}_1, w_1 \equiv_{\mathcal{L}} \mathcal{M}_2$, when for any $\varphi \in \mathcal{L}$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.

Now we consider the saturation condition for WAML. Since we are working with polyadic Kripke frames, it would be instructive to recall the standard saturation condition for polyadic modal logic in [5].

Definition 7. Let $\mathcal{PL}$ be the language for a single N-ary polyadic modal operator ∇ built from Prop. Then, the semantics of ∇ is given by the following clause:

$$(W, R, V), w \models \nabla(\varphi_1, \ldots, \varphi_N)$$

$$\Leftrightarrow \forall \vec{u} \in W^n (R(w, \vec{u}) \to \exists i \in [1, n](W, R, V), \vec{u}[i] \models \varphi_i).$$

We note that the semantics of the dual $\Delta := \neg\nabla\neg$ is

$$(W, R, V), w \models \Delta(\varphi_1, \ldots, \varphi_N)$$

$$\Leftrightarrow \exists \vec{u} \in W^n (R(w, \vec{u}) \wedge \forall i \in [1, n](W, R, V), \vec{u}[i] \models \varphi_i).$$

Now, for any N-Kripke model $\mathcal{M} = (W, R, V)$, we say that it is polyadic saturated (P-saturated for short) just in case for every $w \in W$ and every sequence $\Sigma_1, \ldots, \Sigma_N$ of subsets of $\mathcal{PL}$:

- if for every finite $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_N \subseteq \Sigma_N$ there is $\vec{u} \in W^N$ such that $R(w, \vec{u})$ and $\forall i \in [1, N], \mathcal{M}, \vec{u}[i] \models \Delta_i$,
- then there is $\vec{u} \in W^N$ such that $R(w, \vec{u})$ and $\forall i \in [1, N], \mathcal{M}, \vec{u}[i] \models \Sigma_i$.

By observing the difference between the semantics of ∇ and $\Box$ and the bisimulation condition for $\Box$, we may come to the following definition:

Definition 8. For any N-Kripke model $\mathcal{M} = (W, R, V)$, we say that it is WA-saturated just in case for every $w \in W$ and every sequence $\Sigma_1, \ldots, \Sigma_N$ of subsets of $\mathcal{L}$:

- if for every finite $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_N \subseteq \Sigma_N$ there is $\vec{u} \in W^N$ such that $R(w, \vec{u})$ and $\forall i \in [1, N] \exists j \in [1, N] \mathcal{M}, \vec{u}[i] \models \Delta_j$,
- then there is $\vec{u} \in W^N$ such that $R(w, \vec{u})$ and $\forall i \in [1, N] \exists j \in [1, N], \mathcal{M}, \vec{u}[i] \models \Sigma_j$.

Indeed, it is quite easy to show that two modally equivalent WA-saturated pointed models are bisimilar.

Proposition 1. *Let $\mathcal{M}_1 = (W_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, V_2)$ be WA-saturated N-Kripke models and $w_1 \in W_1$ and $w_2 \in W_2$. Then if $\mathcal{M}_1, w_1 \equiv_{\mathcal{L}} \mathcal{M}_2, w_2$, then $\mathcal{M}_1, w_1 \leftrightarroweq \mathcal{M}_2, w_2$.*

Proof. It suffices to prove that the relation $\equiv_{\mathcal{L}}$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a bisimulation between them.

We focus on the forth condition, since the case for propositional letters is trivially satisfied, and the back condition is completely analogous to the case we prove. Assume that $\mathcal{M}_1, w_1 \equiv_{\mathcal{L}} \mathcal{M}_2, w_2$, $\vec{v_1} \in W^N$, and $R_1(w_1, \vec{v_1})$. Let Σ_i be the set of formulas true at $\vec{v_1}[i]$ for each $i \in [1, N]$.

Obviously, for every sequence of finite subset $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$, we have $\forall i \in [1, N] \exists j \in [1, N]$ (which is just i itself) $\mathcal{M}_1, \vec{v_1}[i] \models \bigwedge \Delta_j$. Hence $\mathcal{M}_1, w_1 \models \Diamond \bigvee_{j \in [1,N]} \bigwedge \Delta_j$. It follows that $\mathcal{M}_2, w_2 \models \Diamond \bigvee_{j \in [1,N]} \bigwedge \Delta_j$ since $\mathcal{M}_1, w_1 \equiv_{\mathcal{L}} \mathcal{M}_2, w_2$, which means there is $\vec{u} \in W_2^N$ such that $\forall i \in [1, N]$, $\mathcal{M}_2$, $\vec{u}[i] \models \bigvee_{j \in [1,N]} \bigwedge \Delta_j$. This means $\forall i \in [1, N] \exists j \in [1, N]$, $\mathcal{M}_2, \vec{u}[i] \models \Delta_j$. Since these Δ_j's are arbitrarily chosen and $\mathcal{M}_2$ is WA-saturated, there is $\vec{v_2} \in W_2^N$ such that $R_2(w_2, \vec{v_2})$ and $\forall i \in [1, N] \exists j \in [1, N] \mathcal{M}_2, \vec{v_2}[i] \models \Sigma_j$, which means for each i there is a j such that $\vec{v_1}[j] \equiv_{\mathcal{L}} \vec{v_2}[i]$. □

However, there is another version of the above saturation condition that is much easier to handle.

