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Propositions Formalized in Chu Spaces*

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Abstract. In the literature, there are many relational semantics of propositional logics each of which, although a satisfaction relation is defined, seems to have a "many-valued" intuition behind. More precisely, this means that a proposition can take one of more than two "truth values" at a state. In this paper, we use a kind of mathematical structures called Chu spaces to model such intuition; and we choose possibility semantics of classical logic, ortho-logic and Holliday's fundamental logic to do case studies. We formalize some informal reasonings about the intuition behind the relational semantics of these three logics, make explicit the underlying assumptions and discover some new consequences of the intuition behind these relational semantics in our setting.

1 Introduction

Relational semantics is a useful tool in studying logics. ([3, 6, 10]) Bi-valence, a feature of classical logic, is kept in relational semantics of many logics. To be precise, given a formula and a point in a relational structure, the satisfaction relation either holds or not between them. Then we get different relational semantics which can characterize different logics by varying the properties of the relation(s) and the requirement on the sets of points which can be interpretations/truth sets of formulas. Arguably, the relation, its properties and the requirement on truth sets of formulas are formal apparatus, whose intuition behind should and can be explained. Moreover, it is worth trying to formalize such explanation. This will result in more concrete models of logics and make explicit the intuition behind logics.

In this paper, we try to implement this process on three logics: (possibility semantics of) classical logic, ortho-logic and Holliday's fundamental logic. We will briefly discuss this process on intuitionistic logic at the end.

The intuition behind the relational semantics of these logics has already been extensively discussed and even partially formalized in the literature. For possibility semantics of classical logic, please refer to [7] and [9]. For ortho-logic, a detailed

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treatment can be found in [2]. For fundamental logic, please refer to [8]. A common feature of these discussions is that they all exploit some kind of three-valueness. In possibility semantics of classical logic, given a possibility and a proposition, at the possibility the proposition could be settled as true, settled as false or neither. The background of ortho-logic is quantum physics. Given a state of a quantum system and a test of a property of the system, the results of performing the test on the state could be always positive, always negative or neither, i.e. some positive and some negative. In fundamental logic, given a state and a proposition, at the state the proposition could be accepted, rejected or neither. By the way, according to these discussions, a formula satisfied at a point corresponds to a proposition settled as true at a possibility in possibility semantics, the results of performing a test at a state being always positive in ortho-logic and a proposition being accepted at a state in fundamental logic, respectively. Moreover, for any two points in a relational structure, the intuition behind the relation holding between them can be somehow explained via the "truth value" or mode of the propositions at the two points.

Here we use Chu spaces ([4]) over a three-element set to try to formalize the informal and implicit three-valued reasoning behind the discussion mentioned above. In a Chu space, both states/possibilities and tests/propositions are considered as primitives and represented by elements from two sets. Moreover, there is an evaluation function which takes a state/possibility and a test/proposition as input and returns an element of a fixed three-element set indicating the "truth value" /mode between the two. We consider such mathematical structures the simplest for modelling the discussion mentioned above with the least built-in assumptions. Then, following [2, 7, 8], we put axioms on Chu spaces and try to formalize the reasoning in these papers. In the meantime, we make explicit the assumptions needed in each step of these reasoning and try to detect any results which may have been bypassed in these papers.

The rest of this paper is organized as follows: In Section 2, we set up the general framework for our analysis of propositions in Chu spaces. Sections 3 to 5 are the three instances of case study, possibility semantics of classical logic, ortho-logic and Holliday's fundamental logic, respectively. These three sections follow the same pattern; and the order is such that the definitions and the properties of the binary relations successively become weaker and, arguably, less intuitive. Section 6 contains the conclusion and some discussion of future work.

2 The Setting

The mathematical structures used in this paper are Chu spaces over a three-element set, which are defined as follows:

Definition 1. A *Chu space (over* 3*)* is a tuple $\mathcal{S} = (X, A, e)$ such that both X and A are non-empty sets and e is a function from $X \times A$ to 3. In set theory, 3 is a three-element set, which we write as $\{\mathbb{T}, \mathbb{I}, \mathbb{F}\}$ here. In particular, $\mathbb{T} \neq \mathbb{F}$.

For any $x, y \in X$, we write $x \sim y$ if and only if, for each $P \in A$, $e(x, P) = e(y, P)$.

For any $P, Q \in A$, we write $P \approx Q$, if and only if, for each $x \in X$, $e(x, P) = e(x, Q)$.

Remark 1. Originally, in $\mathcal{S}$, X is interpreted as a set of objects, A a set of attributes and e an evaluation function indicating how an object relates to an attribute. Here we interpret X as a set of (possible) states, A a set of propositions and e an evaluation function indicating the relation between a state and a proposition. $\mathbb{T}$ can be interpreted as "being settled as true" in [7], "always positive" in [2] or "being accepted" in [8]; $\mathbb{F}$ can be interpreted as "being settled as false' in [7], "always negative" in [2] or "being rejected" in [8]. $\mathbb{I}$ means neither and leaves the space between $\mathbb{T}$ and $\mathbb{F}$.

Please note that we assume *neither* of the following:

1. for any $x, y \in X$, $x \sim y$ implies $x = y$;
2. for any $P, Q \in A$, $P \approx Q$ implies $P = Q$.

Given a Chu space $\mathcal{S} = (X, A, e)$, we will take a binary relation $\sqsubseteq$ on X such that, for any $x, y \in X$, $x \sqsubseteq y$ implies that they satisfy some conditions on their values of e with elements in A. In different semantics, the concrete conditions for $\sqsubseteq$ are based on different intuitions and thus different. For example, $x \sqsubseteq y$ is interpreted as "x is a refinement of y" in possibility semantics ([7]), "x is non-orthogonal to y" in ortho-logic and "x is open to y" in fundamental logic ([8]).

