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Modal Logics over Semi-lattices and Lattices with Alternative Axiomatization*

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Abstract. This paper builds on the previous work starting by X. Wang and Y. Wang (2022, 2023) on modal logics over lattices, exploring further the complex relationship between modal logic and lattice theory. In our initial research, we utilized polyadic hybrid logic with binary modalities $\langle \mathrm{sup} \rangle, \langle \mathrm{inf} \rangle$ to discuss lattices via standard semantics. This paper introduces a focused examination of meet semi-lattices, structures in which not every pair of elements necessarily has a supremum. To address meet semi-lattices, it employs the language of polyadic hybrid logic with unary modality P and binary modality $\langle \mathrm{inf} \rangle$. Subsequently, a complete axiomatization of polyadic hybrid logic over semi-lattices is obtained. In our earlier work, the definition of lattices was primarily based on partial order relations. In the latter part of this paper, an alternative definition of lattices that aligns more with an algebraic perspective is proposed, and the corresponding axiomatic results are provided.

1 Introduction

Lattices have long been foundational to both the algebraic and logical disciplines, tracing back to early works such as that of [10]. The advancements in lattice theory by [6] provided logicians with robust tools for exploring classical and non-classical logics, as demonstrated in the application to quantum logic by [5]. Moreover, lattices have been utilized to structure truth values in many-valued logic, notably in [15], and to form frameworks for families of modal logics ([9]). Despite these comprehensive uses, the intersection of modal logic and lattice structures, particularly using modal logic to capture lattices as Kripke frames, remains underexplored.

The pioneering work about this was introduced by [11]. By using step-by-step method, Burgess obtained a complete tense logic over strict preorders with upper and lower bounds for each pair of elements (not necessarily supremums and infimums), but without greatest or minimal elements. Besides Burgess' work, [13] considered tense logic and its various extensions over *Medvedev frames*, which can be seen as the special finite meet-semilattices. These works focus on the language of tense logic

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with binary modalities F, P. In [1, 2, 3], two binary modalities $\langle\mathrm{sup}\rangle, \langle\mathrm{inf}\rangle$ were introduced in the study of *modal information logic* (MIL). Axiomatizations of MIL with $\langle\mathrm{sup}\rangle$ only have been obtained by [14] over pre-orders and partial orders with quasi-least(or minimal) upper bounds, and over join-semilattices. In [14], Knudstorp employed the non-hybrid language MIL with a single binary modality $\langle\mathrm{sup}\rangle$ to capture these structures. However, this language is too weak to distinguish between pre-order and partial order, as well as quasi-least and minimal upper bound. This weakness in expressive power also complicates the axiomatization of meet-semilattice that requires infinite axioms.

In the study presented by [16, 17, 18], several completeness theorems for lattices over the basic tense language **TL** and a polyadic hybrid language with binary modalities of $\langle\mathrm{sup}\rangle$ and $\langle\mathrm{inf}\rangle$ were introduced. We summarized the completeness results in the following table, where *so-lattices* denote lattices over *strict orders*, and other classes of structures are based on *partial orders*.

Language	System	Frame class	Primitive modalities	Nominals
TL	$\mathbb{SL}$	so-lattice $\mathcal{L}_{sr}$	P, F	None
TL	$\mathbb{L}$	lattice $\mathcal{L}_r$	P, F	None
HLSI	$\mathbb{HLSI}_{\mathrm{L}}$	lattice $\mathcal{L}_t$	$\langle\mathrm{sup}\rangle, \langle\mathrm{inf}\rangle$	Yes
HLSI	$\mathbb{HLSI}_{\mathrm{DL}}$	distributive lattice $\mathcal{L}_d$	$\langle\mathrm{sup}\rangle, \langle\mathrm{inf}\rangle$	Yes
HLSI	$\mathbb{HLSI}_{\mathrm{ML}}$	modular lattice $\mathcal{L}_{mod}$	$\langle\mathrm{sup}\rangle, \langle\mathrm{inf}\rangle$	Yes

In this paper, we will find the proper modal languages and logic systems to capture the following frame class: meet semi-lattices and lattices with alternative definition. In meet semi-lattices not every pair of elements necessarily has a supremum and thus we have no $\langle\mathrm{sup}\rangle$ in our language. We will show that the binary modality $\langle\mathrm{inf}\rangle$, together with the unary modality P, can capture the meet semi-lattices. In [17], the definition of lattices was primarily based on relational understanding. In this paper, we propose another definition of lattices that leans more towards an algebraic perspective, and we provide the corresponding axiomatic results. In this part, the global modality E is needed in our hybrid language. Summarizing our results, we can add the following two rows to the table above:

Language	System	Frame class	Primitive modalities	Nominals
HPI	$\mathbb{TPI}_{\curlywedge}$	meet semi-lattice $\mathcal{L}_{bt}^{\curlywedge}$	$\mathrm{P}, \langle\mathrm{inf}\rangle$	Yes
HLSIE	$\mathbb{HSIE}_L$	lattice $\mathcal{L}_f$	$\langle\mathrm{sup}\rangle, \langle\mathrm{inf}\rangle, \mathrm{E}$	Yes

The rest of the paper is organized as follows. In Section 2, we give the several definition of lattices and semi-lattices based on different perspectives. In particular, we introduce the definition of meet semi-lattice $\mathcal{L}_{bt}^{\curlywedge}$ and the alternative definition lattices $\mathcal{L}_f$. In Section 3 and 4, we give complete axiomatizations for meet semi-lattices $\mathcal{L}_{bt}^{\curlywedge}$ and lattices $\mathcal{L}_f$ respectively by using different modal language.

2 Defining Lattices and Semi-Lattices: Structural Differences

In this section, we will present various definitions of (semi-)lattices, some of which are classical, while others are defined in the sense of Kripke frames.

2.1 Lattices

As is well known, lattices can be defined as posets with special properties, or algebraic structures with two binary operators $\wedge, \vee$. To make a difference between the logical connectives and algebraic operations of *meet* and *join*, in the sequel, we use $\curlywedge$ and $\curlyvee$ to denote the latter. We briefly review the formal definitions.

Definition 1 (Lattice, [7]). A relational structure $\mathcal{L}_r = \langle L, \leq \rangle$ is called a *lattice* iff it satisfies the following axioms:

$$\begin{aligned}
\mathrm{FORef}: &\quad \forall x(x \leq x)\\
\mathrm{FOASym}: &\quad \forall x \forall y(x \leq y \wedge y \leq x \to x = y)\\
\mathrm{FOTrans}: &\quad \forall x \forall y \forall z(x \leq y \wedge y \leq z \to x \leq z)\\
\mathrm{FOSup}: &\quad \forall x \forall y \exists t(x \leq t \wedge y \leq t \wedge \forall z(x \leq z \wedge y \leq z \to t \leq z))\\
\mathrm{FOInf}: &\quad \forall x \forall y \exists t(t \leq x \wedge t \leq y \wedge \forall z(z \leq x \wedge z \leq y \to z \leq t))
\end{aligned}$$

Axioms FORef, $\mathrm{FOTrans}$, and FOASym say that $\leq$ is a partial order (reflexive, transitive and anti-symmetric), and axioms FOSup and FOInf make sure any two elements have a least upper bound (*supremum*) and a greatest lower bound (*infimum*). It is not hard to check that the supremum and the infimum of two elements are unique given the anti-symmetry property of $\leq$, so we can define:

$$x \curlywedge y := \text{ the infimum of } \{x, y\} \qquad x \curlyvee y := \text{ the supremum of } \{x, y\}$$

Denote the class of lattices as $\mathfrak{L}_r = \{\mathcal{L}_r = \langle L, R \rangle \mid \mathcal{L}_r \text{ is a lattice } \}$. To capture $\mathfrak{L}_r$ by modal logic, we use the language of basic tense logic **TL** (classical modal language with two unary modalities F, P). In [16, 17], the **TL**-system $\mathbb{L}$ was constructed and its strong completeness with respect to $\mathfrak{L}_r$ was proven.

Next, we present the classical algebraic definition of a lattice:

Definition 2 ([7]). An algebraic structure $\mathcal{L}_a = \langle L, \curlywedge, \curlyvee \rangle$ is called a *lattice* iff it satisfies the following axioms:

$$\begin{aligned}
\mathrm{FOIde}: &\ \forall x((x \curlyvee x = x) \wedge (x \curlywedge x = x))\\
\mathrm{FOAss}: &\ \forall x \forall y \forall z(((x \curlywedge y) \curlywedge z = x \curlywedge (y \curlywedge z)) \wedge ((x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z)))\\
\mathrm{FOCom}: &\ \forall x \forall y(((x \curlywedge y = y \curlywedge x) \wedge (x \curlyvee y = y \curlyvee x))\\
\mathrm{FOAbs}: &\ \forall x \forall y((x \curlywedge y) \curlyvee y = y) \wedge ((x \curlyvee y) \curlywedge y = y))
\end{aligned}$$

From this algebraic definition, we can recover the partial order by defining $x \leq y := (x \curlywedge y = x)$. These two definitions are equivalent to each other ([7]).