Proposition 2. *For any N-Kripke model $\mathcal{M} = (W, R, V)$, it is WA-saturated iff for every $w \in W$ and every $\Sigma \subseteq \mathcal{L}$: if for every finite $\Delta \subseteq \Sigma$ there is $\vec{u} \in W^N$ such that $R(w, \vec{u})$ and $\forall i \in [1, N] \mathcal{M}, \vec{u}[i] \models \Delta$, then there is $\vec{u} \in W^N$ such that $R(w, \vec{u})$ and $\forall i \in [1, N] \mathcal{M}, \vec{u}[i] \models \Sigma$.*

To better formulate the proof of this proposition, we write $X \subseteq_f Y$ for X being a finite subset of Y and then temporarily expand the language $\mathcal{L}$ to $\mathcal{L}_\infty$ to allow for infinite conjunction, with the grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \bigwedge \Gamma \mid \Box\varphi$$

where $p \in \text{Prop}$ and $\Gamma \subseteq \mathcal{L}$. It is clear that an N-Kripke model $\mathcal{M} = (W, R, V)$ is WA-saturated iff for every $w \in W$ and every sequence $\Sigma_1, \cdots, \Sigma_N$ of sets of formulas in $\mathcal{L}$, if for every $\Delta_1 \subseteq_f \Sigma_1, \cdots \Delta_N \subseteq_f \Sigma_N$, $\mathcal{M}, w \models \Diamond \bigvee_{i=1}^N \bigwedge \Delta_i$, then $\mathcal{M}, w \models \Diamond \bigvee_{i=1}^N \bigwedge \Sigma_i$. What we want to show is that $\mathcal{M}$ is WA-saturated iff for every $w \in W$ and every set $\Sigma \subseteq \mathcal{L}$, if $\mathcal{M}, w \models \Diamond \bigwedge \Delta$ for every finite subset Δ of Σ, then $\mathcal{M}, w \models \Diamond \bigwedge \Gamma$. For this, we use two lemmas.

Lemma 1. *For every sequence $\Sigma_1, \cdots, \Sigma_N$ of sets of formulas in $\mathcal{L}$, $\bigvee_{i=1}^N \bigwedge \Sigma_i$ and $\bigwedge\{\bigvee_{i=1}^N \bigwedge \Delta_i \mid \Delta_1 \subseteq_f \Sigma_1, \cdots \Delta_N \subseteq_f \Sigma_N\}$ are semantically equivalent.*

Proof. For the left-to-right direction, note that $\bigwedge \Sigma_i \models \bigwedge_{i=1}^N \Delta_i$ whenever $\Delta_i \subseteq \Sigma_i$. For the right-to-left direction, suppose that $\bigvee_{i=1}^N \bigwedge \Sigma_i$ is false. So there is a sequence $\varphi_1, \varphi_2, \cdots, \varphi_N$ of formulas such that $\varphi_i \in \Sigma_i$ and each φ_i is false. Then $\bigvee_{i=1}^N \varphi_i$ is false. But this formula is a conjunct of $\bigwedge\{\bigvee_{i=1}^N \bigwedge \Delta_i \mid \Delta_1 \subseteq_f \Sigma_1, \cdots \Delta_N \subseteq_f \Sigma_N\}$. □

Lemma 2. *For every sequence $\Sigma_1, \cdots, \Sigma_N$ of sets of formulas in $\mathcal{L}$, $\Gamma = \{\bigvee_{i=1}^n \bigwedge \Delta_i \mid \Delta_1 \subseteq_f \Sigma_1, \cdots \Delta_N \subseteq_f \Sigma_N\}$ is directed: for any $\alpha, \beta \in \Gamma$, there is a $\gamma \in \Gamma$ s.t. $\gamma \models \alpha \wedge \beta$. By a simple induction, then, for every $\Delta \subseteq_f \Gamma$, there is $\gamma \in \Gamma$ s.t. $\gamma \models \bigwedge \Delta$.*

Proof. Let $\alpha = \bigvee_{i=1}^N \bigwedge A_i$ and $\beta = \bigvee_{i=1}^N \bigwedge B_i$ be two arbitrary formula in Γ. Then $A_i, B_i \subseteq_f \Sigma_i$ for all $i \in [1, N]$. Now let $\gamma = \bigvee_{i=1}^N \bigwedge(A_i \cup B_i)$. Clearly $\gamma \in \Gamma$. To see that $\gamma \models \alpha \wedge \beta$, suppose that γ is true. Then there is $i \in [1, N]$ such that $\bigwedge(A_i \cup B_i)$ is true, which means that $\bigwedge A_i$ and $\bigwedge B_i$ are both true. But $\bigwedge A_i$ is a disjunct of α and $\bigwedge B_i$ is a disjunct of β, so both α and β are true. □

Proof. (of Proposition 2) Let us temporarily call the new condition in Proposition 2 WA'-saturation. Suppose $\mathcal{M}$ is WA-saturated. To show that it is WA'-saturated, pick any $w \in W$ and any $\Sigma \subseteq \mathcal{L}$ and suppose that for any $\Delta \subseteq_f \Sigma$, $\mathcal{M}, w \models \Diamond \bigwedge \Delta$. Now, to use the assumption that $\mathcal{M}$ is WA-saturated, let $\Sigma_1 = \Sigma_2 = \cdots = \Sigma_n = \Sigma$. We can then show the antecedent of the definition of WA-saturation. Pick any $\Delta_i \subseteq_f \Sigma_i$ for all i, let $\Delta = \bigcup_{i=1}^N \Delta_i$. Since Σ_i's are just Σ, $\Delta \subseteq_f \Sigma$. So by supposition, $\mathcal{M}, w \models \Diamond\Delta$. But $\Delta \models \bigwedge_{i=1}^N \Delta_i$ since $\Delta_i \subseteq \Delta$ for all i. Hence $\mathcal{M}, w \models \Diamond \bigvee_{i=1}^N \bigwedge \Delta$, and this shows the antecedent of WA-saturation. Thus, by WA-saturation, $\mathcal{M}, w \models \Diamond \bigvee_{i=1}^N \bigwedge \Sigma_i$. But $\bigvee_{i=1}^N \bigwedge \Sigma_i$ is equivalent to just $\bigwedge \Sigma$ since each Σ_i is just Σ. So $\mathcal{M}, w \models \Diamond \bigwedge \Sigma$ and hence WA'-saturation is shown.