With such a binary relation, we investigate three topics:

First, we characterize in terms of $\sqsubseteq$ the set of the form $\{x \in X \mid e(x, P) = \mathbb{T}\}$ for a $P \in A$. Intuitively, sets of such a form are extensions of propositions. We observe that significant characterizations emerge, when a further property is added on $\mathcal{S}$ such that $\mathbb{T}$ holds between P and as many elements in X as possible. To be precise, it is the non-trivial direction of (A1) in Section 2.3 of [7] which is the following:

(A1') for any $x \in X$ and $P \in A$, if, for each $y \in X$, $y \sqsubseteq x$ implies $e(y, P) \neq \mathbb{F}$, then $e(x, P) = \mathbb{T}$.

We will discuss using our formal setting the intuition behind (A1') in Section 5.3, where the condition on the relation $\sqsubseteq$ is the weakest among the three instances of case study and thus the discussion is the most general.

Second, we give a uniform definition of negation adapted from Notion 5 in [2]:

Definition 2.

1. For each $P \in A$, $Q \in A$ is a *negation* of P, if, for each $x \in X$, both of the following hold:

(a) $e(x, Q) = \mathbb{T}$ if and only if $e(x, P) = \mathbb{F}$;
(b) $e(x, Q) = \mathbb{F}$ if and only if $e(x, P) = \mathbb{T}$.

2. $\mathcal{S}$ is *negation-closed*, if each $P \in A$ has a negation.

Remark 2. Item (a) is (A3) in Section 2.5 of [7].

Note:

1. For any $P, Q \in A$, P is a negation of Q, if and only if Q is a negation of P.
2. For any $P, Q, Q' \in A$, if both Q and Q' are negations of P, then $Q \approx Q'$.

The intuition behind this definition is that the role of $\mathbb{T}$ and $\mathbb{F}$ in a proposition and that in its negation switches.

This definition of negation seems to have built in the double negation law. As can be seen from the proofs below, Item (a) is not needed in characterizing negations in many semantics.[1] Hence we may drop Item (a) and get a weaker definition of negation that may not satisfy the double negation law.

Following the idea in the literature, we will show that, combining with the non-local condition (A1'), this arguably local definition shares the features of different concrete and non-local definitions of negation in different semantics. Moreover, negation-closedness adds symmetry between $\mathbb{T}$ and $\mathbb{F}$ in a Chu space in the sense of the following result.

Proposition 1. *Let $\mathcal{S}$ be negation-closed. The following are equivalent:*

(A1') *For any $x \in X$ and $P \in A$, if, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{F}$, then $e(x, P) = \mathbb{T}$.*

(A2') *For any $x \in X$ and $P \in A$, if, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$, then $e(x, P) = \mathbb{F}$.*

Here (A2') is the non-trivial direction of (A2) in Section 2.3 of [7].

Proof. **From (A1') to (A2')**: Let $x \in X$ and $P \in A$ be arbitrary. Since $\mathcal{S}$ is negation-closed, let Q be a negation of P. Assume that, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$. Since Q is a negation of P, by Item (b) in the definition, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, Q) \neq \mathbb{F}$. By (A1') $e(x, Q) = \mathbb{T}$, so by Item (a) in the definition $e(x, P) = \mathbb{F}$.

From (A2') to (A1'): Let $x \in X$ and $P \in A$ be arbitrary. Since $\mathcal{S}$ is negation-closed, let Q be a negation of P. Assume that, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{F}$. Since Q is a negation of P, by Item (a) in the definition, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, Q) \neq \mathbb{T}$. By (A2') $e(x, Q) = \mathbb{F}$, so by Item (b) in the definition $e(x, P) = \mathbb{T}$. □

[1]This is because we use (A1'), which we think is intuitive and useful. If we use (A2') introduced in Proposition 1, Item (a), instead of Item (b), is needed.

Third, we give a uniform definition of disjunction adapted from [7]:

Definition 3.

1. For $P, Q \in A$, $R \in A$ is a *disjunction* of P and Q, if, for each $x \in X$, $e(x, R) = \mathbb{F}$ if and only if $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$.[2]
2. $\mathcal{S}$ is *disjunction-closed*, if any $P, Q \in A$ have a disjunction.

Remark 3. For any $P, Q, R, R' \in A$, it is possible that both R and R' are disjunctions of P and Q but $R \not\approx R'$. Let $X = \{x, y\}$ and $A = \{P, Q, R, R'\}$:

	P	Q	R	R'
x	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
y	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{T}$	$\mathbb{I}$

This is no longer the case if we add axioms to $\mathcal{S}$ (cf. Propositions 4, 7 and 10 below).

Following the idea in the literature, we will show that, combining with the non-local condition (A1'), this arguably local definition shares the features of different concrete and non-local definitions of disjunction in different semantics, provided that $\mathbb{T}$ and $\mathbb{F}$ are symmetric enough in a Chu space.

Finally, since we have clear intuition, we briefly give a uniform definition of conjunction adapted from [7]:

Definition 4.

1. For $P, Q \in A$, $R \in A$ is a *conjunction* of P and Q, if, for each $x \in X$, $e(x, R) = \mathbb{T}$ if and only if $e(x, P) = \mathbb{T}$ and $e(x, Q) = \mathbb{T}$.[3]
2. $\mathcal{S}$ is *conjunction-closed*, if any $P, Q \in A$ have a conjunction.

Remark 4. For any $P, Q, R, R' \in A$, it is possible that both R and R' are conjunctions of P and Q but $R \not\approx R'$.

3 Possibility Semantics of Classical Logic

In this section, we formalize the intuition behind possibility semantics of classical logic, according to [7] and [9]. For convenience, we fix a Chu space $\mathcal{S} = (X, A, e)$.

