Studies in Logic, Vol. 17, No. 6 (2024): 59–75
PII: 1674-3202(2024)-06-0059-17

Stable Domination and Generic Stability of Linear Algebraic Groups over $\mathbb{C}[[t]]$*

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Abstract. $\mathbb{C}((t))$ is the formal Laurent series over the field $\mathbb{C}$ of complex numbers. It is a henselian valued field, and its valuation ring, denoted by $\mathbb{C}[[t]]$, is the formal power series over $\mathbb{C}$. Let K be any model of $\text{Th}(\mathbb{C}((t)))$ with $\mathcal{O}_K$ its valuation ring and $\boldsymbol{k}$ its residue field. Then $\boldsymbol{k}$ is algebraically closed and $\mathcal{O}_K$ is elemenatry equivalent to $\mathbb{C}[[t]]$.
We first describe the definable subsets of $\mathcal{O}_K$, showing that every definable subset X of $\mathcal{O}_K$ is either res-finite or res-cofinite, i.e., the residue $\text{res}(X)$ of X, is either finite or cofinite in $\boldsymbol{k}$. Moreover, X is res-finite iff $\mathcal{O}_K \backslash X$ is res-cofinite. Applying this result, we show that $\text{GL}(n, \mathcal{O}_K)$, the group of invertible n by n matrices over the valuation ring, is stably dominated via the residue map. As a consequence, we conclude that $\text{GL}(n, \mathcal{O}_K)$ is generically stable, generalizing Y. Halevi's result, where K is an algebraically closed valued field.

1 Introduction

The notion of generically stable types was introduced by Hrushovski and Pillay to describe the "stable-like" behavior in NIP environment. ([11]) Briefly, a type over a monster model $\mathbb{M}$, called a global type, is *generically stable over a small submodel* M if it is finitely satisfiable in and definable over M. A global type is *generically stable* if it is finitely satisfiable in and definable over some small submodel. The theory of algebraically closed valued fields, denoted by ACVF, is considered a typical "stable-like" NIP theory, since the non-trivial generically stable types exist. As a contrast, for the "purely unstable" NIP theories, say p-adically closed fields (pCF), there is no non-trivial generically stable type.

In this paper, we study the structures which are elementarily equivalent to $\mathbb{C}((t))$, the field of formal Laurent series over the complex numbers. It is natural to consider $\text{Th}(\mathbb{C}((t)))$ as a mixture of the "stable-like" theory ACVF and the "purely unstable" theory pCF. To see this, let $K \models \text{Th}(\mathbb{C}((t)))$, $K_1 \models$ACVF and $K_2 \models p$CF, with residue fields $\boldsymbol{k}$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$; value groups Γ, Γ_1 and Γ_2, respectively. Then $\boldsymbol{k} \equiv \boldsymbol{k}_1$ and $\Gamma \equiv \Gamma_2$. On the other side, according to Ax-Kochen-Ershov ([1, 7]), the theory of

Received 2023-09-04

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*The research is supported by The National Social Science Fund of China (Grant No. 20CZX050).

a henselian valued field is completely determined by the theories of its residue field and its value group if its residue field has characteristic 0. Let $K \equiv \mathbb{C}((t))$. We study the definable subsets of $\mathcal{O}_K$, where $\mathcal{O}_K$ is the valuation ring of K. Our first result is:

Theorem 1. *Let* $\mathrm{res} : \mathcal{O}_K \to \boldsymbol{k}$ *be the residue map,* $X \subseteq \mathcal{O}_K$ *a definable set. Then there exists a finite* $Z \subseteq \boldsymbol{k}$ *such that either* $X \subseteq \mathrm{res}^{-1}(Z)$ *or* $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq X$.

Let T be an NIP theory, $\mathbb{M} \models T$ a monster model. Recall from [9] that a definable set $D \subseteq \mathbb{M}^k$ is *stably embedded* if for any formula $\phi(\bar{x}, \bar{y})$, there is a formula $\psi(\bar{x}, \bar{z})$ such that for all tuple $\bar{a}$ there is tuple $\bar{d}$ from D such that

$$\{\bar{x} \in D^r \,|\, \mathbb{M} \models \phi(\bar{x}, \bar{a})\} = \{\bar{x} \in D^r \,|\, \mathbb{M} \models \psi(x, \bar{d})\}$$

Recall from [14] that a definable $D \subseteq \mathbb{M}^k$ is *stable* if there is no formula $\phi(\bar{x}, \bar{y})$ and a sequence of tuples $(\bar{a}_i, \bar{b}_i)_{i \in \omega}$ with $\bar{a}_i \in D^r$ for some r, such that

$$\mathbb{M} \models \phi(\bar{a}_i, \bar{b}_j) \iff i \leq j$$

Let $A \subseteq \mathbb{M}$ and $p \in S_n(\mathbb{M})$ an A-definable global type. Let q be another A-definable type and f an A-definable function. The type p is *dominated by q via f* if all $A \subseteq B$,

$$\bar{a} \models p|B \iff \bar{a} \models p|A \text{ and } f(\bar{a}) \models q|B,$$

We say p is *stably dominated* if there exists a stable, stably embedded definable set D such that q is concentrated on D and p is dominated by q via f.

Let $G \subseteq \mathbb{M}^n$ be a definable group, and $S_G(\mathbb{M})$ the space of all complete types over $\mathbb{M}$ which concentrate on G. A global type $p \in S_G(\mathbb{M})$ is called *G-generic* if p has a bounded orbit under the action of G, namely, $G \cdot p = \{g \cdot p \,|\, g \in G\}$ has cardinality $< |\mathbb{M}|$.

Let $\mathbb{K}$ be a monster model of a valued field, with $\mathcal{O}_{\mathbb{K}}$ its valuation ring and $\boldsymbol{k}_{\mathbb{K}}$ its residue field. We now consider $\mathbb{K}$ as a two sorted structure $(\mathbb{K}, \boldsymbol{k}_{\mathbb{K}})$, then the residue map $\mathrm{res} : \mathcal{O}_{\mathbb{K}} \to \boldsymbol{k}_{\mathbb{K}}$ is a $\varnothing$-definable function. Now we assume that $\mathbb{K} \equiv \mathbb{C}((t))$, then the residue sort $\boldsymbol{k}_{\mathbb{K}}$ is stable and stably embedded since it is an algebraically closed field. Let $\mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ be the group of invertible n by n matrices over $\mathcal{O}_{\mathbb{K}}$. Then we get stable domination of these groups:

Theorem 2. *Let G be* $\mathcal{O}_{\mathbb{K}}$ *or* $\mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$*, then*

(i) *G has a unique global G-generic type* $p_{G,\mathbb{K}} \in S_G(\mathbb{K})$.
(ii) $p_{G,\mathbb{K}}$ *is dominated by* $q = \mathrm{res}(p_{G,\mathbb{K}})$ *via the residue map, where* $\mathrm{res}(p_{G,\mathbb{K}})$ *is the image of* $p_{G,\mathbb{K}}$ *under the residue map. In fact, we have*

$$a \models p_{G,\mathbb{K}} \iff \mathrm{res}(a) \models q.$$

Remark 1. Since $p_{G,\mathbb{K}}$ is the unique global G-generic type, it is G-invariant. As a consequence, $G = G^{00}$.

Recall from [11] that a definable group G is *generically stable* if there is a generically stable type $p \in S_G(\mathbb{M})$ which is G-generic.

Let $\mathbb{K}$ be a monster model of ACVF, Y. Halevi showed that $\mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ is generically stable. ([8], Example 5.1.1) Applying the stable domination, we generalize Y. Halevi's result to the case where $\mathbb{K}$ is a monster model of $\mathrm{Th}(\mathbb{C}((t)))$:

Theorem 3. *Let G be $\mathcal{O}_{\mathbb{K}}$ or $\mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$, then G is generically stable, witnessed by $p_{G,\mathbb{K}}$.*

Applying the following Fact:

Fact 4 ([12], Corollary 4.5)**.** Let G be an algebraic group, N an algebraic subgroup. Let H be a definable subgroup of G in an algebraically closed valued field, with H generically stable. Then $H \cap N$ is generically stable.