Now suppose that $\mathcal{M}$ is WA'-saturated. To show that it is WA-saturated, pick any $w \in W$ and any $\Sigma_i \subseteq \mathcal{L}$ for all i and assume that whenever $\Delta_i \subseteq_f \Sigma_i$ for all i, $\mathcal{M}, w \models \Diamond \bigvee_{i=1}^N \bigwedge \Delta_i$. Now let $\Gamma = \{\bigvee_{i=1}^N \bigwedge \Delta_i \mid \Delta_i \subseteq_f \Sigma_i, i \in [1, N]\}$. Then the assumption shows that for every $\gamma \in \Gamma$, $\mathcal{M}, w \models \Diamond\gamma$. But by Lemma 2, for any $\Delta \subseteq_f \Gamma$, there is $\gamma \in \Gamma$ such that $\gamma \models \Delta$. Hence $\mathcal{M}, w \models \Diamond\Delta$ for every $\Delta \subseteq_f \Gamma$, and this fulfils the antecedent of WA'-saturation on Γ. So it follows that $\mathcal{M}, w \models \Diamond \bigwedge \Gamma$. But by Lemma 1, $\bigwedge \Gamma$ is equivalent to $\bigvee_{i=1}^N \bigwedge \Sigma_i$. So $\mathcal{M}, w \models \bigvee_{i=1}^N \bigwedge \Sigma_i$, and this completes the proof of WA-saturation. □

Given the above proposition, our official notion of saturation for weakly aggregative modal logic is simply that whenever every finite subset of Σ is "diamond-satisfiable", then Σ is itself "diamond-satisfiable". Note that syntactically this is exactly the same for normal modal logics.

To see that this is the correct saturation condition, we should also verify that first-order saturation implies modal saturation. Recall that we can always treat any N-Kripke model as a first-order structure with one $N+1$-ary relation and one unary predicate P for each $p \in \mathrm{Prop}$. Then, each $\varphi \in \mathcal{L}$ can be translated to a first-order formula $ST_x(\varphi)$ with one free variable x (for any choice of variable x) as in [12].

Proposition 3. *If an N-Kripke model $\mathcal{M} = (W, R, V)$ is countably saturated in the first-order sense, then it is also WA-saturated.*

Proof. Suppose that $\mathcal{M} = (W, R, V)$ is a countably saturated model. Let $w \in W$ and $\Sigma \subseteq \mathcal{L}$. Consider the following set of first-order formula where w is taken as a constant denoting itself and $x_1, \ldots, x_N$ are variables:

$$\Gamma = \{R(w, x_1, \ldots, x_n)\} \cup \{ST_{x_i}(\varphi) \mid i \in [1, N], \varphi \in \Sigma\}.$$

To show that $\mathcal{M}$ is WA-saturated, assume that for any $\Delta \subseteq_f \Sigma$, $\mathcal{M}, w \models \Diamond \bigwedge \Delta$, and we only need to show that $\mathcal{M}, w \models \Diamond \bigwedge \Sigma$. The assumption immediately entails that Γ is a finitely satisfiable N-type with one parameter. By countable saturation, Γ is realized by some $\vec{v} \in W^N$. But this precisely means that $\mathcal{M}, w \models \Diamond \bigwedge \Sigma$. □

4 Canonical Extension and Ultrafilter Extension

In this section, we consider the modal way to obtain saturated models: ultrafilter extension and its algebraic generalization, canonical extension. This is a general method of constructing an N-WA frame from any N-WA algebra. But for completeness, we first consider a special class of N-WA algebras that directly correspond to N-WA frames without using ultrafilters. For the definition of standard algebraic notions, see [5] and [9] for example.

Given an N-Kripke frame F, we can easily construct its complex algebra $\mathrm{cmp}(F)$. To obtain a duality between the N-Kripke frames and the corresponding complex algebras, we characterize the complex algebras intrinsically.

Definition 9. Let $\mathcal{B} = (B, \Box)$ be an N-WA algebra, and we will always use $\Diamond$ for the dual operator of $\Box$. We say that $\mathcal{B}$ is perfect if

- B is atomic and lattice-complete;
- for any filter $Q \subseteq B$, $\bigwedge\{\Box a \mid a \in Q\} \leq \Box \bigwedge Q$.

Note that the second condition is equivalent to that

- for any ideal $I \subseteq B$, $\Diamond \bigvee I \leq \bigvee\{\Diamond a \mid a \in I\}$.

We call this condition the *complete primeness of* $\Diamond$.

For any perfect N-WA algebra $\mathcal{B} = (B, \Box)$, define $\mathrm{AF}(\mathcal{B}) = (W, R)$ where W is the set of atoms of $\mathcal{B}$ and R is defined as follows:

- for any $w \in W$ and $\vec{v} \in W^N$, $R(w, \vec{v})$ iff $w \leq \Diamond \bigvee \mathrm{set}(\vec{v})$.

Proposition 4. *For any N-Kripke frame $F = (W, R)$, $\mathrm{cmp}(F)$ is a perfect N-WA algebra, and if F is further an N-WA frame, then $\mathrm{AF}(\mathrm{cmp}(F))$ is isomorphic to F.*

Proof. To see that $\mathrm{cmp}(F)$ is perfect, the only non-trivial item to show is that for any ideal $I \subseteq \wp(W)$, $\Diamond \bigcup I \subseteq \bigcup\{\Diamond X \mid X \in I\}$. So suppose $x \in \Diamond \bigcup I$. This means there is $\vec{y} \in W^N$ such that $R(x, \vec{y})$ and for any $i \in [1, N]$, $\vec{v}[i] \in \bigcup I$. Thus we have $\{X_i\}_{i \in [1,N]} \subseteq I$ such that $\vec{v}[i] \in X_i$. But since I is an ideal, $X := \bigcup_{i \in [1,N]} X_i$ is an element of the ideal I, and clearly $\mathrm{set}(\vec{y}) \subseteq X$. So indeed there is $X \in I$ such that $x \in \Diamond X$. So $x \in \bigcup\{\Diamond X \mid X \in I\}$. This finishes the proof that $\mathrm{cmp}(F)$ is perfect.