Define a binary relation $\triangleleft$ on X as follows:

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[2] This is (A5) in Section 2.5 of [7].

[3] This is (A4) in Section 2.5 of [7].

Definition 5. $\lhd$ is the binary relation on X such that, for any $x, y \in X$, $x \lhd y$ if and only if both of the following hold:

1. for each $P \in A$, $e(y, P) = \mathbb{T}$ implies that $e(x, P) = \mathbb{T}$;
2. for each PnA, $e(y, P) = \mathbb{F}$ implies that $e(x, P) = \mathbb{F}$.

Remark 5. Note that:

1. $\lhd$ is reflexive and transitive.
2. The roles of $\mathbb{T}$ and $\mathbb{F}$ are symmetric in this definition.

Let $\sqsubseteq$ be a (not necessarily proper) subset of $\lhd$ which is reflexive and transitive.

3.1 Proposition

Here we show that the notion of propositions in our setting coincides with that in [7]. The result is (C1) in Section 2.4 of [7]. Our proof is different, for we do not use (A2') or other stronger axioms. Moreover, we find that these can also be characterized by persistence in intuitionistic logic (Item (iii) in the following proposition).

Proposition 2. *Let $\mathcal{S}$ satisfy* (A1').
For any $x \in X$ and $P \in A$, the following are equivalent:

(i) $e(x, P) = \mathbb{T}$*;*
(ii) *for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $z \sqsubseteq y$ and $e(z, P) = \mathbb{T}$;*
(iii) *for each $y \in X$, if $y \sqsubseteq x$, $e(y, P) = \mathbb{T}$.*

Proof. **From (i) to (ii)**: Let $y \in X$ be such that $y \sqsubseteq x$. By (i) and Item 1 in Definition 5 $e(y, P) = \mathbb{T}$. Moreover, by reflexivity $y \sqsubseteq y$. Hence y is the required z.

From (ii) to (iii): Let $y \in X$ satisfy $y \sqsubseteq x$. Suppose (towards a contradiction) that $e(y, P) \neq \mathbb{T}$. By (A1') there is a $y' \in X$ such that $y' \sqsubseteq y$ and $e(y', P) = \mathbb{F}$. Consider y'. Since $y' \sqsubseteq y$ and $y \sqsubseteq x$, $y' \sqsubseteq x$. Moreover, since $e(y', P) = \mathbb{F}$, for each $z \in X$, $z \sqsubseteq y'$ implies that $e(z, P) = \mathbb{F}$ and thus $e(z, P) \neq \mathbb{T}$. This contradicts (ii). Hence $e(y, P) = \mathbb{T}$.

From (iii) to (i): By reflexivity $x \sqsubseteq x$. By (iii) $e(x, P) = \mathbb{T}$. □

According to the analysis in [7], Item (i) is equivalent to Item (iii) plus a condition called refinability. We show that in our setting refinability always holds because of (A1'), so Item (i) is equivalent to Item (iii) alone.

Lemma 1. *Let $\mathcal{S}$ satisfy* (A1'). *For any $x \in X$ and $P \in A$, if $e(x, P) \neq \mathbb{T}$, then there is a $y \in X$ such that $y \sqsubseteq x$ and, for each $z \in X$, $z \sqsubseteq y$ implies that $e(z, P) \neq \mathbb{T}$.*

Proof. Assume that $e(x, P) \neq \mathbb{T}$. By (A1') there is a $y \in X$ such that $y \sqsubseteq x$ and $e(y, P) = \mathbb{F}$. Hence for each $z \in X$, $z \sqsubseteq y$ implies that $e(z, P) = \mathbb{F}$ and thus $e(z, P) \neq \mathbb{T}$. □

3.2 Negation and Disjunction

In this section, we investigate negation and disjunction in possibility semantics.

First we show that the notion of negations in our setting coincides with that in [7]. The equivalence between (i) and (iii) below is mentioned without detailed proof as (C4) in [7]. Moreover, we find that negation can also be characterized by an apparently stronger condition (Item (ii) in the following proposition).

Proposition 3. *Let $\mathcal{S}$ satisfy* (A1'). *For any $x \in X$ and $P, Q \in A$ such that Q is a negation of P, the following are equivalent:*

(i) $e(x, Q) = \mathbb{T}$*;*
(ii) *for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) = \mathbb{F}$;*
(iii) *for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$.*

Proof. **From (i) to (ii)**: Assume that $e(x, Q) = \mathbb{T}$. Let $y \in X$ satisfy $y \sqsubseteq x$. Since $y \sqsubseteq x$, $e(y, Q) = \mathbb{T}$. Since Q is a negation of P, by Item (a) in the definition $e(y, P) = \mathbb{F}$.

From (ii) to (iii): Since $\mathbb{T} \neq \mathbb{F}$, it holds.

From (iii) to (i): Assume that, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$. Since Q is a negation of P, by Item (b) in the definition, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, Q) \neq \mathbb{F}$. By (A1') $e(x, Q) = \mathbb{T}$. □

Remark 6. The equivalence between (i) and (iii) does not need Item (a) in the definition of negation. From the proof, it is obvious that the direction from (iii) to (i) does not need it. In the direction from (i) to (iii), we can derive from $e(y, Q) = \mathbb{T}$ that $e(y, Q) \neq \mathbb{F}$, and then get $e(y, P) \neq \mathbb{T}$ by Item (b) in the definition.