Y. Halevi was able to show that:

Fact 5 ([8], Corollary 5.1.2)**.** Let $\mathbb{K}$ be a monster model of ACVF. If N is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{K})$, then $N \cap \mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ is generically stable.

It is reasonable to arise a question that whether or not Fact 5 holds in the case where $\mathbb{K} \models \mathrm{Th}(\mathbb{C}((t)))$.

The paper is organized as follows. For the rest of this section, we recall some basic facts around valued fields and $\mathbb{C}((t))$ and introduce notations we use. In section 2, we prove that every definable subset of the valuation ring $\mathcal{O}_K$ is either res-finite or res-cofinite. In section 3, we give a generically stable type $p_{trans,\mathbb{K}}$, which witnesses the generic stability of $\mathcal{O}_{\mathbb{K}}$ and $\mathbb{U}$. From this onwards, in section 4, we show that for every n, $\mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$ has a unique $\mathrm{GL}(n, \mathcal{O}_{\mathbb{K}})$-generic and generically stable type that is dominated by its image under the residue map.

1.1 Preliminaries

Let K be a field, $(\Gamma, >)$ an ordered abelian group, and $v : K \to \Gamma \cup \{\infty\}$, where $\infty > \Gamma$. We say (K, v) or K is a valued field, if v is an onto map satisfying the following for all $x, y \in K$:

(i) $v(x) = \infty$ iff $x = 0$.
(ii) $v(x + y) \geq \min(v(x), v(y))$.
(iii) $v(xy) = v(x) + v(y)$.

We call v its valuation map and Γ its value group. We denote $\mathcal{O}_K := \{x \in K | \, v(x) \geq 0\}$ the valuation ring of K, $\mathfrak{m}_K := \{x \in K | \, v(x) > 0\}$ the unique maximal ideal of $\mathcal{O}_K$, and $\boldsymbol{k} = \mathcal{O}_K/\mathfrak{m}_K$ the residue field of K. The residue map is the natural projection

$$\mathrm{res} : \mathcal{O}_K \to \boldsymbol{k}, \;\; x \mapsto x/\mathfrak{m}_K.$$

For each $x \in K$ we call $v(x)$ the valuation of x. For each $x \in \mathcal{O}_K$ we call $\operatorname{res}(x)$ the residue of x. If $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathcal{O}_K[x]$, then by $\operatorname{res}(f)$ we mean the polynomial

$$\operatorname{res}(a_n)x^n + \cdots + \operatorname{res}(a_1)x + \operatorname{res}(a_0) \in \boldsymbol{k}[x].$$

We call $\mathcal{O}_K$ henselian, if for any $f \in \mathcal{O}_K[x]$ and $\alpha \in \mathcal{O}_K$ satisfying

$$f(\alpha) \in \mathfrak{m}_K \text{ and } f'(\alpha) \notin \mathfrak{m}_K,$$

there exists $a \in \mathcal{O}_K$ such that $f(a) = 0$ and $a \equiv \alpha \; mod \; \mathfrak{m}_K$, where $f'(x)$ is the derivative of $f(x)$. We say K is a henselian valued field if $\mathcal{O}_K$ is henselian.

For any $a \in \mathcal{O}_K$ and $f(x) \in \mathcal{O}_K[x]$, $\operatorname{res}(f(a)) = \operatorname{res}(f)(\operatorname{res}(a))$. For any subset X of $\mathcal{O}_K$, by $\operatorname{res}(X)$ we mean the set $\{\operatorname{res}(a) |\; a \in X\}$. It is easy to see that $\operatorname{res}(A \cup B) = \operatorname{res}(A) \cup \operatorname{res}(B)$ and $\operatorname{res}(A \cap B) \subseteq \operatorname{res}(A) \cap \operatorname{res}(B)$ for any $A, B \subseteq \mathcal{O}_K$.

Let $K^* = K \backslash \{0\}$ be the multiplicative group. Then the set of n-th powers $P_n(K^*) = \{a^n |\; a \in K^*\}$ is a subgroup of K^*, which is definable in the language of rings.

From now on, K will denote a henselian valued field with an algebraically closed residue field of characteristic 0, i.e. $\boldsymbol{k}$ is algebraically closed of characteristic 0.

Lemma 1. *For any $c \in K$, if $v(c) = 0$, then $c \in P_n(K^*)$ for any $n \in \mathbb{N}^{>0}$.*

Proof. We consider the polynomial $f(x) = x^n - c$ where $c \in K$ and $v(c) = 0$. As $\boldsymbol{k}$ is algebraically closed, and $\operatorname{res}(c)$ is nonzero, $x^n - \operatorname{res}(c)$ has a nonzero root in $\boldsymbol{k}$. So there is $b \in K$ such that $v(b) = 0$ and $\operatorname{res}(f(b)) = (\operatorname{res}(b))^n - \operatorname{res}(c) = 0$.

Thus $f(b) = b^n - c \in \mathfrak{m}_K$. Since $\operatorname{char}(\boldsymbol{k}) = 0$, the derivative $f'(b) = nb^{n-1} \notin \mathfrak{m}_K$. Now K is henselian, so $f(x)$ has a root in $b + \mathfrak{m}_K$, and thus $c \in P_n(K^*)$. This completes the proof. □

Corollary 1. *For any $b \in K^*$, $b \in P_n(K^*)$ iff $v(b) \in n\Gamma = \{n\gamma |\; \gamma \in \Gamma\}$.*

Proof. Suppose that $b \in P_n(K^*)$, then there is $a \in K^*$ such that $b = a^n$. So $v(b) = v(a^n) = nv(a) \in n\Gamma$.

Conversely, suppose that $b \in K^*$ such that $v(b) = nv(a)$ for some $a \in K^*$. Then $v(ba^{-n}) = v(b) - nv(a) = 0$. By Lemma 1, we see that $ba^{-n} \in P_n(K^*)$. Since $a^{-n} \in P_n(K^*)$, we have $b \in P_n(K^*)$ as required. □

Fact 6. ([6], Theorem 2.9) The residue field $\boldsymbol{k}$ can be lifted. Namely, there is a subfield E of $\mathcal{O}_K$ such that $\operatorname{res} : E \to \boldsymbol{k}$ is an isomorphism. So we can consider $\boldsymbol{k}$ as a subfield of $\mathcal{O}_K$.

Remark 2. Note that the lift E of $\boldsymbol{k}$ in Fact 6 is not unique. According to the discussion before Theorem 2.9 of [6], Fact 6 has a stronger variant: Let $E_0 \subseteq \mathcal{O}_K$ be a lift of a subfield $\boldsymbol{k}_0$ of $\boldsymbol{k}$, then E_0 can be extended to a lift of $\boldsymbol{k}$.

1.2 Model Theory of $\mathbb{C}((t))$

Let $\mathbb{C}((t))$ be the field of formal Laurent series over the field $\mathbb{C}$ of complex numbers. Elements of $\mathbb{C}((t))$ are of the form $\sum_{i=n}^{\infty} a_i t^i$, where t is a variable, $n \in \mathbb{Z}$, each $a_i \in \mathbb{C}$. $\mathbb{C}((t))$ is a henselian valued field with the valuation map:

$$v : \mathbb{C}((t)) \to \mathbb{Z} \cup \{\infty\}, \sum_{i=n}^{\infty} a_i t^i \mapsto \min\{i \mid i \geq n \text{ and } a_i \neq 0\}.$$

The valuation ring of $\mathbb{C}((t))$ is $\mathbb{C}[[t]] = \{\sum_{i=n}^{\infty} a_i t^i \mid n \in \mathbb{N}\}$, and the residue field of $\mathbb{C}((t))$ is $\mathbb{C}$ (see [6] for details).