Now let (W', R') be $\mathrm{AF}(\mathrm{cmp}(F))$. Obviously $W' = \{\{x\} \mid x \in W\}$, so the natural isomorphism should be $f : W \to W' :: f(x) = \{x\}$. Now we only need to show that, given that F is N-WA: for any $x \in W$ and $\vec{y} \in W^N$,

- $R(x, \vec{y})$ iff $x \in \Diamond\mathrm{set}(\vec{y})$.

The left-to-right direction is trivial given how $\Diamond$ is defined as a function from $\wp(W)$ to $\wp(W)$. For the other direction, note that if $x \in \Diamond\mathrm{set}(\vec{y})$, then there is $\vec{z} \in W^N$ such that $R(x, \vec{z})$ and $\mathrm{set}(\vec{z}) \subseteq \mathrm{set}(\vec{y})$. But the requirement for N-WA frame precisely says that this implies that $R(x, \vec{y})$. $\Box$

Proposition 5. *For any perfect N-WA algebra $\mathcal{B} = (B, \Box)$, $\mathrm{AF}(\mathcal{B})$ is an N-WA frame, and $\mathrm{cmpAF}(\mathcal{B})$ is isomorphic to $\mathcal{B}$.*

Proof. That $\mathrm{AF}(\mathcal{B})$ is an N-WA frame is easy to verify from the definition of R and the monotonicity of $\Diamond$. It is also a standard result in Boolean algebra that there is a Boolean isomorphism f from $\mathrm{cmp}(\mathrm{AF}(\mathcal{B}))$ to $\mathcal{B}$ where $f(X) = \bigvee X$. Now we show that $f(\Diamond_R X) = \Diamond f(X)$. It is enough to show that for any atom $w \in W$, the set of atoms of B, $w \leq f(\Diamond_R X)$ iff $w \leq \Diamond f(X)$.

If $w \leq f(\Diamond_R X)$, then $w \in \Diamond_R X$. So there is $\vec{v} \in W^N$ such that $R(w, \vec{v})$ and $\mathrm{set}(\vec{v}) \subseteq X$. $R(w, \vec{v})$ means that $w \leq \Diamond \bigvee \mathrm{set}(\vec{v})$, and $\mathrm{set}(\vec{v}) \subseteq X$ means that $\bigvee \mathrm{set}(\vec{v}) \leq \bigvee X = f(X)$. Since $\Diamond$ is monotone, $w \leq \Diamond f(X)$.

If $w \leq \Diamond f(X)$, then $w \leq \Diamond \bigvee X$. Let $I = \{\bigvee X_0 \mid X_0 \subseteq_f X\}$ where $\subseteq_f$ means "is a finite subset of". Then I is an ideal in B and $\bigvee I = \bigvee X$. By the complete primeness of $\Diamond$, $w \leq \bigvee\{\Diamond \bigvee X_0 \mid X_0 \subseteq_f X\}$. Since w is an atom, there is $X_0 \subseteq_f X$ such that $w \leq \Diamond \bigvee X_0$. Now let Y be an element in the non-empty set $\{Y \in \wp(X_0) \mid w \leq \Diamond \bigvee Y\}$ that has the smallest cardinality. First, $|Y|$ is non-empty as $\Diamond\bot = \bot$. Now we claim that $|Y| \leq N$. Suppose not, then let $\{y_0, \dots, y_N\}$ be a subset of Y

with $N+1$ elements. Then note that $Y = \bigwedge_{i<j}(\bigvee(Y \setminus \{y_i\}) \vee \bigvee(Y \setminus \{y_j\}))$. If we write the K_N axiom for N-WA algebra in terms of $\Diamond$, we see that

$$\Diamond(\bigwedge_{i<j}(\bigvee(Y \setminus \{y_i\}) \vee \bigvee(Y \setminus \{y_j\}))) \leq \bigvee_i \Diamond \bigvee(Y \setminus \{y_i\}).$$

So there is $i \in [0, N]$ such that $w \leq \Diamond \bigvee(Y \setminus \{y_i\})$, contradicting that Y is one of the smallest subset of X_0 such that $w \leq \Diamond \bigvee Y$. So $|Y| \leq N$, and we can easily pick a $\vec{v}$ as an surjection from $[1, N]$ to Y so that set$(\vec{v}) = y$, and $w \leq \Diamond \bigvee$ set$(\vec{v})$. Then $R(w, \vec{v})$. Recall that $Y \subseteq X_0 \subseteq X$. So $w \in \Diamond_R X$, and thus $w \leq \bigvee \Diamond_R X = f(\Diamond_R X)$. □

One can extend the above duality into a full categorical duality, noting that the right morphism between N-Kripke frames that would correspond to homomorphisms between N-WA algebras is the following p-morphism:

Definition 10. Let $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ be N-Kripke frames. A function $f : W_1 \to W_2$ is a p-morphism (for WAML) just in case:

- for any $x \in W_1$ and $\vec{y} \in W_1^N$, if $R_1(x, \vec{y})$, then there is $\vec{z} \in W_2^N$ such that $R_2(f(x), \vec{z})$ and set$(\vec{z}) \subseteq f[$set$(\vec{y})]$;
- for any $x \in W_1$ and $\vec{z} \in W_2^N$, if $R_2(f(x), \vec{z})$, then there is $\vec{y} \in W_1^N$ such that $R_1(x, \vec{y})$ and set$(\vec{y}) \subseteq f^{-1}[set(\vec{z})]$;

Given the above definition, the following is easy to verify.

Proposition 6. *Let $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ be N-Kripke frames. Then, for any function $f : W_1 \to W_2$, its dual $f^{-1} : \wp(W_2) \to \wp(W_1)$ is a homomorphism from $\mathrm{cmp}(F_2)$ to $\mathrm{cmp}(F_1)$ iff f is a p-morphism. Thus, p-morphisms from F_1 to F_2 and homomorphisms from $\mathrm{cmp}(F_2)$ to $\mathrm{cmp}(F_1)$ are in 1-to-1 correspondence.*

Now we consider the ultrafilter way of defining an N-WA frame from an N-WA algebra, which, if applied to $\mathrm{cmp}(F)$, produces the ultrafilter extension of F. For generality and later application, we work with general frames carrying a distinguished field of sets.