Therefore, it seems that (ii) is equivalent to (i) as well as (iii), only if we use both Item (a) and Item (b) to define the notion of negation, which builds in the double negation law. If we use the weaker definition of negation which only has Item (b), (ii) will be strictly stronger than (i) and (iii). Consider the following example where $X = \{x, y\}$, $A = \{P, Q\}$ and

	P	Q
x	$\mathbb{I}$	$\mathbb{T}$
y	$\mathbb{F}$	$\mathbb{T}$

Then we show that the notion of disjunctions in our setting coincides with that in [7]. The proof of the direction from (i) to (ii) is essentially the informal proof of (C5) in [7]. Here we also need (A2') in Proposition 1, which is the following:

(A2') For any $x \in X$ and $P \in A$, if, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$, then $e(x, P) = \mathbb{F}$.

Proposition 4. *Let $\mathcal{S}$ satisfy* (A1') *and* (A2').

For any $P, Q, R \in A$, the following are equivalent:

(i) *R is a disjunction of P and Q;*

(ii) *for each $x \in X$, $e(x, R) = \mathbb{T}$ if and only if, for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds.*

In particular, this equivalence holds, if $\mathcal{S}$ is negation-closed.

Proof. **From (i) to (ii)**: First assume that $e(x, R) = \mathbb{T}$. Let $y \in X$ be such that $y \sqsubseteq x$. By definition $e(y, R) = \mathbb{T}$. Then $e(y, R) \neq \mathbb{F}$. By (i) $e(y, P) \neq \mathbb{F}$ or $e(y, Q) \neq \mathbb{F}$. By (A2') there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds.

Second assume that, for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds. Let $y \in X$ be such that $y \sqsubseteq x$. By the assumption there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds. By definition $e(y, P) \neq \mathbb{F}$ or $e(y, Q) \neq \mathbb{F}$. By (i) $e(y, R) \neq \mathbb{F}$. By (A1') $e(x, R) = \mathbb{T}$.

From (ii) to (i): First assume that $e(x, R) = \mathbb{F}$. Let $y \in X$ be such that $y \sqsubseteq x$. By definition $e(y, R) = \mathbb{F}$. Then $e(y, R) \neq \mathbb{T}$. By (ii) there is a $z \in X$ such that $z \sqsubseteq y$ and, for each $u \in X$, if $u \sqsubseteq z$, then $e(u, P) \neq \mathbb{T}$ and $e(u, Q) \neq \mathbb{T}$. By Proposition 2 $e(y, P) \neq \mathbb{T}$ and $e(y, Q) \neq \mathbb{T}$. By (A2') $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$.

Second assume that $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$. Let $y \in X$ satisfy $y \sqsubseteq x$. By definition $e(y, P) = \mathbb{F}$ and $e(y, Q) = \mathbb{F}$. Suppose (towards a contradiction) that $e(y, R) = \mathbb{T}$. On the one hand, by reflexivity $y \sqsubseteq y$. By (ii) there is a $u \in X$ such that $u \sqsubseteq y$ and at least one of $e(u, P) = \mathbb{T}$ and $e(u, Q) = \mathbb{T}$ holds. On the other hand, since $u \sqsubseteq y \sqsubseteq x$, by transitivity $u \sqsubseteq x$. By the assumption and Item 2 in Definition 5 $e(u, P) = \mathbb{F}$ and $e(u, Q) = \mathbb{F}$. We get a contradiction. Hence $e(y, R) \neq \mathbb{T}$. By (A2') $e(x, R) = \mathbb{F}$. □

4 Ortho-logic

In this section, we formalize the intuition behind ortho-logic, following [2]. For convenience, we fix a Chu space $\mathcal{S} = (X, A, e)$.

Define a binary relation $\triangleleft$ on X as follows:

Definition 6. $\triangleleft$ is the binary relation on X such that, for any $x, y \in X$, $x \triangleleft y$ if and only if both of the following hold:

1. for each $P \in A$, $e(y, P) = \mathbb{T}$ implies that $e(x, P) \neq \mathbb{F}$;
2. for each $P \in A$, $e(y, P) = \mathbb{F}$ implies that $e(x, P) \neq \mathbb{T}$.

Remark 7. Note that:

1. $\lhd$ is reflexive and symmetric.
2. The roles of $\mathbb{T}$ and $\mathbb{F}$ are symmetric in this definition.
3. Comparing with the definition in the previous section, $e(x, P) = \mathbb{T}$ is weakened to $e(x, P) \neq \mathbb{F}$ and $e(x, P) = \mathbb{F}$ is weakened to $e(x, P) \neq \mathbb{T}$.

Let $\sqsubseteq$ be a (not necessarily proper) subset of $\lhd$ which is reflexive and symmetric.

4.1 Proposition

We show that the notion of propositions in our setting coincides with that in [6].

Proposition 5. *Let $\mathcal{S}$ satisfy* (A1').
For any $x \in X$ and $P \in A$, the following are equivalent:

(i) $e(x, P) = \mathbb{T}$*;*
(ii) *for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $z \sqsubseteq y$ and $e(z, P) = \mathbb{T}$.*

Proof. From (i) to (ii): Let $y \in X$ be such that $y \sqsubseteq x$. By (i) and Item 1 in Definition 6 $e(x, P) = \mathbb{T}$. Moreover, by symmetry $x \sqsubseteq y$. Hence x is the required z.

From (ii) to (i): Assume that, for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $z \sqsubseteq y$ and $e(z, P) = \mathbb{T}$. Suppose (towards a contradiction) that $e(x, P) \neq \mathbb{T}$. By (A1') there is a $y \in X$ such that $y \sqsubseteq x$ and $e(y, P) = \mathbb{F}$. By the assumption there is a $z \in X$ such that $z \sqsubseteq y$ and $e(z, P) = \mathbb{T}$. Since $z \sqsubseteq y$, $e(y, P) \neq \mathbb{F}$, contradicting that $e(y, P) = \mathbb{F}$. □

4.2 Negation and Disjunction

In this section, we study negation and disjunction in ortho-logic. First we show that the notion of negations in our setting coincides with that in ortho-logic ([6]).