Note that in mathematics, binary functions are essentially ternary relations that satisfy specific conditions. Therefore, we consider the Kripke frame $\mathcal{M} = \langle W, R_{\text{sup}}, R_{\text{inf}}\rangle$, where R_{sup} and R_{inf} are *arbitrary* ternary relations on W such that $R_{\text{inf}}xyz$ iff $x = y \wedge z$, $R_{\text{sup}}xyz$ iff $x = y \vee z$. This allows us to use the binary modalities $\langle\text{sup}\rangle$ and $\langle\text{inf}\rangle$ to characterize the ternary relations R_{sup} and R_{inf} respectively.

There are two ways to understand what makes the frame $\mathcal{M} = \langle W, R_{\text{sup}}, R_{\text{inf}}\rangle$ a lattice. The first is based on partial order relations and the second on the functionality of the ternary relations. Let us see the first one:

Note that in a lattice, binary partial order relations can be defined by binary functions: if $x = y \wedge z$ (i.e., $R_{\text{inf}}xyz$ holds), then $x \leq y$; if $x = y \vee z$ (i.e., $R_{\text{sup}}xyz$ holds), then $y \leq x$. This method can be generalized to any ternary relation. Define the relations R, R' as follows:

$$\begin{array}{rcl} xRy & \iff & \text{there is } z \in W \text{ such that } R_{\text{inf}}xyz; \\ xR'y & \iff & \text{there is } z \in W \text{ such that } R_{\text{sup}}xyz. \end{array}$$

If R and R' are inverse relations, and if R is a partial order, then we can define a lattice structure in the following way: For any two points x, y in the frame, there exist z, t such that $R_{\text{sup}}zxy$ and $R_{\text{inf}}txy$, and z, t are respectively the least upper bound and greatest lower bound of x and y. Formally:

Definition 3 ([17]). A frame $\mathcal{F}_t = \langle W, R_{\text{sup}}, R_{\text{inf}}\rangle$ is called a *lattice* iff it satisfies the following axioms:

$$\begin{array}{rl} \text{FORef:} & \forall x(xRx) \\ \text{FOASym:} & \forall x\forall y(xRy \wedge yRx \to x = y) \\ \text{FOTrans:} & \forall x\forall y\forall z(xRy \wedge yRz \to xRz) \\ \text{FOSym:} & \forall x\forall y(xRy \leftrightarrow yR'x) \\ \text{FOSupm:} & \forall x\forall y\forall z(R_{\text{sup}}zxy \to (xRz \wedge yRz \wedge \forall t(xRt \wedge yRt \to zRt))) \\ \text{FOInfm:} & \forall x\forall y\forall z(R_{\text{inf}}zxy \to (zRx \wedge zRy \wedge \forall t(tRx \wedge tRy \to tRz))) \\ \text{FOEsi:} & \forall x\forall y\exists z\exists t(R_{\text{sup}}zxy \wedge R_{\text{inf}}txy) \end{array}$$

where $xRy := \exists z R_{\text{inf}}xyz$, $xR'y := \exists z R_{\text{sup}}xyz$. We use $\mathcal{L}_t = \langle L, R_{\text{sup}}, R_{\text{inf}}\rangle$ to denote such lattice structures and use $\mathfrak{L}_t$ to denote the class of them.