Let $L_{ring} = \{0, 1, +, \times\}$ be the language of rings and

$$L_{VR} = L_{ring} \cup \{\,|\,, N\} \cup \{P_n \mid n \in \mathbb{N}^{>0}\}$$

an expansion of L_{ring} for the valuation ring, where the new predicate $x|y$ is interpreted as $v(x) \leq v(y)$, $N(x)$ is interpreted as $v(x) = 1$, and $P_n(x)$ is interpreted as the set of n-th powers. Note that any atomic L_{VR}-formula $\psi(\bar{x})$ in Th($\mathbb{C}((t))$) is equivalent to one of the following four types:

Type (i) $f(\bar{x}) = 0$, where f is a polynomial over $\mathbb{Z}$.
Type (ii) $f(\bar{x})|g(\bar{x})$, where f, g are polynomials over $\mathbb{Z}$. (For convenience, we will write "$v(f(\bar{x})) \leq v(g(\bar{y}))$" for "$f(x)|g(x)$".)
Type (iii) $P_n(f(\bar{x}))$, where f is a polynomial over $\mathbb{Z}$.
Type (iv) $N(f(\bar{x}))$, where f is a polynomial over $\mathbb{Z}$.

The theory Th($\mathbb{C}((t))$) has NIP and quantifier elimination in the language L_{VR} ([5]). So any L_{VR}-formula is equivalent to a Boolean combination of the formulas of **Type (i)-(iv)**.

Remark 3. Let $K \equiv \mathbb{C}((t))$. Since $\mathbb{C}((t)) \models \forall x(N(x) \leftrightarrow (v(x) = v(t)))$, we have $K \models \exists y \forall x(N(x) \leftrightarrow (v(x) = v(y)))$, which means that there is $a \in K$ such that $K \models \forall x(N(x) \leftrightarrow (v(x) = v(a)))$. So any L_{VR}-formula $\psi(\bar{x})$ with parameters from K is equivalent to a Boolean combination of the formulas $\{\phi_i(\bar{x}, \bar{b}_i) \mid i \leq n\}$, where $\phi_i(\bar{x}, \bar{y}_i)$'s are formulas of **Type (i)-(iii)** and $\bar{b}_i$'s are tuples from K.

Remark 4. Let $K \equiv \mathbb{C}((t))$. Then the relation "res$(x) =$ res(y)" on $\mathcal{O}_K$ is definable in the language L_{VR}, in fact, res$(x) =$ res(y) is defined by the formula

$$v(x) \geq 0 \wedge v(y) \geq 0 \wedge v(x - y) > 0.$$

Lemma 2. *Let $K \equiv \mathbb{C}((t))$ and $f(\bar{x}) \in \boldsymbol{k}[\bar{x}]$, where $\bar{x} = (x_1, ..., x_n)$, then the set*

$$\{\bar{a} \in \mathcal{O}_K^n \mid f(\text{res}(\bar{a})) = 0\}$$

is definable in the language L_{VR} with parameters from $\mathcal{O}_K$.

Proof. Let $l : \boldsymbol{k} \to \mathcal{O}_K$ be a lift of the residue field. Then $\text{res}(l(b)) = b$ for all $b \in \boldsymbol{k}$. Let $l(f) \in \mathcal{O}_K[\bar{x}]$ be the lift of f under l. Then

$$\{\bar{a} \in \mathcal{O}_K^n | \ f(\text{res}(\bar{a})) = 0\} = \{\bar{a} \in \mathcal{O}_K^n | \ v(l(f)(\bar{a})) > 0\},$$

which is clearly definable. □

According to Lemma 2, for each $f \in \boldsymbol{k}[\bar{x}]$, it is reasonable to consider "$f(\text{res}(\bar{x})) = 0$" as a formula in the language L_{VR} with parameters from $\mathcal{O}_K$.

1.3 Notations

We use T to denote the complete theory of $\mathbb{C}((t))$ in the language of L_{VR}. Let κ be an arbitrarily large cardinal, and $\mathbb{K}$ a κ-saturated, strongly κ-homogeneous model of T, with valuation ring $\mathcal{O}_{\mathbb{K}}$, residue field $\boldsymbol{k}_{\mathbb{K}}$, and value group $\Gamma_{\mathbb{K}}$. We call an object "small" or "bounded" if it is of cardinality $< \kappa$.

For A a subset of $\mathbb{K}$, an $L_{VR}(A)$-formula is a formula with parameters from A. If $\phi(\bar{x})$ is an $L_{VR}(\mathbb{K})$-formula and $A \subseteq \mathbb{K}$, then $\phi(A)$ is the collection of the realizations of $\phi(\bar{x})$ from A, namely, $\phi(A) = \{\bar{a} \in A^{|\bar{x}|} | \ \mathbb{K} \models \phi(\bar{a})\}$.

From now on, K will denote an elementary small submodel of $\mathbb{K}$. When we speak of a set X definable subset of K, we mean that $X \subseteq K^n$ is defined by some $L_{VR}(K)$-formula. If $X \subseteq K^n$ is a definable subset of K, we use $X(x)$ to denote the formula which defines X, and $S_X(K)$ to denote the space of complete types over K concentrating on X.

When we speak of a K-definable set X, we mean a definable subset of $\mathbb{K}$ defined by an $L_{VR}(K)$-formula. In general, when we speak of a definable object (set, or group) we mean a definable object in $\mathbb{K}$.

Let R be a local ring (or just an integral domain), then $\text{GL}(n, R)$ will denote the group of n^2-tuples $\bar{x}$ from R such that $\det(\bar{x})$ is invertible in R. Clearly, $\text{GL}(n, R)$ is a group definable in R, defined in the language of rings.

$\text{GL}(n, \mathbb{K}) \subseteq \mathbb{K}^{n\times n}$ is the general linear algebraic group over $\mathbb{K}$, consisting of all $n \times n$ invertible matrices over $\mathbb{K}$, defined by the formula "$\det(\bar{x}) \neq 0$". It is easy to see that $\text{GL}(n, \mathcal{O}_{\mathbb{K}})$ is a subgroup of $\text{GL}(n, \mathbb{K})$. Note that $\text{GL}(n, \mathcal{O}_{\mathbb{K}}) \neq \text{GL}(n, \mathbb{K}) \cap \mathcal{O}_{\mathbb{K}}^{n\times n}$ since the latter is NOT a group. Let $\mathbb{U} = \{a \in \mathbb{K} | \ v(a) = 0\}$ be the set of units in $\mathcal{O}_{\mathbb{K}}$, it is easy to see that $\mathbb{U} = \text{GL}(1, \mathcal{O}_{\mathbb{K}})$.

If $X \subseteq \mathbb{K}^n$, then by $\text{res}(X)$, we mean the set $\{\text{res}(\bar{a}) | \ \bar{a} \in X \cap \mathcal{O}_{\mathbb{K}}^n\}$. If $A \subseteq B \subseteq K$ and $p \in S_n(B)$ a complete type over B, then by $p|A$ we mean the restriction $\{\varphi \in p | \ \varphi \text{ is a formula over } A\}$. If $p = \text{tp}(\bar{a}/K)$ and $\bar{a} \in \mathcal{O}_{\mathbb{K}}^n$, then by $\text{res}(p)$ we mean the complete type $\text{tp}(\text{res}(\bar{a})/\boldsymbol{k})$.

Our notations for model theory are standard, and we will assume familiarity with basic notions such as very saturated models (or monster models), partial types, type-definable, finitely satisfiable, etc. We refer reader to [15] as well as [13].

2 Definable Subsets of the Valuation Ring

In this section, we will study the definable subsets of $\mathcal{O}_K$. Let $f(x), g(x) \in K[x]$ be polynomials. Suppose that

$$f(x) = b_0 + \cdots + b_n x^n.$$

Let $e_f \in \{b_0, \cdots, b_n\}$ such that

$$v(e_f) = \min\{v(b_i) \mid i = 0, \cdots, n\}.$$

Let $f^* = f/e_f$, then $f^* \in \mathcal{O}_K[x]$. It is easy to see that f and f^* have the same zeros.

Remark 5. Let $a \in \mathcal{O}_K$ such that $f(a) \in \mathcal{O}_K$. Since res is a homomorphism from $\mathcal{O}_K$ to $\boldsymbol{k}$, we see that $f(a) \neq 0$ whenever $\mathrm{res}(f(a)) \neq 0$. We conclude directly that if $X = \{a \in \mathcal{O}_K \mid f(a) = 0\}$ and $Z = \{u \in \boldsymbol{k} \mid \mathrm{res}(f)(u) = 0\}$, then $X \subseteq \mathrm{res}^{-1}(Z)$ and $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq \mathcal{O}_K \backslash X$.