Proposition 6. *Let $\mathcal{S}$ satisfy* (A1'). *For any $x \in X$ and $P, Q \in A$ such that Q is a negation of P, the following are equivalent:*

(i) $e(x, Q) = \mathbb{T}$*;*
(ii) *for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$.*

Proof. From (i) to (ii): Assume that $e(x, Q) = \mathbb{T}$. Let $y \in X$ satisfy $y \sqsubseteq x$. By definition $e(y, Q) \neq \mathbb{F}$. Since Q is a negation of P, by Item (b) in the definition $e(y, P) \neq \mathbb{T}$.

From (ii) to (i): Assume that, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$. Since Q is a negation of P, Item (b) in the definition, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, Q) \neq \mathbb{F}$. By (A1') $e(x, Q) = \mathbb{T}$. □

Then we show that the notion of disjunctions in our setting coincides with that in ortho-logic. For this, we also need (A2'). In fact, since $\sqsubseteq$ is symmetric, we can also use (A2$^-$) introduced in the next section.

Proposition 7. *Let* $\mathcal{S}$ *satisfy* (A1') *and* (A2')/(A2$^-$).

For any $P, Q, R \in A$*, the following are equivalent:*

(i) R *is a disjunction of* P *and* Q*;*

(ii) *for each* $x \in X$*,* $e(x, R) = \mathbb{T}$ *if and only if, for each* $y \in X$*, if* $y \sqsubseteq x$*, then there is a* $z \in X$ *such that* $z \sqsubseteq y$ *and at least one of* $e(z, P) = \mathbb{T}$ *and* $e(z, Q) = \mathbb{T}$ *holds.*

In particular, this equivalence holds if $\mathcal{S}$ *is negation-closed.*

Proof. From (i) to (ii): First assume that $e(x, R) = \mathbb{T}$. Let $y \in X$ be such that $y \sqsubseteq x$. By definition $e(y, R) \neq \mathbb{F}$. By (i) $e(y, P) \neq \mathbb{F}$ or $e(y, Q) \neq \mathbb{F}$. By (A2') there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds.

Second assume that, for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds. Let $y \in X$ be such that $y \sqsubseteq x$. By the assumption there is a $z \in X$ such that $z \sqsubseteq y$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds. By definition $e(y, P) \neq \mathbb{F}$ or $e(y, Q) \neq \mathbb{F}$. By (i) $e(y, R) \neq \mathbb{F}$. By (A1') $e(x, R) = \mathbb{T}$.

From (ii) to (i): First assume that $e(x, R) = \mathbb{F}$. Let $y \in X$ be such that $y \sqsubseteq x$. By definition $e(y, R) \neq \mathbb{T}$. By (ii) there is a $z \in X$ such that $z \sqsubseteq y$ and, for each $u \in X$, if $u \sqsubseteq z$, then $e(u, P) \neq \mathbb{T}$ and $e(u, Q) \neq \mathbb{T}$. By Proposition 5 $e(y, P) \neq \mathbb{T}$ and $e(y, Q) \neq \mathbb{T}$. By (A2') $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$.

Second assume that $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$. Let $y \in X$ be such that $y \sqsubseteq x$. Suppose (towards a contradiction) that $e(y, R) = \mathbb{T}$. By symmetry $x \sqsubseteq y$. By (ii), there is a $u \in X$ such that $u \sqsubseteq x$ and at least one of $e(u, P) = \mathbb{T}$ and $e(u, Q) = \mathbb{T}$ holds. By Definition 6 at least one of $e(x, P) \neq \mathbb{F}$ and $e(x, Q) \neq \mathbb{F}$ holds, contradicting the assumption. Hence $e(y, R) \neq \mathbb{T}$. By (A2') $e(x, R) = \mathbb{F}$. □

5 Holliday's Fundamental Logic

In this section, we formalize the intuition behind Holliday's fundamental logic, according to [8]. For convenience, we fix a Chu space $\mathcal{S} = (X, A, e)$.

Define a binary relation $\lhd$ on X as follows:

Definition 7. Take a binary relation $\lhd$ on X such that, for any $x, y \in X$, $x \lhd y$ if and only if, for each $P \in A$, $e(y, P) = \mathbb{T}$ implies that $e(x, P) \neq \mathbb{F}$.

Remark 8. Note that:

1. $\triangleleft$ is reflexive.
2. The roles of $\mathbb{T}$ and $\mathbb{F}$ are *not* symmetric in this definition.
3. Comparing with the definition in the previous section, only Item 1 is kept and Item 2 is deleted.

Let $\sqsubseteq$ be a (not necessarily proper) subset of $\triangleleft$ which is reflexive.

5.1 Proposition

We show that the notion of propositions in our setting coincides with that in [8].

Proposition 8. *Let $\mathcal{S}$ satisfy* (A1').
For any $x \in X$ and $P \in A$, the following are equivalent:

(i) $e(x, P) = \mathbb{T}$*;*
(ii) *for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $y \sqsubseteq z$ and $e(z, P) = \mathbb{T}$.*

Proof. From (i) to (ii):
Let $y \in X$ be such that $y \sqsubseteq x$. By (i) and Definition 7 $e(x, P) = \mathbb{T}$. Since $y \sqsubseteq x$, x is the required z.