To capture $\mathfrak{L}_t$, the language of *nominal polyadic modal logic* is used in [17]:

Definition 4. Given a countable set of proposition letters **P**, a countable set of nominals **N** and binary modalities $\langle\text{sup}\rangle$, $\langle\text{inf}\rangle$, the language of hybrid logic with sup and inf (**HLSI**) is defined by the following BNF grammar:

$$\varphi ::= p \in \mathbf{P} \mid i \in \mathbf{N} \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\text{sup}\rangle(\varphi, \varphi) \mid \langle\text{inf}\rangle(\varphi, \varphi).$$

Define the following modalities:

$$[\sup](\psi,\varphi) := \neg\langle\sup\rangle(\neg\psi,\neg\varphi) \qquad [\inf](\psi,\varphi) := \neg\langle\inf\rangle(\neg\psi,\neg\varphi)$$
$$\mathrm{P}\psi := \langle\sup\rangle(\psi,\top) \qquad \mathrm{F}\psi := \langle\inf\rangle(\psi,\top)$$
$$\mathrm{H}\psi := [\sup](\psi,\bot) \qquad \mathrm{G}\psi := [\inf](\psi,\bot)$$

The $\mathbb{HLSI}$-system $\mathbb{HLSI}_{\mathrm{L}}$ for $\mathfrak{L}_t$ is listed here ([17]):

Axioms

TAUT: propositional tautologies

$\mathrm{Dual_b}$: $(\langle\sup\rangle(p,q) \leftrightarrow \neg[\sup](\neg p,\neg q)) \wedge (\langle\inf\rangle(p,q) \leftrightarrow \neg[\inf](\neg p,\neg q))$

$\mathrm{K_{sup}}$: $[\sup](p\to q,r) \to ([\sup](p,r)\to[\sup](q,r))$

$\mathrm{K_{inf}}$: $[\inf](p\to q,r) \to ([\inf](p,r)\to[\inf](q,r))$

$\mathrm{Ref_P}$: $p \to \mathrm{P}p$

Sym: $(p\to \mathrm{GP}p)\wedge(p\to\mathrm{HF}p)$

$\mathrm{Com_{sup}}$: $\langle\sup\rangle(p,q)\to\langle\sup\rangle(q,p)$

$\mathrm{Com_{inf}}$: $\langle\inf\rangle(p,q)\to\langle\inf\rangle(q,p)$

Nom: $\mathrm{PF}(i\wedge p)\leftrightarrow\mathrm{HG}(i\to p)$

Tra: $\mathrm{PFPF}(i\wedge p)\to\mathrm{PF}(i\wedge p)$

4_{F}: $\mathrm{FF}i\to\mathrm{F}i$

Asym: $i\to\mathrm{G}(\mathrm{F}i\to i)$

Con$'$: $\mathrm{FP}i$

$\mathrm{sup_E}$: $\mathrm{P}i\wedge\mathrm{P}j\to\mathrm{P}\langle\sup\rangle(i,j)$

$\mathrm{inf_E}$: $\mathrm{F}i\wedge\mathrm{F}j\to\mathrm{F}\langle\inf\rangle(i,j)$

$\mathrm{sup_U}$: $k\wedge\langle\sup\rangle(i,j)\to\mathrm{HG}(\langle\sup\rangle(i,j)\to k)$

$\mathrm{inf_U}$: $k\wedge\langle\inf\rangle(i,j)\to\mathrm{HG}(\langle\inf\rangle(i,j)\to k)$

Rules

$$\mathrm{MP}\ \frac{\psi,\psi\to\varphi}{\varphi} \qquad \mathrm{NEC_{sup}}\ \frac{\vdash\varphi}{\vdash[\sup](\varphi,\psi)} \qquad \mathrm{NEC_{inf}}\ \frac{\vdash\varphi}{\vdash[\inf](\varphi,\psi)} \qquad \mathrm{USUB}\ \frac{\vdash\varphi(p,i)}{\vdash\varphi[\psi/p,j/i]}$$

Nominal-Rules

$$\mathrm{NAME}\ \frac{\vdash j\to\varphi}{\vdash\varphi} \qquad \mathrm{PASTE_{sup}}\ \frac{\vdash\mathrm{PF}(i\wedge\langle\sup\rangle(j,k))\wedge\mathrm{PF}(j\wedge\gamma)\wedge\mathrm{PF}(k\wedge\varphi)\to\psi}{\vdash\mathrm{PF}(i\wedge\langle\sup\rangle(\gamma,\varphi))\to\psi}$$

$$\mathrm{PASTE_{inf}}\ \frac{\vdash\mathrm{PF}(i\wedge\langle\inf\rangle(j,k))\wedge\mathrm{PF}(j\wedge\gamma)\wedge\mathrm{PF}(k\wedge\varphi)\to\psi}{\vdash\mathrm{PF}(i\wedge\langle\inf\rangle(\gamma,\varphi))\to\psi}$$

In NAME*, j does not occur in φ; in* $\mathrm{PASTE_{sup}}$ *and* $\mathrm{PASTE_{inf}}$*, j,k are distinct, $j\neq i$, $k\neq i$ and do not occur in γ,φ,ψ.*

Theorem 1 ([17]). $\mathbb{HLSI}_L$ *is sound and strongly complete with respect to the class of lattices* $\mathfrak{L}_t$.