Lemma 3. *Suppose that $a \in \mathcal{O}_K$. If $\mathrm{res}(f^*(a)) \neq 0$, then $v(f(a)) = v(e_f)$.*

Proof. Clearly, $v(f(a)) = v(e_f) + v(f^*(a))$. Since $\mathrm{res}(f^*(a)) \neq 0$, we have $v(f^*(a)) = 0$. So $v(f(a)) = v(e_f)$ as required. □

Corollary 2. *Let $g \in K[x]$. If $X = \{a \in \mathcal{O}_K \mid v(f(a)) \leq v(g(a))\}$ then there is a finite set $Z \subseteq \boldsymbol{k}$ such that either $X \subseteq \mathrm{res}^{-1}(Z)$ or $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq X$.*

Proof. Suppose that $g(x) = c_0 + \cdots + c_m x^m$. Take $e_g \in \{c_0, \cdots, c_m\}$ such that $v(e_g) = \min\{v(c_j) \mid j = 0, \cdots, m\}$. Let $g^* = g/e_g$ and

$$Z = \{c \in \boldsymbol{k} \mid (\mathrm{res}(f^*)(c) = 0) \vee (\mathrm{res}(g^*)(c) = 0)\}.$$

Then Z is finite. By Lemma 3, if $v(e_f) \leq v(e_g)$, then $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq X$. If $v(e_f) > v(e_g)$, then $X \subseteq \mathrm{res}^{-1}(Z)$. □

Lemma 4. *If $X = \{a \in \mathcal{O}_K \mid K \models P_n(f(a))\}$, then there is a finite set $Z \subseteq \boldsymbol{k}$ such that either $X \subseteq \mathrm{res}^{-1}(Z)$ or $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq X$.*

Proof. Let $Z = \{c \in \boldsymbol{k} \mid \mathrm{res}(f^*)(c) = 0\}$. By Lemma 3, $v(f(a)) = v(e_f)$ whenever $a \notin \mathrm{res}^{-1}(Z)$. By Corollary 1, if $v(e_f) \in n\Gamma$, then $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq X$, otherwise, $X \subseteq \mathrm{res}^{-1}(Z)$. □

Let $Z \subseteq \boldsymbol{k}$ and $X \subseteq \mathcal{O}_K$, then it is easy to see that $X \subseteq \mathrm{res}^{-1}(Z)$ implies $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq (\mathcal{O}_K \backslash X)$, and $\mathrm{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq X$ implies $\mathcal{O}_K \backslash X \subseteq \mathrm{res}^{-1}(Z)$. Summarising Remark 5, Corollary 2, and Lemma 4, we conclude that

Corollary 3. *Suppose that X is either $\varphi(K, \bar{a})$ or $\neg\varphi(K, \bar{a})$, where $\bar{a}$ is a tuple from K and $\varphi(x, \bar{y})$ is an L_{VR}-formula of* ***Type (i)-(iii)****. Then there is a finite set $Z \subseteq \boldsymbol{k}$ such that either $X \cap \mathcal{O}_K \subseteq \operatorname{res}^{-1}(Z)$ or $\operatorname{res}^{-1}(\boldsymbol{k}\backslash Z) \subseteq X \cap \mathcal{O}_K$. In particular, $\operatorname{res}(X \cap \mathcal{O}_K)$ is finite iff $\operatorname{res}(\mathcal{O}_K \backslash X)$ is cofinite in $\boldsymbol{k}$.*

Theorem 7. *Let $Y \subseteq \mathcal{O}_K$ be definable, there is a finite set $Z \subseteq \boldsymbol{k}$ such that $\operatorname{res}^{-1}(\boldsymbol{k}\backslash Z) \subseteq Y$ if $\operatorname{res}(Y)$ is infinite. In particular, $\operatorname{res}(Y)$ is cofinite in $\boldsymbol{k}$ iff $\operatorname{res}(\mathcal{O}_K \backslash Y)$ is finite.*

Proof. By quantifier elimination and Remark 3, we may assume that Y is a finite boolean combination of sets given in Corollary 3. Suppose that $Y = \bigcap_{i=1}^{r} \bigcup_{j=1}^{s} Y_{i,j}$, where each $Y_{i,j}$ is either either $\varphi_{i,j}(K, \bar{a}_{i,j})$ or $\neg\varphi_{i,j}(K, \bar{a}_{i,j})$, where each $\bar{a}_{i,j}$ is a tuple from K and each $\varphi_{i,j}(x, \bar{y}_{i,j})$ is an L_{VR}-formula of **Type (i)-(iii)**. By Corollary 3, for each $Y_{i,j}$, there is a finite set $Z_{i,j} \subseteq \boldsymbol{k}$ such that either $Y_{i,j} \subseteq \operatorname{res}^{-1}(Z_{i,j})$ or $\operatorname{res}^{-1}(\boldsymbol{k}\backslash Z_{i,j}) \subseteq \mathcal{O}_K \backslash Y_{i,j}$. Let $Z = \bigcup_{i \leq r, j \leq s} Z_{i,j}$, then we have that either $Y_{i,j} \subseteq \operatorname{res}^{-1}(Z)$ or $\operatorname{res}^{-1}(\boldsymbol{k}\backslash Z) \subseteq \mathcal{O}_K \backslash Y_{i,j}$ for each $i \leq r$ and $j \leq s$. Clearly, $\operatorname{res}(Y) \subseteq \bigcap_{i=1}^{r} \bigcup_{j=1}^{s} \operatorname{res}(Y_{i,j})$.

If $\operatorname{res}(Y)$ is infinite, then for each $i \leq r$ there is $j(i) \leq s$ such that $\operatorname{res}(Y_{i,j(i)})$ is infinite, hence is cofinite in $\boldsymbol{k}$. Thus $\operatorname{res}^{-1}(\boldsymbol{k}\backslash Z)$ is contained in $Y_{i,j(i)}$ for each $i \leq r$, we conclude that $\operatorname{res}^{-1}(\boldsymbol{k}\backslash Z)$ is contained in Y. □

Definition 1. We call a definable subset X of K *res-finite* (resp. *res-cofinite*) if $\operatorname{res}(X \cap \mathcal{O}_K)$ is finite (resp. cofinite). We call an $L_{VR}(K)$ formula $\varphi(x)$ *res-finite* (resp. *res-cofinite*) if $\varphi(K)$ is res-finite (resp. res-cofinite).

Let $\psi(x)$ be an $L_{VR}(K)$-fromula. By Theorem 7, $\psi(K)$ res-finite iff $\neg\psi(K)$ is res-cofinite.

Corollary 4. *Let X be a definable subset of $\mathcal{O}_K$, K_0 a subfield of K. If X is res-cofinite, then $X \cap K_0 \neq \varnothing$.*

Proof. By Theorem 7, there is a cofinite subset $Z^* \subseteq \boldsymbol{k}$ such that $\operatorname{res}^{-1}(Z^*) \subseteq X$. Let $\boldsymbol{k}_0$ be the residue field of K_0. Then $\boldsymbol{k}_0$ is a subfield of $\boldsymbol{k}$. Clearly, $Z^* \cap \boldsymbol{k}_0$ is nonempty as $\boldsymbol{k}_0$ is infinite. Take any $u \in Z^* \cap \boldsymbol{k}_0$ and $\tilde{u} \in \mathcal{O}_{K_0}$ such that $\operatorname{res}(\tilde{u}) = u$, then $\tilde{u} \in X \cap K_0$. □

3 Stable Domination and Generic Stability of $\mathcal{O}_{\mathbb{K}}$ and $\mathbb{U}$

Recall that by a definable group in $\mathbb{K}$, we mean a definable set G and a definable map $\cdot : G \times G \to G$ such that $(G, \cdot)$ is a group. We call an $L_{VR}(\mathbb{K})$-formula $\varphi(\bar{x})$ a G-formula if $\mathbb{K} \models \forall \bar{x}(\varphi(\bar{x}) \to G(\bar{x}))$.