From (ii) to (i): Assume that, for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $y \sqsubseteq z$ and $e(z, P) = \mathbb{T}$. Suppose (towards a contradiction) that $e(x, P) \neq \mathbb{T}$. By (A1') there is a $y \in X$ such that $y \sqsubseteq x$ and $e(y, P) = \mathbb{F}$. By the assumption there is a $z \in X$ such that $y \sqsubseteq z$ and $e(z, P) = \mathbb{T}$. Since $e(y, P) = \mathbb{F}$ and $y \sqsubseteq z$, by definition $e(z, P) \neq \mathbb{T}$, contradicting that $e(z, P) = \mathbb{T}$. ☐

5.2 Negation and Disjunction

In this section, we study negation and disjunction in fundamental logic. First we show that the notion of negations in our setting coincides with that in [8].

Proposition 9. *Let $\mathcal{S}$ satisfy* (A1'). *For any $x \in X$ and $P, Q \in A$ such that Q is a negation of P, the following are equivalent:*

(i) $e(x, Q) = \mathbb{T}$*;*
(ii) *for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$.*

Proof. From (i) to (ii): Assume that $e(x, Q) = \mathbb{T}$. Let $y \in X$ satisfy $y \sqsubseteq x$. By definition $e(y, Q) \neq \mathbb{F}$. Since Q is a negation of P, by Item (b) in the definition $e(y, P) \neq \mathbb{T}$.

From (ii) to (i): Assume that, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, P) \neq \mathbb{T}$. Since Q is a negation of P, by Item (b) in the definition, for each $y \in X$, if $y \sqsubseteq x$, then $e(y, Q) \neq \mathbb{F}$. By (A1') $e(x, Q) = \mathbb{T}$. ☐

Then we show that the notion of disjunctions in our setting coincides with that in [8]. For this, we also need (A2$^-$), a variant of (A2').

(A2$^-$) for any $x \in X$ and $P \in A$, if, for each $y \in X$, $x \sqsubseteq y$ implies that $e(y, P) \neq \mathbb{T}$, then $e(x, P) = \mathbb{F}$.

It is the non-trivial direction of the intuitive meaning of rejecting a proposition explained in Remark 4.2 in [8]. The direction from (i) to (ii) in the following result is also mentioned without proof in the last sentence of this remark.

Proposition 10. *Let $\mathcal{S}$ satisfy* (A1') *and* (A2$^-$).
For any $P, Q, R \in A$, the following are equivalent:

(i) *R is a disjunction of P and Q;*
(ii) *for each $x \in X$, $e(x, R) = \mathbb{T}$ if and only if, for each $y \in X$, if $y \sqsubseteq x$, then there is a $z \in X$ such that $y \sqsubseteq z$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds.*

Proof. **From (i) to (ii)**: First assume that $e(x, R) \neq \mathbb{T}$. By (A1') there is a $y \in X$ such that $y \sqsubseteq x$ and $e(y, R) = \mathbb{F}$. By (i) $e(y, P) = \mathbb{F}$ and $e(y, Q) = \mathbb{F}$. Then, for each $z \in X$ such that $y \sqsubseteq z$, $e(z, P) \neq \mathbb{T}$ and $e(z, Q) \neq \mathbb{T}$.

Second assume that $e(x, R) = \mathbb{T}$. Let $y \in X$ satisfy $y \sqsubseteq x$. By definition $e(y, R) \neq \mathbb{F}$. By (i) $e(y, P) \neq \mathbb{F}$ or $e(y, Q) \neq \mathbb{F}$. In either cases, by (A2$^-$) there is a $z \in X$ such that $y \sqsubseteq z$ and at least one of $e(z, P) = \mathbb{T}$ and $e(z, Q) = \mathbb{T}$ holds.

From (ii) to (i): First assume that $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$. Let $y \in X$ satisfy $x \sqsubseteq y$. Consider x. We have $x \sqsubseteq y$ and, for each $z \in X$, if $x \sqsubseteq z$, then by the assumption and definition $e(z, P) \neq \mathbb{T}$ and $e(z, Q) \neq \mathbb{T}$. By (ii) $e(y, R) \neq \mathbb{T}$. By (A2$^-$) $e(x, R) = \mathbb{F}$.

Second assume that $e(x, R) = \mathbb{F}$. Let $y \in X$ satisfy $x \sqsubseteq y$. Then $e(y, R) \neq \mathbb{T}$. By (ii) there is a $z \in X$ such that $z \sqsubseteq y$ and, for each $u \in X$, $z \sqsubseteq u$ implies that $e(u, P) \neq \mathbb{T}$ and $e(u, P) \neq \mathbb{T}$. By Proposition 8 $e(y, P) \neq \mathbb{T}$ and $e(y, Q) \neq \mathbb{T}$. By (A2$^-$) $e(x, P) = \mathbb{F}$ and $e(x, Q) = \mathbb{F}$. □

5.3 (A1'), (A2$^-$) and Maximality

In this subsection, we discuss the intuition behind (A1') and (A2$^-$) via some formal results. We start with some definitions about $\mathbb{T}$-$\mathbb{F}$ pair.

Definition 8. Let $\mathcal{S} = (X, A, e)$ be a Chu space.

1. A $\mathbb{T}$-$\mathbb{F}$ *pair* in $\mathcal{S}$ is an ordered pair $(U, V) \in \wp(X) \times \wp(X)$[4] such that, for any $x \in U$ and $y \in V$, $y \not\sqsubseteq x$.

[4]For a set A, $\wp(A)$ denotes the power set of A.

2. A $\mathbb{T}$-$\mathbb{F}$ pair (U, V) is $\mathbb{T}$-*maximal*, if, for each $\mathbb{T}$-$\mathbb{F}$ pair (U', V'), $U \subseteq U'$ and $V \subseteq V'$ imply that $U = U'$.
3. A $\mathbb{T}$-$\mathbb{F}$ pair (U, V) is $\mathbb{F}$-*maximal*, if, for each $\mathbb{T}$-$\mathbb{F}$ pair (U', V'), $U \subseteq U'$ and $V \subseteq V'$ imply that $V = V'$.
4. A $\mathbb{T}$-$\mathbb{F}$ pair (U, V) is *maximal*, if, for each $\mathbb{T}$-$\mathbb{F}$ pair (U', V'), $U \subseteq U'$ and $V \subseteq V'$ imply that $U = U'$ and $V = V'$.