Definition 11. An N-general frame is a triple (W, R, A) where (W, R) is an N-Kripke frame and A is a field of sets (with intersection and complementation as its Boolean operation) on W that is also closed under $\Box_R$ (or equivalently, $\Diamond_R$). For any N-general frame $\mathbf{F} = (W, R, A)$, we write $\mathbf{F}^+$ for its internal N-WA algebra $(A, \Box_R)$, and write $\mathbf{F}_-$ for its N-Kripke frame base (W, R).

Definition 12. Let $\mathcal{B} = (B, \Box)$ be an N-WA algebra. We define the ultrafilter N-general frame $\mathrm{UF}(\mathcal{B}) = (Ult(B), R_\Box^{Ult}, \widehat{B})$ as follows:

- $Ult(B)$ is the set of ultrafilters of B;
- $\forall u \in Ult(B)$ and $\vec{v} \in Ult(B)^N$, $R_\Box^{Ult}(u, \vec{v})$ iff for any $a \in \bigcap_{i\in[1,N]} \vec{v}[i]$, $u \ni \Diamond a$;
- $\widehat{B} = \{\widehat{b} \mid b \in B\}$ where $\widehat{b} = \{u \in Ult(B) \mid u \ni b\}$.

Now we have to make sure that $\mathrm{UF}(\mathcal{B})$ is indeed an N-WA algebra. As usual, we can show that $\mathcal{B}$, in fact, is isomorphic to the internal algebra $\mathrm{UF}(\mathcal{B})^+$.

Proposition 7. *Let $\mathcal{B} = (B, \Box)$ be an N-WA algebra. Then $\mathrm{UF}(\mathcal{B})$ is an N-general frame and $\mathcal{B}$ is isomorphic to $\mathrm{UF}(\mathcal{B})$ by the operation $\widehat{\cdot}$.*

Proof. The only non-routine item to show is that for any $b \in B$, $\widehat{\Diamond b} = \Diamond\widehat{b}$, where the $\Diamond$ on the left-hand side is given by the algebra $\mathcal{B} = (B, \Box)$, and the $\Diamond$ on the right-hand side is defined by $R_\Box^{Ult}$ in the frame $\mathrm{UF}(\mathcal{B})$.

Pick any $u \in \widehat{\Diamond b}$. We want to show that there are $\vec{v} \in Ult(B)^N$ such that $R_\Box^{Ult}(u, \vec{v})$ and $\mathrm{set}(\vec{v}) \subseteq \widehat{b}$. Let $\Diamond^{-1}u = \{a \in B \mid \Diamond a \in u\}$. Unpacking some definitions, we want to show that

$$\exists \vec{v} \in Ult(B)^N, (\bigcap_i \vec{v}[i] \subseteq \Diamond^{-1}u \text{ and } b \in \bigcap_i \vec{v}[i]).$$

Since $u \in \widehat{\Diamond b}$, $b \in \Diamond^{-1}u$, and by the algebraic properties of $(B, \Box)$ and in particular the monotonicity of $\Diamond$, $\Diamond^{-1}u$ is an upward-closed set (upset). Consider all filters of B that are subsets of $\Diamond^{-1}u$. The principal filter $\uparrow b$ generated by b is clearly one of them. The precondition for Zorn's lemma is also satisfied. So let Q be a filter of B that is maximal among those contained in $\Diamond^{-1}u$ and also contains b. Q must be a proper filter since $\Diamond\bot = \bot$ and $\bot \notin \Diamond^{-1}u$. This means Q has at least 1 ultrafilter extension. Now we show that Q has at most N ultrafilter extensions.

Suppose not, and let $v_0, v_1, v_2, \ldots, v_N$ be distinct ultrafilters in $Ult(B)$ that extend Q. It is routine to see that there are $a_0, \ldots, a_N \in B$ such that $a_i \in v_i$ and $a_i \wedge a_j = \bot$ whenever $i \neq j$. Now, for any $q \in Q$, note that $q = \bigwedge_{i<j\in[0,N]}((q \wedge \neg a_i) \vee (q \wedge \neg a_j))$. This means $\Diamond \bigwedge_{i<j\in[0,N]}((q \wedge \neg a_i) \vee (q \wedge \neg a_j))$ is in u. If we write the special axiom of N-WA algebras in terms of $\Diamond$, we see that

$$\Diamond \bigwedge_{i<j\in[0,N]} ((q \wedge \neg a_i) \vee (q \wedge \neg a_j)) \leq \bigvee_{i\in[0,N]} \Diamond(q \wedge \neg a_i).$$

So $\bigvee_{i\in[0,N]} \Diamond(q \wedge \neg a_i) \in u$. Since u is an ultrafilter, there is an $g(q) \in [0, N]$ such that $(q \wedge \neg a_{g(q)}) \in \Diamond^{-1}(u)$. Since this q is chosen arbitrarily from Q, essentially we found a function $g : Q \to [0, N]$ such that $(q \wedge \neg a_{g(q)}) \in \Diamond^{-1}(u)$. But this shows that we can extend Q non-trivially inside $\Diamond^{-1}u$. Since Q is a filter and the range of g is a finite set, there is $l \in [0, N]$ such that $g^{-1}(l)$ is a dense subset of Q in the sense that for any $a \in Q$, there is $b \in Q$ such that $b \leq a$ and $g(b) = l$. But then

$\{q \wedge \neg a_l \mid q \in Q\}$ is a subset of $\Diamond^{-1}u$ and can be extended to a filter $Q' \subseteq \Diamond^{-1}u$ by merely taking its upset closure since itself is downward directed. Clearly, $\neg a_l$ is in Q' but not in Q, since otherwise v_l cannot be an ultrafilter extending Q. So we contradicted the maximality of Q.