In this section, K will also be an arbitrary elementary submodel of $\mathbb{K}$. Suppose that G is definable over K. The type space $S_G(K)$ of G over K can be also consider as the space of ultrafilters of the algebra of G-formulas over K. If $g \in G(K)$ and $\varphi(\bar{x})$ is a G-formula, then by $g \cdot \varphi$, we mean the formula $\varphi(g^{-1} \cdot \bar{x})$. It is easy to see that $(g \cdot \varphi)(K) = g \cdot (\varphi(K))$. If $g \in G(K)$ and $p \in S_G(K)$, then $g \cdot p = \{g \cdot \varphi |\, \varphi \in p\}$. It is easy to see that $g \cdot p \in S_G(K)$. Let

$$p_{\text{trans},K}(x) = \{v(x) = 0\} \cup \{h(\text{res}(x)) \neq 0 |\; h(x) \in \boldsymbol{k}[x]\}.$$

We see from Remark 4 that $p_{\text{trans},K}$ is a partial type over K.

Lemma 5. *Let $\varphi(x)$ be an $L_{VR}(K)$-formula, then $p_{\text{trans},K} \models \varphi(x)$ iff it is res-cofinite.*

Proof. Suppose that φ is res-finite. Then $\text{res}(\varphi(K) \cap \mathcal{O}_K) = \{u_1, \cdots, u_n\}$ is finite. Let $h(x) = \prod_{i=1}^n (x - u_i)$, then $h(x) \in \boldsymbol{k}[x]$ and

$$\{a \in \mathcal{O}_K |\; h(\text{res}(a)) \neq 0\} \cap \varphi(K) = \varnothing.$$

Namely, $\big(v(x) = 0) \wedge (h(\text{res}(x)) \neq 0)\big)$ is inconsistent with $\varphi(x)$. Since

$$\big(v(x) = 0) \wedge (h(\text{res}(x)) \neq 0)\big) \in p_{\text{trans},K},$$

we conclude that $p_{\text{trans},K} \not\models \varphi(x)$.

Suppose that φ is res-cofinite. Then there is a finite set $Z = \{u_1, \cdots, u_n\} \subseteq \boldsymbol{k}$ such that $\text{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq \varphi(K)$. Let $h(x) = \prod_{i=1}^n (x - u_i)$, then

$$\{a \in K \big(v(a) = 0) \wedge (h(\text{res}(a)) \neq 0)\} \subseteq \text{res}^{-1}(\boldsymbol{k} \backslash Z) \subseteq \varphi(K).$$

so we have

$$K \models \forall x \Big(\big(v(x) = 0) \wedge (h(\text{res}(x)) \neq 0)\big) \to \varphi(x) \Big),$$

which implies that $p_{\text{trans},K} \models \varphi(x)$. □

By Theorem 7, an $L_{VR}(K)$-formula $\varphi(x)$ is res-finite (resp. res-cofinite) iff $\neg\varphi(x)$ is res-cofinite (resp. res-finite). So we see from Lemma 5 that $p_{\text{trans},K}$ determines a complete type over K, abusing the notation, this complete type is also denoted by $p_{\text{trans},K}$.

The residue field $\boldsymbol{k}$ of K is algebraically closed. It is well-known that the theory of algebraically closed fields has quantifier elimination in the language L_{ring}. So every definable subset of $\boldsymbol{k}$ is finite or cofinite. Let $q_{\text{trans},\boldsymbol{k}} \in S_1(\boldsymbol{k})$ be the unique transcendental type over $\boldsymbol{k}$, namely, $q_{\text{trans},\boldsymbol{k}} = \{f(x) \neq 0 |\; f \in \boldsymbol{k}[x]\}$. It is easy to see that $q_{\text{trans},\boldsymbol{k}}$ is precisely $\text{res}(p_{\text{trans},K})$: for any $\tilde{a} \models p_{\text{trans},K}$, $\text{res}(\tilde{a}) \models q_{\text{trans},\boldsymbol{k}}$.

Theorem 8. *$p_{\text{trans},K}$ is dominated by $q_{\text{trans},\boldsymbol{k}}$ via the residue map.*

Proof. Suppose that $a \models q_{\text{trans},\boldsymbol{k}}$ and $\tilde{a} \in \text{res}^{-1}(a)$. Let $\varphi(x)$ be an $L_{VR}(K)$-formula. By Theorem 7, $\text{res}(\varphi(\mathbb{K}) \cap \mathcal{O}_{\mathbb{K}})$ is a definable subset of $\boldsymbol{k}_{\mathbb{K}}$. Let $\psi(x)$ be an L_{ring}-formula with parameters from $\boldsymbol{k}$ such that $\psi(\boldsymbol{k}_{\mathbb{K}}) = \text{res}(\varphi(\mathbb{K}) \cap \mathcal{O}_{\mathbb{K}})$. Suppose that $\tilde{a} \models \varphi(x)$, then $a \in \psi(\boldsymbol{k}_{\mathbb{K}})$. Since a realizes the transcendental type over $\boldsymbol{k}$, we see that $\psi(\boldsymbol{k})$ is cofinite. We conclude that $\tilde{a}$ realizes every res-cofinite L_{VR}-formula over K, so $\tilde{a}$ realizes $p_{\text{trans},K}$ by Lemma 5. □

Lemma 6. *Let G be either $(\mathcal{O}_{\mathbb{K}}, +)$ or $(\mathbb{U}, \times)$, then $p_{\text{trans},\mathbb{K}}$ is G-invariant, consequently, $p_{\text{trans},\mathbb{K}}$ is a global G-generic type.*

Proof. Let $\mathfrak{G}_a$ and $\mathfrak{G}_m$ be the additive group $(\boldsymbol{k}_{\mathbb{K}}, +)$ and the multiplicative group $(\boldsymbol{k}_{\mathbb{K}} \backslash \{0\}, \times)$ respectively. Since $\boldsymbol{k}_{\mathbb{K}}$ is strongly minimal, $q_{\text{trans},\boldsymbol{k}_{\mathbb{K}}}$ is invariant under both the actions of $\mathfrak{G}_a$ and $\mathfrak{G}_m$.

Note that $\text{res} : \mathcal{O}_{\mathbb{K}} \to \boldsymbol{k}_{\mathbb{K}}$ is a ring homomorphism with $\text{res}(\mathcal{O}_{\mathbb{K}}) = \mathfrak{G}_a$ and $\text{res}(\mathbb{U}) = \mathfrak{G}_m$. We see from stable domination that $p_{\text{trans},\mathbb{K}}$ is G-invariant. □

We now show that $p_{\text{trans},\mathbb{K}} \in S_1(\mathbb{K})$ is a generically stable type.

Lemma 7. *$p_{\text{trans},K}$ is finitely satisfiable in every elementary submodel of K.*

Proof. Let K_0 be an elementary submodel of K and $\varphi(x)$ an $L_{VR}(K)$-formula. Suppose that $\varphi(x) \in p_{\text{trans},K}$. By Lemma 5, $\varphi(x)$ is res-cofinite, and by Corollary 4, $\varphi(K) \cap K_0 \neq \varnothing$. This completes the proof. □

Lemma 8. *$p_{\text{trans},K}$ is definable over $\varnothing$.*

Proof. For each L_{VR} formula $\varphi(x, \bar{y})$, let

$$D_\varphi = \{\bar{b} \in K^{|\bar{y}|} \mid \varphi(x, \bar{b}) \in p_{\text{trans},K}\}.$$

To see the definability of $p_{\text{trans},K}$, we need to show that: For each L_{VR} formula $\varphi(x, \bar{y})$, D_φ is $\varnothing$-definable. By quantifier elimination, we only need to check the formulas of **Type (i)-(iv)**. (Note that Remark 3 does not apply since our parameter set is just $\text{dcl}(\varnothing)$, rather than a model)

Let $f(x, \bar{y}) = g_0(\bar{y}) + g_1(\bar{y})x + \cdots + g_n(\bar{y})x^n$, where $g_0, ..., g_n \in \mathbb{Z}[\bar{y}]$. By Remark 5, we see that

$$\{\bar{b} \in K^{|\bar{y}|} \mid (f(x, \bar{b}) = 0) \in p_{\text{trans},K}\} = \{\bar{b} \in K^{|\bar{y}|} \mid g_0(\bar{y}) = ... = g_n(\bar{b}) = 0\}.$$

Obviously, $\varnothing$-definable. So D_φ is $\varnothing$-definable for any formula $\varphi(x, \bar{y})$ of **Type (i)**.