Proposition 11. *For each $P \in A$, the following are equivalent:*

(A1'(P)) *For each $x \in X$, if, for each $y \in X$, $y \sqsubseteq x$ implies $e(y, P) \neq \mathbb{F}$, then $e(x, P) = \mathbb{T}$.*

($\mathbb{T}$-Max(P)) $\big(\{x \in X \mid e(x, P) = \mathbb{T}\}, \{x \in X \mid e(x, P) = \mathbb{F}\}\big)$ *is a $\mathbb{T}$-maximal $\mathbb{T}$-$\mathbb{F}$ pair.*

Proof. From (A1'(P)) to ($\mathbb{T}$-Max(P)): First prove that it is a $\mathbb{T}$-$\mathbb{F}$ pair. Let $x, y \in X$ such that $e(x, P) = \mathbb{T}$ and $e(y, P) = \mathbb{F}$. By definition $y \not\sqsubseteq x$.

Second prove $\mathbb{T}$-maximality. Assume that (U, V) is a $\mathbb{T}$-$\mathbb{F}$ pair such that $\{x \in X \mid e(x, P) = \mathbb{T}\} \subseteq U$ and $\{x \in X \mid e(x, P) = \mathbb{F}\} \subseteq V$.

Let $x \in U$. For each $y \in X$ satisfying $y \sqsubseteq x$, since $x \in U$ and (U, V) is a $\mathbb{T}$-$\mathbb{F}$ pair, $y \notin V$ and thus $y \notin \{x \in X \mid e(x, P) = \mathbb{F}\}$, so $e(y, P) \neq \mathbb{F}$. By (A1'(P)) $e(x, P) = \mathbb{T}$.

From ($\mathbb{T}$-Max(P)) to (A1'(P)): Assume that, for each $y \in X$, $y \sqsubseteq x$ implies $e(y, P) \neq \mathbb{F}$. Then $\big(\{x \in X \mid e(x, P) = \mathbb{T}\} \cup \{x\}, \{x \in X \mid e(x, P) = \mathbb{F}\}\big)$ is a $\mathbb{T}$-$\mathbb{F}$ pair. Since $\{x \in X \mid e(x, P) = \mathbb{T}\} \subseteq \{x \in X \mid e(x, P) = \mathbb{T}\} \cup \{x\}$, by ($\mathbb{T}$-Max(P)) $e(x, P) = \mathbb{T}$. □

Corollary 1. *The following are equivalent:*

(A1') *For any $x \in X$ and $P \in A$, if, for each $y \in X$, $y \sqsubseteq x$ implies $e(y, P) \neq \mathbb{F}$, then $e(x, P) = \mathbb{T}$.*

($\mathbb{T}$-Max) *For each $P \in A$, $\big(\{x \in X \mid e(x, P) = \mathbb{T}\}, \{x \in X \mid e(x, P) = \mathbb{F}\}\big)$ is a $\mathbb{T}$-maximal $\mathbb{T}$-$\mathbb{F}$ pair.*

Proposition 12. *For each $P \in A$, the following are equivalent:*

($A2^-$(P)) *For each $x \in X$, if, for each $y \in X$, $x \sqsubseteq y$ implies $e(y, P) \neq \mathbb{T}$, then $e(x, P) = \mathbb{F}$.*

($\mathbb{F}$-Max(P)) $\big(\{x \in X \mid e(x, P) = \mathbb{T}\}, \{x \in X \mid e(x, P) = \mathbb{F}\}\big)$ *is an $\mathbb{F}$-maximal $\mathbb{T}$-$\mathbb{F}$ pair.*

Proof. From ($A2^-$(P)) to ($\mathbb{F}$-Max(P)): The proof that it is a $\mathbb{T}$-$\mathbb{F}$ pair is the same as that in Proposition 11. For $\mathbb{F}$-maximality, assume that (U, V) is a $\mathbb{T}$-$\mathbb{F}$ pair such that $\{x \in X \mid e(x, P) = \mathbb{T}\} \subseteq U$ and $\{x \in X \mid e(x, P) = \mathbb{F}\} \subseteq V$.

Let $x \in V$. For each $y \in X$ satisfying $x \sqsubseteq y$, since $x \in V$ and (U, V) is a $\mathbb{T}$-$\mathbb{F}$ pair, $y \notin U$ and thus $y \notin \{x \in X \mid e(x, P) = \mathbb{T}\}$, so $e(y, P) \neq \mathbb{T}$. By (A2$^-$(P)) $e(x, P) = \mathbb{F}$.

From ($\mathbb{F}$-Max(P)) to (A2$^-$(P)): Assume that, for each $y \in X$, $x \sqsubseteq y$ implies $e(y, P) \neq \mathbb{T}$. Then $\big(\{x \in X \mid e(x, P) = \mathbb{T}\}, \{x \in X \mid e(x, P) = \mathbb{F}\} \cup \{x\}\big)$ is a $\mathbb{T}$-$\mathbb{F}$ pair. Since $\{x \in X \mid e(x, P) = \mathbb{F}\} \subseteq \{x \in X \mid e(x, P) = \mathbb{F}\} \cup \{x\}$, by ($\mathbb{F}$-Max(P)) $e(x, P) = \mathbb{F}$. □

Corollary 2. *The following are equivalent:*

(A2$^-$) *For any $x \in X$ and $P \in A$, if, for each $y \in X$, if $x \sqsubseteq y$, then $e(y, P) \neq \mathbb{T}$, then $e(x, P) = \mathbb{F}$.*

($\mathbb{F}$-Max) *For each $P \in A$, $\big(\{x \in X \mid e(x, P) = \mathbb{T}\}, \{x \in X \mid e(x, P) = \mathbb{F}\}\big)$ is an $\mathbb{F}$-maximal $\mathbb{T}$-$\mathbb{F}$ pair.*

Corollary 3. *The following are equivalent:*

(i) *$\mathcal{S}$ satisfies* (A1') *and* (A2$^-$).