Given the preceding argument, there is $n \in [1, N]$ such that all the ultrafilters extending Q can be listed as $v_1, \dots, v_n$. Since these are *all* the possible extensions, Q must be equal to $\bigcap\{v_1, \dots, v_n\}$. Thus, let $\vec{v} \in Ult(W)^N$ enumerate these v_i's with possible repetition, and we have that $\bigcap_i \vec{v}[i] = Q \subseteq \Diamond^{-1}u$ and $b \in Q = \bigcap_i \vec{v}[i]$.

Now pick any $u \in \Diamond\widehat{b}$. This means there is $\vec{v} \in Ult(B)^N$ such that $\bigcap_i \vec{v}[i] \subseteq \Diamond^{-1}u$ and $b \in \bigcap_i \vec{v}[i]$. Connecting these two claims, $b \in \Diamond^{-1}u$ and thus $u \in \widehat{\Diamond b}$. $\Box$

Now we observe that $\mathrm{UF}(\mathcal{B})$ is saturated in the right way.

Proposition 8. *For any N-WA algebra $\mathcal{B} = (B, \Box)$, $\mathrm{UF}(\mathcal{B}) = (Ult(B), R_\Box^{Ult}, \widehat{B})$ is such that for any $u \in Ult(B)$ and any filter S of $\widehat{B}$, if $u \in \Diamond X$ for any $X \in S$, then $u \in \Diamond \bigcap S$. Thus, if a model $\mathcal{M} = (Ult(B), R_\Box^{Ult}, V)$ is a model such that for any $p \in \mathrm{Prop}$, $V(p) \in \widehat{B}$, then $\mathcal{M}$ is WA-saturated.*

Proof. Let $Q = \{b \in B \mid \widehat{b} \in S\}$. Since for any $X \in S, u \in \Diamond X$, by reversing the isomorphism $\widehat{\cdot}$, for any $b \in Q$, $b \in \Diamond^{-1}u$. Observe also that Q is a filter in B. So Q is a filter contained in $\Diamond^{-1}u$. By repeating the proof for Proposition 7, we see that we can extend Q to a maximal filter Q' among those contained in $\Diamond^{-1}u$ and find $\vec{v} \in Ult(B)^N$ such that (1) $\vec{v}$ enumerates with possible repetition all ultrafilters extending Q' and (2) $R_\Box^{Ult}(u, \vec{v})$. Then for any $i \in [1, N]$, $\vec{v}[i]$ is an ultrafilter extending Q. This means for any $q \in Q$, any $i \in [1, N]$, $\vec{v}[i] \in \widehat{q}$. In other words, $\mathrm{set}(\vec{v}) \in \bigcap S$ as S is just $\{\widehat{q} \mid q \in Q\}$. Thus, $u \in \Diamond \bigcap S$. $\Box$

Thus, we arrive at the right definition of ultrafilter extension:

Definition 13. Let $F = (W, R)$ be an N-Kripke frame. Then define $\mathrm{ue}(F)$ to be $\mathrm{UF}(\mathrm{cmp}(F))_- = (Ult(\wp(W)), R_{\Box_R}^{Ult})$. Let $\mathcal{M} = (W, R, V)$ be an N-Kripke model. Then define $\mathrm{ue}(\mathcal{M}) = (Ult(\wp(W)), R_{\Box_R}^{Ult}, \widehat{V})$ where for any $p \in \mathrm{Prop}$, $\widehat{V}(p) = \widehat{V(p)}$.

5 A Goldblatt-Thomason Theorem for WAML

In this section, we apply our ultrafilter/canonical extension construction to show the analogue of Goldblatt-Thomason theorem for weakly aggregative modal logic. As is well known, the most efficient way to prove this is the algebraic method. Hence we assume a basic understanding of the algebraic treatment of modal logic, including the following immediate consequence of the classic Birkhoff theorem.

Proposition 9 (Birkhoff)**.** *Let $\mathcal{K}$ be a class of N-WA algebras and* L *the set of formulas valid on every element of $\mathcal{K}$. Then, an N-WA algebra $\mathcal{B}$ validates* L *iff $\mathcal{B}$ is a homomorphic image of a subalgebra of a product of some algebras in $\mathcal{K}$.*

The following two facts on the duality functor UF are needed.

Proposition 10. *Let $\mathcal{B}_1 = (B_1, \Box_1)$ and $\mathcal{B}_2 = (B_2, \Box_2)$ be N-WA algebras, and f an homomorphism from $\mathcal{B}_1$ to $\mathcal{B}_2$. Then $\widehat{f} : Ult(B_2) \to Ult(B_1)$ defined by $\widehat{f}(u_2) = \{b \in B_1 \mid f(b) \in u_2\}$ is a continuous p-morphism from* $\mathrm{UF}(\mathcal{B}_2) = (Ult(B_2), R_2^{Ult}, \widehat{B_2})$ *to* $\mathrm{UF}(\mathcal{B}_1) = (Ult(B_1), R_1^{Ult}, \widehat{B_1})$ *in the sense that*

- *$\widehat{f}$ is a p-morphism from* $\mathrm{UF}(\mathcal{B}_2)_- = (Ult(B_2), R_2^{Ult})$ *to* $\mathrm{UF}(\mathcal{B}_1)_- = (Ult(B_1), R_1^{Ult})$*;*
- *for any $S \in \widehat{B_1}$, $\widehat{f}^{-1}[S] \in \widehat{B_2}$.*

Proof. It is routine to check that, with $\bar{\cdot}$ denoting the inverse of $\widehat{\cdot}$, $\widehat{f}^{-1}[S] = \widehat{f(\overline{S})}$, or equivalently, for any $b \in B_1$, $\widehat{f}^{-1}[\widehat{b}] = \widehat{f(b)}$. So the second point is verified.