Suppose that $\varphi(x, \bar{y})$ is of the **Type (ii)**, namely, of the form

$$v(f(x, \bar{y})) \leq v(g(x, \bar{y})),$$

where

$$f(x, \bar{y}) = g_0(\bar{y}) + g_1(\bar{y})x + \cdots + g_n(\bar{y})x^n,$$

and

$$g(x, \bar{z}) = h_0(\bar{y}) + h_1(\bar{y})x + \cdots + h_m(\bar{y})x^m.$$

By Lemma 3, for any $\bar{b} \in K^{|\bar{y}|}$,

$$\varphi(x, \bar{b}) \in p_{0,\text{trans},K} \iff \min\{v(g_i(\bar{b}))|\ i \leq n\} \leq \min\{v(h_j(\bar{b}))|\ j \leq m\}.$$

Clearly,

$$D_\varphi = \{\bar{b} \in K^{|\bar{y}|}|\ \min\{v(g_i(\bar{b}))|\ i \leq n\} \leq \min\{v(h_j(\bar{b}))|\ j \leq m\}\}$$

is definable over $\varnothing$.

Suppose that $\varphi(x, \bar{y})$ is of the **Type (iii)**, namely, of the form $P_m(f(x, \bar{y}))$, where

$$f(x, \bar{y}) = g_0(\bar{y}) + g_1(\bar{y})x + \cdots + g_n(\bar{y})x^n$$

By Corollary 1 and Lemma 3, for each $\bar{b} \in \mathbb{K}^{|\bar{y}|}$, we have that

$$P_n(f(x, \bar{b})) \in p_{0,\text{trans},K} \iff \bigvee_{i=0}^{n} \left(\bigwedge_{j=0}^{n} \big(v(g_i(\bar{b})) \leq v(g_j(\bar{b}))\big) \wedge P_m(g_i(\bar{b})) \right).$$

Clearly,

$$\left\{ \bar{b} \in K^{|\bar{y}|} \mid \bigvee_{i=0}^{n} \left(\bigwedge_{j=0}^{n} \big(v(g_i(\bar{b})) \leq v(g_j(\bar{b}))\big) \wedge P_m(g_i(\bar{b})) \right) \right\}$$

is definable over $\varnothing$.

Suppose that $\varphi(x, \bar{y})$ is of the **Type (iv)**, namely, of the form $N(f(x, \bar{y}))$, where

$$f(x, \bar{y}) = g_0(\bar{y}) + g_1(\bar{y})x + \cdots + g_n(\bar{y})x^n.$$

By Lemma 3, for each $\bar{b} \in \mathbb{K}^{|\bar{y}|}$, we have that

$$N(f(x, \bar{b})) \in p_{0,\text{trans},K} \iff \bigvee_{i=0}^{n} \left(\bigwedge_{j=0}^{n} \big(v(g_i(\bar{b})) \leq v(g_j(\bar{b}))\big) \wedge N(g_i(\bar{b})) \right).$$

Clearly,

$$\left\{ \bar{b} \in K^{|\bar{y}|} \mid \bigvee_{i=0}^{n} \left(\bigwedge_{j=0}^{n} \big(v(g_i(\bar{b})) \leq v(g_j(\bar{b}))\big) \wedge N(g_i(\bar{b})) \right) \right\}$$

is definable over $\varnothing$. □

Remark 6. It is easy to see from the proof of Lemma 8 that: For any elementary extension K_1 of K, p_{trans,K_1} is definable over $\varnothing$, and is the unique heir of $p_{\text{trans},K}$ over K_1.

Summarizing Lemma 7, Lemma 8, and Lemma 6, we have that

Theorem 9. *$(\mathcal{O}_{\mathbb{K}},+)$ and $(\mathbb{U},\times)$ are generically stable, and $p_{\text{trans},\mathbb{K}}$ is a witness.*

Fact 10 (See Lemma 9.7 and Lemma 9.12 of [3])**.** Let $X\subseteq\mathbb{K}^n$ be a definable set, E a type-definable equivalence relation on X, defined over K. If $|X/E|$ is small/bounded, then for every $a,b\in X$, $a/E=b/E$ whenever $\text{tp}(a/K)=\text{tp}(b/K)$.

Let G be a group definable in $\mathbb{K}$, H a subgroup of G, and $A\subseteq\mathbb{K}$ a set of parameters. We call H a type-definable subgroup of G over A if $H\le G$ is defined by a partial type over A. Suppose that $H\le G$ is a type-definable subgroup over K with small/bounded index, namely, $|G/H|<|\mathbb{K}|$, then the above fact says that each $p\in S_G(K)$ determines a coset of H, i.e. , for any $g_1,g_2\in G$, $\text{tp}(g_1/K)=\text{tp}(g_2/K)$ implies that $g_1H=g_2H$. Since T has NIP, by [10], G has the smallest type-definable subgroup of bounded index, written G^{00}, which is type-definable over $\varnothing$ and called the *type-definable connected component* of G. The following fact is a folklore:

Fact 11. Let G be a group definable in $\mathbb{K}$. If there is a global type $p\in S_G(\mathbb{K})$ which is G-invariant, namely, $g\cdot p=p$ for all $g\in G$, then $G=G^{00}$.

Proof. Let H be a type-definable subgroup of G over A of bounded index. Let K_0 be an elementary small submodel of K such that $A\subseteq K_0$ and K_0 meets every coset of H. Let $p_0\in S_G(K_0)$ be the restriction of p to K_0. Then $g\cdot p_0=p_0$ for all $g\in G(K_0)$. Assume that p is contained in some coset of H, say gH. If H is a proper subgroup of G, then there is $g'\in G$ such that $g'gH\neq gH$. Since $g'\cdot p$ is contained in $g'gH$, we see that $p\neq g'\cdot p$. A contradiction. □

We conclude from Lemma 6 and Fact 11 that:

Corollary 5. *Let G be $\mathcal{O}_{\mathbb{K}}$ or $\mathbb{U}$, then $G=G^{00}$.*

4 Stable Domination and Generic Stability of $\text{GL}(n,\mathcal{O}_{\mathbb{K}})$

In this section, we identify an n^2-tuple $\bar{a}\in\mathbb{K}^{n\times n}$ with an $n\times n$ matrix. Recall that

$$\text{GL}(n,\mathcal{O}_{\mathbb{K}})=\{\bar{g}\in\mathcal{O}_{\mathbb{K}}^{n\times n}|\ \det(\bar{g})\in\mathbb{U}\}$$

is the group of invertible $n\times n$ matrices over $\mathcal{O}_{\mathbb{K}}$, and

$$\text{GL}(n,\boldsymbol{k}_{\mathbb{K}})=\{\bar{a}\in\boldsymbol{k}_{\mathbb{K}}^{n\times n}|\ \det(\bar{a})\neq 0\}$$

is the $n\times n$ general linear algebraic group over $\boldsymbol{k}_{\mathbb{K}}$. Let G denote the group $\mathrm{GL}(n,\mathcal{O}_{\mathbb{K}})$ and $\mathfrak{G}$ denote the group $\mathrm{GL}(n,\boldsymbol{k}_{\mathbb{K}})$. Clearly, G is an $\varnothing$-definable group in the home sort, and $\mathfrak{G}$ is an $\varnothing$-definable group in the residue sort. Suppose that $\bar{g}=(g_{i,j})\in\mathcal{O}_{\mathbb{K}}^{n\times n}$, by $\mathrm{res}(\bar{g})$ we mean the matrix $(\mathrm{res}(g_{i,j}))\in\boldsymbol{k}_{\mathbb{K}}^{n\times n}$. It is easy to see that the map:

$$\bar{g}\mapsto\mathrm{res}(\bar{g}), G\to\mathfrak{G}$$

is an onto group homomorphism. We denote this homomorphism by res for convenience.

The following fact is easy to verify.

Fact 12. Let $M_0\prec M_1\prec M_2\prec M_3$ be structures over a language L. Let $b\in M_3$ and $a\in M_2$.