(Max) *For each $P \in A$, $\big(\{x \in X \mid e(x, P) = \mathbb{T}\}, \{x \in X \mid e(x, P) = \mathbb{F}\}\big)$ is a maximal $\mathbb{T}$-$\mathbb{F}$ pair.*

Proof. Note that (Max) is equivalent to ($\mathbb{T}$-Max) and ($\mathbb{F}$-Max) together. □

Finally, (A2') in possibility semantics can be analyzed in the same way. The key is to observe that (A2') follows from (A2$^-$), so we may assume (A2$^-$) in our analysis of possibility semantics. Please note that in the following the definition of $\sqsubseteq$ is the one in Section 3.

Lemma 2. *Let $\mathcal{S} = (X, A, e)$ be a Chu space. (A2$^-$) implies (A2') (introduced in Proposition 1.)*

Proof. Let $x \in X$ and $P \in A$ be arbitrary. Assume that, for each $y \in X$, $y \sqsubseteq x$ implies $e(y, P) \neq \mathbb{T}$. Note that, for each $y \in X$ satisfying $x \sqsubseteq y$, $e(y, P) \neq \mathbb{T}$; otherwise, by definition $e(x, P) = \mathbb{T}$ and by reflexivity $x \sqsubseteq x$, contradicting the assumption. By (A2$^-$) $e(x, P) = \mathbb{F}$. □

6 Conclusion and Future Work

In this paper, we use Chu spaces to formalize the discussion about the intuition behind the relational semantics of three logics, namely, possibility semantics of classical logic, ortho-logic and Holliday's fundamental logic, in the literature. ([2, 7, 8, 9])

1. The definition of Chu spaces is from [4]. Many other definitions are formalizations of their informal counterparts in the literature. These include: Definition 2 (from [2]), Definitions 3, 4 and 5 (from [7]), Definition 6 (from [2, 6]) and Definition 7 (from [8]). The conditions (A1') and (A2') are both inspired by [7], and (A2$^-$) is inspired by [8].
2. Some results are formalizations of their informal counterparts in the literature. These includes:

   (a) the equivalence between (i) and (ii) in Proposition 2;
   (b) the equivalence between (i) and (iii) in Proposition 3;
   (c) the direction from (i) to (ii) in Proposition 4;
   (d) Propositions 8 and 9, as well as the direction from (i) to (ii) in Proposition 10, are mentioned and considered straightforward in [8].

   The analysis of ortho-logic here follows the pattern in Section 2 in [7].
3. The directions from (ii) to (i) in Propositions 4 and 10 complete the analysis in [7] and [8].
4. Section 5.3 is completely original and is the main contribution of this paper. We propose the axiom (Max) whose intuition is arguably clear and simple: for each proposition P, minimize the set $\{x \in X \mid e(x, P) = \mathbb{I}\}$. And we prove that it is equivalent to (A1') and (A2$^-$) together, which are sufficient, and possibly necessary, to derive all the results we want in all three semantics under consideration.

The investigation in this paper is highly tentative. The additional "truth value" $\mathbb{I}$ introduces great flexibility with which it is still not clear how to deal properly. Currently it seems that Chu spaces satisfying (Max) is a general and proper setting. However, many questions are still waiting for studying. Here we briefly mention three examples.

The first one is what is the precise relation between the original two-valued relational settings and our three-valued setting. It may be helpful to follow the method in [1] and study the functors between the categories formed by the different kinds of mathematical structures. In fact, the use of $\sqsubseteq$, instead of $\triangleleft$, facilitates the definition of such functors.

The second one is what is intuitionistic logic in our setting. Intuitively the definitions of $\sqsubseteq$ and $\triangleleft$ in possibility semantics of classical logic and intuitionistic logic are the same in our setting: in negation-closed Chu spaces, the relations preserve truth. Moreover, as is mentioned in Item 4 above, the notions of propositions in these two semantics can be uniformly characterized by persistence. Hence it is not clear at present and is interesting that in our setting where the difference between possibility semantics of classical logic and intuitionistic logic lies. Two directions are worth considering. One is to investigate the properties of the weak negation defined using only Item (b); and the other is to follow [5] and [7] and investigate the counterpart of a nucleus in our setting.

Third, considering the generality of our setting, it is an important direction for future work to consider whether para-consistent logics can be handled in our setting. The key to make $\lhd$ and $\sqsubseteq$ no longer reflexive; for this, we may need to change e from a function to a relation.

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命题在 Chu 空间中的形式化

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摘　要

文献中的许多命题逻辑的关系语义尽管都以满足关系为基础，但它们背后似乎有着“多值”的直观。更准确地说，这意味着在一个状态下，一个命题可以拥有两个以上“真值”中的一个。在本文中，我们使用一种被称为 Chu 空间的数学结构来为这种直观建立数学模型；我们选择经典逻辑的可能性语义、正交逻辑以及 Holliday 的基本逻辑作为案例研究。我们对与这三种逻辑的关系语义背后的直观相关的一些非形式推理进行了形式化，明确了隐含的假设，并发现了在我们的理论框架中这些逻辑背后的直观具有一些新的后果。

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