To check that $\widehat{f}$ is a p-morphism, first take any $u \in Ult(B_2)$ and $\vec{v} \in Ult(B_2)^N$ such that $R_2^{Ult}(u, \vec{v})$. This means that $\bigcap_i \vec{v}[i] \subseteq \Diamond_2^{-1} u$. Let $\widehat{f}\vec{v}$ be the N-tuple where $\widehat{f}\vec{v}[i] = \widehat{f}(\vec{v}[i])$. Now we note that $\bigcap_i \widehat{f}\vec{v}[i] \subseteq \Diamond_1^{-1} \widehat{f}(u)$. This is because for any $b \in B_1$ such that $b \in \widehat{f}\vec{v}[i]$ for all i, $f(b) \in \vec{v}[i]$ for all i, meaning that $f(b) \in \bigcap_i \vec{v}[i]$. So $f(b) \in \Diamond_2^{-1} u$ and thus $\Diamond_2 f(b) \in u$. Since f is a homomorphism, $f(\Diamond_1 b) \in u$ and thus $b \in \Diamond_1^{-1} \widehat{f}(u)$. By definition of R_1^{Ult}, we have $R_1^{Ult}(\widehat{f}(u), \widehat{f}\vec{v})$, and this is enough for the forward direction requirement of p-morphism.

Now suppose there is $\vec{w} \in Ult(B_1)^N$ such that $R_1^{Ult}(\widehat{f}(u), \vec{w})$. Let $P = \bigcap_i \vec{w}[i]$, which is an intersection of N ultrafilters and thus a filter in B_1. Then let $Q = f[P]$, the f image of P. It is a standard exercise to show that, since f is a homomorphism, Q is a filter in B_2, and an ultrafilter v in B_2 extends Q iff $\widehat{f}(v)$ is an ultrafilter extending P iff $\widehat{f}(v) \in \mathrm{set}(\vec{w})$. Thus, all we need to show is that there is $\vec{v} \in Ult(B_2)$ such that $R_2^{Ult}(u, \vec{v})$ and $Q \subseteq \bigcap_i \vec{v}[i]$. By saturation, it is enough to see that $Q \subseteq \Diamond_2^{-1} u$. But clearly $P \subseteq \Diamond_1^{-1} \widehat{f}(u)$. So for any $b \in P$, $\Diamond_1 b \in \widehat{f}(u)$, $f(\Diamond_1 b) \in u$, $\Diamond_2 f(b) \in u$, $f(b) \in \Diamond_2^{-1} u$. So indeed $Q \subseteq \Diamond_2^{-1} u$. □

Fact 11. Let $\mathcal{K}$ be a set of N-WA frames. Then the complex algebra of the disjoint union of all frames in $\mathcal{K}$ is isomorphic to the product of the complex algebras of the individual elements of $\mathcal{K}$.

Finally, the following connection between saturated ultrapower and ultrafilter extension is needed.

Proposition 12. *Let $F = (W, R)$ be an N-WA frame. Then* $\mathrm{ue}(F) = \mathrm{UF}(\mathrm{cmp}(F))_- = (Ult(\wp(W)), R^{Ult})$ *is a p-morphic image of an ultrapower of F.*

Proof. We extend the natural first-order language for $F = (W, R)$ with new unary predicates $\{P_X \mid X \in \wp(W)\}$ where P_X's interpretation is precisely X. Then, with a suitably chosen ultrafilter, by ultrapower we obtain a countably saturated first-order structure $\overline{F} = (\overline{W}, \overline{R}, \langle\overline{X}\rangle_{X\in\wp(W)})$ (here $\overline{X}$ is the interpretation of P_X).

Define function g on $\overline{W}$ by $g(s) = \{X \in \wp(W) \mid s \in \overline{X}\}$. By Łoś's theorem, $\overline{\cdot}$ is a Boolean homomorphism from $\wp(W)$ to $\wp(\overline{W})$, and thus $g(s) \in Ult(\wp(W))$. Conversely, for any $u \in Ult(\wp(W))$, by countable saturation, $\{P_X(x) \mid X \in u\}$ is a consistent 1-type realized in $\overline{F}$. Thus g is a surjection from $\overline{W}$ to $Ult(\wp(W))$.

Now we show that g is a p-morphism from $(\overline{W}, \overline{R})$ to $(Ult(\wp(W)), R^{Ult})$. Pick any $s \in \overline{W}$. If $\overline{R}(s, \vec{t})$. Let $Q = \bigcap_i g(\vec{t}[i])$. By Łoś's theorem, for any $X \in g(s)$ and $Y \in Q$, $X \cap \Diamond_R Y$ must be non-empty in (W, R), since s in $(\overline{W}, \overline{R}, \langle\overline{X}\rangle_{X\in\wp(X)})$ satisfies

$$P_X(x) \wedge \exists y_1 \ldots \exists y_n(R(x, y_1, \ldots, y_N) \wedge P_Y(y_1) \wedge \cdots \wedge P_Y(y_N))$$

and some w in $(W, R, \langle X\rangle_{X\in\wp(W)})$ must satisfies it as well. But since $g(s)$ is an ultrafilter, this means for any $Y \in Q$, $\Diamond_R Y$ is in fact in $g(s)$. In other words, $Q \subseteq \Diamond_R^{-1} g(s)$. By saturation of $(Ult(\wp(W)), R^{Ult})$, there is $\vec{v} \in Ult(B)^N$ such that $R^{Ult}(g(s), \vec{v})$ and $Q \subseteq \bigcap_i \vec{v}[i]$. Since Q is the intersection of the ultrafilters $g(\vec{t}[i])$ for $i \in [1, N]$, each $\vec{v}[i]$ must equal one of the $g(\vec{t}[j])$. That is, set$(\vec{v}[i]) \subseteq g[set(\vec{t})]$. This checks the forward direction for g being a p-morphism.