(i) If both $\mathrm{tp}(b/M_2)$ and $\mathrm{tp}(a/M_1)$ are definable over M_0, then $\mathrm{tp}(a,b/M_1)$ is definable over M_0.

(ii) If both $\mathrm{tp}(b/M_2)$ and $\mathrm{tp}(a/M_1)$ are finitely satisfiable in M_0, then $\mathrm{tp}(a,b/M_1)$ is finitely satisfiable in M_0.

Now we consider the small submodel K. Let $K=K_0\prec K_1\prec\cdots\prec K_{n^2}$ be an elementary chain such that $g_i\in K_i$ and $g_i\models p_{\text{trans},K_{i-1}}$ for each $i=1,\cdots,n^2$. Let $\bar{g}^*=(g_1,...,g_{n^2})$. Let $\boldsymbol{k}_i$ be the residue field of K_i for $i=0,...,n^2$. Since $\mathrm{res}(g_i)$ is transcendental over $\boldsymbol{k}_i$, for each $i=1,\cdots,n^2$, the transcendence degree $\mathrm{trdeg}(\mathrm{res}(\bar{g}^*)/\boldsymbol{k})$ (or, equivalently, algebraic dimension $\dim(\mathrm{res}(\bar{g}^*)/\boldsymbol{k})$) of $\mathrm{res}(\bar{g}^*)$ over $\boldsymbol{k}$ is n^2, we see that $\mathrm{res}(\det(\bar{g}^*))=\det(\mathrm{res}(\bar{g}^*))\neq 0$, so $\det(g^*)\in\mathbb{U}$, and thus $\bar{g}^*\in G$. Let $p_{G,K}\in S_G(K)$ be the type realized by $\bar{g}^*$.

Lemma 9. *Let $p_{G,K}$ be as the above, then $p_{G,K}$ is definable over and finitely satisfiable in every small submodel of K.*

Proof. Let $M\prec K$. Then by Lemma 7 and Lemma 8, $\mathrm{tp}(g_i/K_{i-1})=p_{\text{trans},K_{i-1}}$ is definable over and finitely satisfiable in M for each $i=1,\cdots,n^2$. Applying Fact 12 and induction on $i\leq n^2$, we conclude that $p_{G,K}$ is definable over and finitely satisfiable in M. □

Note that since $p_{\text{trans},K}$ has a unique heir over any set $A\supseteq K$, we see that $p_{G,K}$ is realized by any tuple $\bar{h}=(h_1,\cdots,h_{n^2})\in G$ such that $h_1\models p_{\text{trans},K}$, $h_2\models$ the unique heir of $p_{\text{trans},K}$ over $K\cup\{h_1\},\cdots$, and $h_{n^2}\models$ the unique heir/coheir of $p_{\text{trans},K}$ over $K\cup\{h_1,\cdots,h_{n^2-1}\}$.

Let H be a group definable in an ω-stable structure $\mathcal{U}$. We say that $p\in S_H(\mathcal{U})$ is *generic* if the Morley rank of p equals to the Morley rank of H. Note that "generic" coincides with "H-generic" in ω-stable theories. If H is definably-connected, namely, H has no proper definable subgroup of finite index, then $S_H(\mathcal{U})$ contains a unique generic type, and $p \in S_H(\mathcal{U})$ is generic iff p is H-invariant (see Chapter 2 of [2] for details).

Now the residue field $\boldsymbol{k}$ algebraically closed field, hence is ω-stable, and the transcendence degree of a tuple $\bar{a}$ over $\boldsymbol{k}$ coincides with the Morley rand of $\mathrm{tp}(\bar{a}/\boldsymbol{k})$. Let $q_{\mathfrak{G},\boldsymbol{k}} \in S_{\mathfrak{G}}(\boldsymbol{k})$ be a generic type. Since $\mathfrak{G} \subseteq \boldsymbol{k}_{\mathbb{K}}^{n^2}$ is an irreducible algebraic group over $\varnothing$, it is definably-connected. So $q_{\mathfrak{G},\boldsymbol{k}}$ is the unique generic type in $S_{\mathfrak{G}}(\boldsymbol{k})$. Clearly, $q_{\mathfrak{G},\boldsymbol{k}}$ is $\mathfrak{G}(\boldsymbol{k})$-invariant. Moreover, for each $\bar{a} \in \mathfrak{G}$,

$$\bar{a} \models q_{\mathfrak{G},\boldsymbol{k}} \iff \mathrm{trdeg}(\bar{a}/\boldsymbol{k}) = n^2,$$

which implies that $q_{\mathfrak{G},\boldsymbol{k}} = \mathrm{res}(p_{G,K})$, namely, $q_{\mathfrak{G},\boldsymbol{k}}$ is realized by $\mathrm{res}(\bar{g}^*)$ for any $\bar{g}^* \models p_{G,K}$.

Theorem 13. *$p_{G,K}$ is dominated by $q_{\mathfrak{G},\boldsymbol{k}}$ via the residue map. Namely, for any $\bar{a} \in G$,*

$$\bar{a} \models p_{G,K} \iff \mathrm{res}(\bar{a}) \models q_{\mathfrak{G},\boldsymbol{k}}$$

To prove Theorem 13, we need to prove the following Lemmas. Let E be any field, then by E^{alg}, we mean the (field-theoretic) algebraic closure of E. If $E \subseteq K$, we say that E is *algebraically closed in K* if $E^{\mathrm{alg}} \cap K = E$.

Lemma 10. *Suppose that $K_0 \subseteq K$ is algebraically closed in K. If there is $u \in K_0$ such that $v(u) = 1$, then $K_0 \prec K$.*

Proof. Clearly, K_0 is a valued subfield of K with valuation ring $\mathcal{O}_{K_0} = K_0 \cap \mathcal{O}_K$. It is also easy to see that K_0 is henselian since K_0 is algebraically closed in K, and K is henselian. We claim that the residue field $\boldsymbol{k}_0$ of K_0 is algebraically closed, i.e. an elementary substructure of $\boldsymbol{k}$. By Fact 6, $\boldsymbol{k}_0$ has a lift E_0 in $\mathcal{O}_{K_0}$. By Remark 2, E_0 can be extended to a lift E of $\boldsymbol{k}$, so we can consider $\boldsymbol{k}_0$ as a subfield $\boldsymbol{k}$. Since K_0 is algebraically closed in K, we see that $\boldsymbol{k}_0$ is algebraically closed in $\boldsymbol{k}$, which implies that $\boldsymbol{k}_0$ is an algebraically closed field.

Secondly, we show that Γ_0, the value group of K_0, is an elementary substructure of Γ. Let $X \subseteq \Gamma$ be a nonempty set definable over Γ_0 in the language of Presburger arithmetic. It suffices to show that $X \cap \Gamma_0 \neq \varnothing$. By [4], we may assume that X is of the form

$$X = \{\eta \in \Gamma |\ \alpha < \eta < \beta \wedge D_n(\eta - \gamma)\},$$

where $\alpha, \beta, \gamma \in \Gamma_0$, and $D_n(x)$ is the predicate for "x is divisible by n. Take any $a, b, u \in K_0$ such that $v(a) = \alpha, v(b) = \beta$ and $v(u) = 1$. If $\beta - \alpha \in \mathbb{N}^{>0}$, then there is $i < \beta - \alpha$ such that $v(u^i a) = i + v(a) \in X \cap \Gamma_0$. Otherwise, assume that $\beta - \alpha > \mathbb{N}$. Take any $c \in K_0$ such that $v(c) = \gamma$. There is $i < n$ such that $v(u^i a/c) = i + v(a) - v(c)$ is divisible by n. Let $\eta = v(u^i a)$, then $\eta \in X \cap \Gamma_0$.