From an algebraic perspective, there is another straightforward way to define the relational structure $\langle W, R_{\sup}, R_{\inf}\rangle$ as a lattice: $R_{\sup}$ and $R_{\inf}$ act as binary functions and satisfy the algebraic definitions of a lattice. Formally:

Definition 5. A relational structure $\mathcal{M}_t = \langle M, R_{\text{sup}}, R_{\text{inf}}\rangle$ is *functional* iff $R_{\text{sup}}, R_{\text{inf}}$ are binary functions, i.e., the following hold:

$$\begin{aligned} \text{FOFunE: } & \forall x \forall y \exists z \exists t (R_{\text{sup}} zxy \wedge R_{\text{inf}} txy) \\ \text{FOFunU}_{\text{sup}}\text{: } & \forall x \forall y \forall z \forall t (R_{\text{sup}} zxy \wedge R_{\text{sup}} txy \to z = t) \\ \text{FOFunU}_{\text{inf}}\text{: } & \forall x \forall y \forall z \forall t (R_{\text{inf}} zxy \wedge R_{\text{inf}} txy \to z = t) \end{aligned}$$

Denote the functional structure as $\mathcal{M}_f = \langle M, \curlyvee, \curlywedge\rangle$ where $\curlyvee := x \curlyvee y = z \iff R_{\text{sup}} zxy$ holds; $\curlywedge := x \curlywedge y = z \iff R_{\text{inf}} zxy$ holds.

Then lattice can be defined as follows:

Definition 6. A relational structure $\mathcal{M}_t = \langle W, R_{\text{sup}}, R_{\text{inf}}\rangle$ is called a *lattice* iff it is functional and satisfies the axioms listed in Def. 2 where $\curlyvee, \curlywedge$ are defined as before.

We use $\mathcal{L}_f = \langle L, R_{\text{sup}}, R_{\text{inf}}\rangle$ to denote such lattice structure and use $\mathfrak{L}_f$ to denote the class of them. In the remainder of this paper, when the context is clear, "lattice" refers to the lattice $\mathcal{L}_f$ defined by Def. 6.

It is not hard to check that Def. 3 and Def. 6 define the same class of structures, i.e., $\mathfrak{L}_t = \mathfrak{L}_f$. In [17], we proposed the hybrid polyadic logic over $\mathfrak{L}_t$. In Section 4, we will give the hybrid polyadic logic with global modality E over lattices $\mathfrak{L}_f$, which can be seen as an alternative modal axiomatization of lattices.

2.2 Semi-lattices

At first we give the standard definition of $\curlywedge$-semi-lattices:

Definition 7 ([7]). Call a partial order a relational structure $\mathcal{M} = \langle W, R\rangle$ a $\curlywedge$-*semi-lattice* if R is a partial order $\leq$ and:

$$\text{for all } a, b \in W, \text{ there exists } \max\{c \in W \mid c \leq a, c \leq b\}.$$

Denote $a \curlywedge b = \max\{c \in W \mid c \leq a, c \leq b\}$ as the *infimum* of $\{a, b\}$.

Denote the $\curlywedge$-semi-lattice as $\mathcal{L}_{\curlywedge}$. Similar to lattices, $\curlywedge$-semi-lattices also have an algebraic definition:

Definition 8 ([7]). Let $\curlywedge$ be a binary function symbol, and call the algebraic structure $\mathcal{M}_a = \langle W, \curlywedge\rangle$ a $\curlywedge$-*semi-lattice* if, for all $a, b, c \in W$:

$$\begin{aligned} a \curlywedge a &= a \\ a \curlywedge b &= b \curlywedge a \\ a \curlywedge (b \curlywedge c) &= (a \curlywedge b) \curlywedge c \end{aligned}$$

In a $\curlywedge$-semi-lattice, we can define the partial order $a \leq b :=$ there is c such that $a = b \curlywedge c$.

In the next section, we will discuss how to use the polyadic tense logic (with unary modality P and binary modality $\langle\text{inf}\rangle$) to characterize the class of $\curlywedge$-semi-lattices.