For the other direction, suppose $R^{Ult}(g(s), \vec{v})$. Let $Q = \bigcap_i \vec{v}[i]$. Then by the definition of R^{Ult}, $\Diamond_R Y \in g(s)$ for any $Y \in Q$. Consider the N-type with s as a parameter:

$$\tau = \{R(s, y_1, \ldots, y_n)\} \cup \bigcup_{i=1}^{N} \{P_Y(y_i) \mid Y \in Q\}.$$

We claim that every finite subset δ of τ is satisfiable in $\overline{F}$. Since Q is a filter, it is enough to focus on δ of the form, with some $Y \in Q$,

$$\{R(s, y_1, \ldots, y_n), P_Y(y_1), P_Y(y_2), \ldots, P_Y(y_N)\}.$$

This is satisfiable in $\overline{F}$ iff $\exists y_1 \ldots y_N(R(s, y_1, \ldots, y_N) \wedge \bigwedge_i P_Y(y_i))$ is true in $\overline{F}$. But note that in F, the following sentence is true merely by the interpretation of $P_{\Diamond_R Y}$ and P_Y:

$$\forall x(P_{\Diamond_R Y}(x) \leftrightarrow \exists y_1 \ldots y_N(R(x, y_1, \ldots, y_N) \wedge \bigwedge_i P_Y(y_i))).$$

By Łoś's theorem, the same is true in $\overline{F}$. Since $\Diamond_R Y \in g(s)$ by assumption, $P_{\Diamond_R Y}(s)$ is true in $\overline{F}$, and hence δ is satisfiable. So τ is finitely satisfiable, and by saturation satisfiable, say by $\vec{t}[1], \ldots, \vec{t}[N]$. Then $\overline{R}(s, \vec{t})$ and for each $i \in [1, N]$, $g(\vec{t}[i])$ is an ultrafilter extending Q, which means $g(\vec{t}[i])$ must be one of the $\vec{v}[i]$'s. In other words, set$(\vec{t}) \subseteq g^{-1}[set(\vec{v})]$. This concludes the proof that g is a p-morphism. □

Now we are ready for the Goldblatt-Thomason theorem.

Theorem 13. *Let $\mathcal{K}$ be a class of N-WA frames that is closed under ultrapower, disjoint union, generated subframes, and p-morphic image, and reflects ultrafilter extension in the sense that if* $\mathrm{ue}(F) \in \mathcal{K}$ *then* $F \in \mathcal{K}$*. Then* $\mathcal{K}$ *is modally definable: let* L *be the set of formulas valid on each frame in* $\mathcal{K}$*, then any* N*-WA frame* F *that validates* L *is already in* $\mathcal{K}$*.*

Proof. Let $\mathrm{cmp}(\mathcal{K})$ be the $\{\mathrm{cmp}(tF) \mid F \in \mathcal{K}\}$. Then by Birkhoff's theorem, for any N-WA frame F, if F validates L, then $\mathrm{cmp}(F)$ is a homomorphic image of a subalgebra of a product of some algebras in $\mathrm{cmp}(\mathcal{K})$. Since $\mathcal{K}$ is closed under disjoint union, $\mathrm{cmp}(F)$ is a homomorphic image of a subalgebra of some element in $\mathrm{cmp}(\mathcal{K})$. More precisely, there is an N-WA algebra $\mathcal{B}$ and a $H \in \mathcal{K}$ and f, g such that f is a surjective homomorphism from $\mathcal{B}$ to $\mathrm{cmp}(F)$ and g is an injective homomorphism from $\mathcal{B}$ to $\mathrm{cmp}(H)$. Dualizing everything by UF, we see that $\widehat{f}$ is an injective p-morphism from $\mathrm{ue}(F) = \mathrm{UF}(\mathrm{cmp}(F))_-$ to $\mathrm{UF}(\mathcal{B})_-$ and $\widehat{g}$ is a surjective p-morphism from $\mathrm{ue}(H) = \mathrm{UF}(\mathrm{cmp}(H))_-$ to $\mathrm{UF}(\mathcal{B})_-$. In other words, $\mathrm{ue}(F)$ is (isomorphic) to a generated subframe of a p-morphic image of $\mathrm{ue}(H)$. Now $\mathrm{ue}(H) \in \mathcal{K}$ since it is a p-morphic image of an ultrapower of H and $\mathcal{K}$ is closed in these operations. So $\mathrm{ue}(F)$ is also in $\mathcal{K}$. Then $F \in \mathcal{K}$. □

6 Conclusion

We conclude with several prominent unaddressed questions. First, we believe the analogue of Fine's canonicity can be proved without much difficulty. But more interesting would be finding a large fragment of formulas for which automated first-order correspondence and canonicity can be proven. Let $\mathcal{D}(\wedge)$ be the fragment of $\mathcal{L}$ that is generated by $\Diamond$ and $\wedge$ from Prop. It seems clear that any formula of the form $\varphi \to \psi$ where $\varphi \in \mathcal{D}(\wedge)$ and ψ is a positive formula is amenable to minimal valuation technique of computing a first-order correspondence. Then, a general canonicity theorem can be proven. But we further conjecture that the fragment $\mathcal{D}(\wedge)$ can be replaced by $\mathcal{D}(\wedge, \vee)$ where we can also use $\vee$ in addition to $\Diamond$ and $\wedge$.

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弱聚合模态逻辑的饱和模型与超滤展开

丁一峰　　刘佶鑫

摘　要

弱聚合模态逻辑（WAML）是一类弱于正规模态逻辑系统 K 的逻辑。这类逻辑在基于多元关系的克里普克框架上的语义定义如下：$\Box\varphi$ 在一个可能世界上为真当且仅当该世界的后继序组中有某个世界使得 φ 为真。本文研究了 WAML 相对于这类语义学的饱和模型和超滤展开，并应用两者证明了 WAML 的 Goldblatt-Thomason 定理。

丁一峰　北京大学哲学系
yf.ding@pku.edu.cn
刘佶鑫　四川大学哲学系
xuetian997@scu.edu.cn