Now we see that $\boldsymbol{k}_0 \prec \boldsymbol{k}$ and $\Gamma_0 \prec \Gamma$, according to the Ax-Kochen-Ershov-results, $K_0 \prec K$ (see Theorem 4′ of [5] for details). □

Lemma 11. *Let $E \prec \boldsymbol{k}_{\mathbb{K}}$ such that $E \succ \boldsymbol{k}$. Then there exists $K_1 \prec \mathbb{K}$ such that $K_1 \succ K$ and the residue field $\boldsymbol{k}_1$ of K_1 is E.*

Proof. By Fact 6, we consider E as a subfield of $\mathbb{K}$. Let K_1 be the algebraic closure of $E \cup K$ in $\mathbb{K}$. By Lemma 10, $K_1 \prec \mathbb{K}$. Since $K \subseteq K_1$, we see that $K \prec K_1$. Let $\boldsymbol{k}_1$ be the residue field of K_1. We now verify that $\boldsymbol{k}_1 = E$. Clearly, $E \subseteq \boldsymbol{k}_1$. Conversely, consider $\boldsymbol{k}_1$ a subfield of K_1 and take any $a \in \boldsymbol{k}_1$. Let F be the field generated by $E \cup K$. If $a \notin F$, then there is non-constant $f(x) = c_0 + \cdots c_n x^n \in F[x]$ such that $f(a) = 0$. Let $\lambda = \min\{v(c_i) | \ i = 0, \cdots, n\}$ and $f^* = f/\lambda$. Then res(f^*) is a nonzero polynomial over E and

$$\text{res}(f^*)(a) = \text{res}(f^*)(\text{res}(a)) = \text{res}(f^*(a)).$$

Since $f^*(a) = f(a) = 0$, res$(f^*(a)) = 0$. We conclude that a is algebraic over E. Since E is algebraically closed, $a \in E$. This completes the proof. □

Proof of Theorem 13 Let $\bar{a} = (a_1, \cdots, a_{n^2}) \models q_{\mathfrak{G},\boldsymbol{k}}$. Let $E_0 = \boldsymbol{k}$, and each E_i is the algebraic closure of $E_{i-1}(a_{i-1})$ for $i = 1, ..., n^2$. By Lemma 11, there is an elementary chain $K = K_0 \prec K_1 \prec ... \prec K_{n^2}$ such that $\boldsymbol{k}_i = E_i$, where $\boldsymbol{k}_i$ is the residue field of K_i. Since a_i is transcendental over $\boldsymbol{k}_{i-1}$, we have that $a_i \models q_{\text{trans},\boldsymbol{k}_{i-1}}$ for each $i \leq n^2$. Suppose that $\bar{a}^* = (a_1^*, \cdots, a_{n^2}^*) \in \text{res}^{-1}(\bar{a})$. Then by Theorem 8, we see that $a_i^* \models p_{\text{trans},K_{i-1}}$ for $i = 1, \cdots, n^2$. So $\bar{a}^* \models p_{G,K}$ as required. □

Corollary 6. *$p_{G,K}$ is $G(K)$-invariant, namely, $g \cdot p_{G,K} = p_{G,K}$ for all $g \in G(K)$. In particular, $p_{G,\mathbb{K}}$ is a witness of the generic stability of G.*

Proof. Let $g \in G(K)$, then res$(g) \cdot q_{\mathfrak{G},\boldsymbol{k}} = q_{\mathfrak{G},\boldsymbol{k}}$ since $q_{\mathfrak{G},\boldsymbol{k}}$ is $\mathfrak{G}(\boldsymbol{k})$-invariant. We see that if $h \models g \cdot p_{G,K}$, then

$$\text{res}(h) \models \text{res}(g) \cdot q_{\mathfrak{G},\boldsymbol{k}} = q_{\mathfrak{G},\boldsymbol{k}}.$$

Since $p_{G,K}$ is dominated by $q_{\mathfrak{G},\boldsymbol{k}}$ via the residue map, $g \cdot p_{G,K} = p_{G,K}$ as required. □

Corollary 7. *$p_{G,\mathbb{K}}$ is the unique witness of the generic stability of G.*

Proof. Suppose that $r \in S_G(\mathbb{K})$ witnesses the generic stability of G, then $r_0 =$ res(r) is also a witness of the generic stability of $\mathfrak{G}$. Since $\mathfrak{G}$ has only one such witness, we see that $r_0 = q_{\mathfrak{G},\boldsymbol{k}}$. Since $p_{G,K}$ is dominated by $q_{\mathfrak{G},\boldsymbol{k}}$ via the residue map, $r = p_{G,\mathbb{K}}$ as required. □

We conclude from Fact 11, Corollary 6, and Corollary 7 that:

Corollary 8. $G = G^{00}$.

References

[1] J. Ax and S. Kochen, 1965, “Diophantine problems over local fields i”, *American Journal of Mathematics*, **87(3)**: 605–630.

[2] E. Bouscaren, 1998, *Model Theory and Algebraic Geometry: An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture*, Berlin, Heidelberg: Springer.

[3] E. Casanovas, 2011, *Simple Theories and Hyperimaginaries*, New York: Cambridge University Press.

[4] R. Cluckers, 2003, “Presburger sets and p-minimal fields”, *The Journal of Symbolic Logic*, **68(1)**: 153–162.

[5] F. Delon, 1978–1979, “Types sur C((x))”, *Groupe d'étude de théories stables*, **2(5)**: 1–29.

[6] L. van den Dries, 2014, “Lectures on the model theory of valued fields”, in H. D. Macpherson and C. Toffalori (eds.), *Model Theory in Algebra, Analysis and Arithmetic: Cetraro, Italy 2012*, pp. 55–157, Berlin, Heidelberg: Springer.

[7] Y. Ershov, 1965, “On elementary theories of local fields”, *Algebra i Logika*: 5–30.

[8] Y. Halevi, 2019, “On stably pointed varieties and generically stable groups in acvf”, *Annals of Pure and Applied Logic*, **170(2)**: 180–217.

[9] E. Hrushovski and F. Loeser, 2016, *Non-Archimedean Tame Topology and Stably Dominated Types*, Princeton, NJ: Princeton University Press.

[10] E. Hrushovski, Y. Peterzil and A. Pillay, 2008, “Groups, measures, and the nip”, *Journal of the American Mathematical Society*, **21(2)**: 563–596.

[11] E. Hrushovski and A. Pillay, 2011, “On nip and invariant measures”, *Journal of the European Mathematical Society*, **13**: 1005–1061.

[12] E. Hrushovski and S. Rideau-Kikuchi, 2017, “Valued fields, metastable groups”, https://arxiv.org/abs/1709.08801.

[13] D. Marker, 2002, *Model Theory: An Introduction*, New York: Springer.

[14] A. Onshuus and Y. Peterzil, 2007, “A note on stable sets, groups, and theories with nip”, *Mathematical Logic Quarterly*, **53(3)**: 295–300.

[15] B. Poizat, 2000, *A Course in Model Theory*, New York: Springer.

$\mathbb{C}[[t]]$ 上的线性代数群的稳定支配与泛稳定性

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摘　要

$\mathbb{C}((t))$ 是复数域 $\mathbb{C}$ 上的形式洛朗级数，它是一种亨泽尔赋值域，其赋值环为 $\mathbb{C}$ 上的形式幂级数，记作 $\mathbb{C}[[t]]$。令 K 为 Th($\mathbb{C}((t))$) 的一个模型，$\mathcal{O}_K$ 表示 K 的赋值环，$\boldsymbol{k}$ 表示 K 的剩余域，则 $\mathcal{O}_K$ 和 $\boldsymbol{k}$ 也分别与 $\mathbb{C}[[t]]$ 和 $\mathbb{C}$ 初等等价。

本文首先刻画了 $\mathcal{O}_K$ 的可定义子集，证明了 $\mathcal{O}_K$ 的可定义子集 X 或者是剩余-有限的，或者是剩余-余有限的，即 X 在剩余映射 res 下的像总是 $\boldsymbol{k}$ 的有限子集或者余有限子集。此外，剩余-有限集在 $\mathcal{O}_K$ 中的补集恰好是剩余-余有限集。基于这个性质，我们证明了 $\mathcal{O}_K$ 上的 n 阶可逆矩阵群 GL$(n,\mathcal{O}_K)$ 被其剩余映射稳定支配。作为一个推论，我们证明了 GL$(n,\mathcal{O}_K)$ 具有泛稳定性，该结果推广了 Y. Halevi 在代数闭赋值域中证明的一个定理。

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