Remark 1. In [17], to characterize the lattice $\mathcal{L}_t = \langle L, R_{\text{sup}}, R_{\text{inf}}\rangle$, we use $\langle\text{sup}\rangle$ and $\langle\text{inf}\rangle$ as initial symbols in the language, and then define $\mathrm{P}\psi := \langle\text{sup}\rangle(\psi, \top)$ and $\mathrm{F}\psi := \langle\text{inf}\rangle(\psi, \top)$ to describe the partial order and its inverse relation within the lattice structure.

In $\curlywedge$-semi-lattice, we can define F by $\langle\text{inf}\rangle$ to describe the partial order as before. Moreover, we still need a modality to describe the inverse of the partial order, hence we need to add a new unary modality P to our language because we no longer have $\langle\text{sup}\rangle$ syntactically.

3 Hybrid Polyadic Modal Logic over Semi-lattices

Definition 9. Given a countable set of proposition letters **P**, a countable set of nominals **N** and binary modality $\langle\text{inf}\rangle$, the language of hybrid logic with past and inf (**HPI**) is defined by the following BNF grammar:

$$\varphi ::= p \in \mathbf{P} \mid i \in \mathbf{N} \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathrm{P}\varphi \mid \langle\text{inf}\rangle(\varphi, \varphi).$$

Define the following modalities:

$$[\text{inf}](\psi, \varphi) := \neg\langle\text{inf}\rangle(\neg\psi, \neg\varphi) \qquad \mathrm{F}\psi := \langle\text{inf}\rangle(\psi, \top)$$
$$\mathrm{H}\psi := \neg\mathrm{P}\neg\psi \qquad \mathrm{G}\psi := \neg\mathrm{F}\neg\psi$$

The Kripke model of **HPI** is $\mathcal{M}_{bt} = \langle W, R, R_{\text{inf}}, V\rangle$[1] where R_{inf} is a ternary relation in W and $V = V_{\mathbf{P}} \cup V_{\mathbf{N}}$, $V_{\mathbf{P}} : \mathbf{P} \to \mathcal{P}(W)$, $V_{\mathbf{N}} : \mathbf{N} \to W$. The Kripke semantics is defined as follows:

$\mathcal{M}, s \vDash i$	$\iff$	$s = V_{\mathbf{N}}(i)$
$\mathcal{M}, s \vDash \mathrm{P}\varphi$	$\iff$	there is $t \in W$ such that tRs and $\mathcal{M}, t \vDash \varphi$
$\mathcal{M}, s \vDash \langle\text{inf}\rangle(\varphi, \psi)$	$\iff$	there are $t, u \in W$ such that $R_{\text{inf}}stu$, $\mathcal{M}, t \vDash \varphi$ and $\mathcal{M}, u \vDash \psi$

The **HPI**-frame is denoted as $\mathcal{F}_{bt} = \langle W, R, R_{\text{inf}}\rangle$[2], and thus we can represent the **HPI**-model as $\mathcal{M}_{bt} = \langle \mathcal{F}_{bt}, V\rangle$. The definition of truth and validity in **HPI**-model or frame are as the same as in classical modal logic. We call an **HPI**-model $\mathcal{M}_t$ *named* if $V_{\mathbf{N}}$ is a surjection, meaning that every world in W has a name.

[1] Here, "bt" indicates that the model $\mathcal{M}$ includes both a binary relation and a ternary relation.

[2] Note that for a general relational frame $\langle W, R, R_{\text{inf}}\rangle$, R and R_{inf} may not be related. In a lattice, R can be defined by R_{inf}, and the modality P characterizes the inverse relation R^{-1} of R. This is also why the truth value of $\mathrm{P}\varphi$ is defined in this way.

Here we give the definition of $\wedge$-semi-lattices based on **HPI**-frame $\mathcal{F}_{bt} = \langle W, R, R_{\text{inf}}\rangle$:

Definition 10. A frame $\mathcal{F}_{bt} = \langle W, R, R_{\text{inf}}\rangle$ is called a $\wedge$-*semi-lattice* iff it satisfies the following axioms:

$$
\begin{aligned}
\text{FORef: } & \forall x(xRx) \\
\text{FOASym: } & \forall x\forall y(xRy \wedge yRx \to x = y) \\
\text{FOTrans: } & \forall x\forall y\forall z(xRy \wedge yRz \to xRz) \\
\text{FOSym: } & \forall x\forall y(xRy \leftrightarrow \exists z R_{\text{inf}}xyz) \\
\text{FOEi: } & \forall x\forall y\exists z(R_{\text{inf}}zxy) \\
\text{FOInfm: } & \forall x\forall y\forall z(R_{\text{inf}}zxy \to (zRx \wedge zRy \wedge \forall t(tRx \wedge tRy \to tRz)))
\end{aligned}
$$

In the following , we use $\mathcal{L}^{\wedge}_{bt} = \langle L, R, R_{\text{inf}}\rangle$ to denote such $\wedge$-semi-lattice structures and use $\mathfrak{L}^{\wedge}_{bt}$ to denote the class of them.

The goal of this section is to identify a suitable **HPI**-axiomatic system to characterize $\mathfrak{L}^{\wedge}_{bt}$. The method used is similar to that described in [17] for characterizing the class of lattices $\mathfrak{L}_t$. Before axiomatizing the hybrid polyadic modal logic over $\wedge$semi-lattices, we first define a class of *lower connected frames* and axiomatize it. The axiomatization of lattices will be based on such a weaker system. After that, we show that by adding *pure formulas* as axioms, we can obtain the desaired completeness theorem over semi-lattices ([8]).

3.1 Lower connected frame and lower connected system

Definition 11 (Lower connected frame)**.** An $\mathcal{F}_{bt} = \langle W, R, R_{\text{inf}}\rangle$ is called *lower connected* if it satisfies the follows:

- $\forall x(xRx)$;
- $\forall x\forall y\forall z(xRy \leftrightarrow \exists z R_{\text{inf}}xyz)$;
- $\forall x\forall y\exists z(zRx \wedge zRy)$;
- $\forall x\forall y\forall z(R_{\text{inf}}xyz \to R_{\text{inf}}xzy)$.

Denote the lower connected frame as $\mathcal{F}^{lc}_{bt}$. A lower connected model $\mathcal{M}^{lc}_{bt}$ is a model based on a lower connected frame. Let $\mathfrak{F}^{lc}_{bt}$ be the class of all lower connected $\mathcal{F}^{lc}_{bt}$.

A lower connected frame is much weaker than a lattice. Intuitively, it is a frame $\mathcal{F}_{bt}$ such that:

- R is reflexive;
- R can be defined by R_{inf} : xRy iff there is z such that $R_{\text{inf}}xyz$ holds;
- every two elements in the frame have a lower bound;
- R_{inf} is “symmetric” in the later two arguments.

From the definition of lower connected frame, it is easy to check:

Proposition 1. *For any* $\mathcal{F}^{lc}_{bt} \in \mathfrak{F}^{lc}_{bt}$, $s \in \mathcal{F}^{lc}_{bt}, \varphi \in$ **HPI** *and valuation* V*:*

$$\mathcal{F}^{lc}_{bt}, V, s \vDash \mathrm{F}\varphi \iff \textit{there is } t \in W \textit{ such that } sRt \textit{ and } \mathcal{F}^{lc}_{bt}, V, t \vDash \varphi.$$

Proof. Suppose $\mathcal{F}^{lc}_{bt}, V, s \vDash \mathrm{F}\varphi$. By the definition of F, there are $t, u \in W$ such that $R_{\inf}stu$ and $\mathcal{F}^{lc}_{bt}, V, t \vDash \varphi$. Since $\mathcal{F}^{lc}_{bt}$ is a lower connected frame, we have sRt.

Suppose there is $t \in W$ such that sRt and $\mathcal{F}^{lc}_{bt}, V, t \vDash \varphi$. Then there is u such that $R_{\inf}stu$ holds. Moreover, we have $\mathcal{F}^{lc}_{bt}, V, u \vDash \top$. By definition, $\mathcal{F}^{lc}_{bt}, V, s \vDash \langle\inf\rangle(\varphi, \top)$, which means $\mathcal{F}^{lc}_{bt}, V, s \vDash \mathrm{F}\varphi$. □

Remark 2. In such lower connected frames, the global modality E and the satisfaction operator $@_i$ ([12]) in hybrid logic can be defined using P and F. The intuitive idea is that in lower connected frame, any two points can be connected by there lower bound. Formally:

Proposition 2. *In a lower connected model* $\mathcal{M}^{lc}_{bt}$*, the globally existential modality* E *and hybrid modalities* $@_i$ *can be defined by* F, P*, i.e., for any* $s \in \mathcal{M}^{lc}_{bt}$ *and any* $\varphi \in$ **HPI***:*

$$\begin{array}{lll} \mathcal{M}^{lc}_{bt}, s \vDash \mathrm{PF}\varphi & \textit{iff} & \textit{for some } u, \mathcal{M}^{lc}_{bt}, u \vDash \varphi \\ \mathcal{M}^{lc}_{bt}, s \vDash \mathrm{PF}(i \wedge \varphi) & \textit{iff} & \mathcal{M}^{lc}_{bt}, u \vDash \varphi \textit{ for } V_{\mathbf{N}}(i) = u. \end{array}$$

Next we give an axiom system for the class of **HPI**-lower connected frames $\mathfrak{F}^{lc}_{bt}$. The lower connected **HPI**-system $\mathbb{HPIlc}$ is defined as follows:

Axioms

$$\begin{array}{rl} \mathrm{TAUT}: & \text{propositional tautologies} \\ \mathrm{Dual_H}: & \mathrm{P}p \leftrightarrow \neg\mathrm{H}\neg p \\ \mathrm{Dual_{inf}}: & \langle\inf\rangle(p, q) \leftrightarrow \neg[\inf](\neg p, \neg q) \\ \mathrm{K_H}: & \mathrm{H}(p \to q) \to (\mathrm{H}p \to \mathrm{H}q) \\ \mathrm{K_{inf}}: & [\inf](p \to q, r) \to ([\inf](p, r) \to [\inf](q, r)) \\ \mathrm{Ref_P}: & p \to \mathrm{P}p \\ \mathrm{Sym}: & (p \to \mathrm{GP}p) \wedge (p \to \mathrm{HF}p) \\ \mathrm{Com_{inf}}: & \langle\inf\rangle(p, q) \to \langle\inf\rangle(q, p) \\ \mathrm{Nom}: & \mathrm{PF}(i \wedge p) \leftrightarrow \mathrm{HG}(i \to p) \\ \mathrm{Tra}: & \mathrm{PFPF}(i \wedge p) \to \mathrm{PF}(i \wedge p) \end{array}$$

Rules

$$\mathrm{MP}\ \frac{\psi, \psi \to \varphi}{\varphi} \qquad \mathrm{NEC_H}\ \frac{\vdash \varphi}{\vdash \mathrm{H}\varphi} \qquad \mathrm{NEC_{inf}}\ \frac{\vdash \varphi}{\vdash [\inf](\varphi, \psi)} \qquad \mathrm{USUB}\ \frac{\vdash \varphi(p, i)}{\vdash \varphi[\psi/p, j/i]}$$

Nominal-Rules

$$\mathrm{NAME}\ \frac{\vdash j \to \varphi}{\vdash \varphi} \qquad \mathrm{PASTE_P}\ \frac{\vdash \mathrm{PF}(i \wedge \mathrm{P}j) \wedge \mathrm{PF}(j \wedge \varphi) \to \psi}{\vdash \mathrm{PF}(i \wedge \mathrm{P}\varphi) \to \psi}$$

$$\mathrm{PASTE_{inf}}\ \frac{\vdash \mathrm{PF}(i \wedge \langle\inf\rangle(j, k)) \wedge \mathrm{PF}(j \wedge \gamma) \wedge \mathrm{PF}(k \wedge \varphi) \to \psi}{\vdash \mathrm{PF}(i \wedge \langle\inf\rangle(\gamma, \varphi)) \to \psi}$$

In NAME*, j does not occur in φ; in* $\mathrm{PASTE_P}$*, j is distinct from i that does not occur in φ or ψ; in* $\mathrm{PASTE_{inf}}$*, j, k are distinct, not equal to i and do not occur in γ, φ, ψ.*

$\mathrm{Ref_P}$ indicates that the induced relation R is reflexive. Sym represents the standard tense axiom, capturing that R_P is the inverse of R. Nom can be divided into two parts: In $\mathrm{HG}(i \to p) \to \mathrm{PF}(i \wedge p)$, when substituting p with i, we obtain $\mathrm{PF}i$, which implies that every pair of points in the frame has a lower bound. The implication $\mathrm{PF}(i \wedge p) \to \mathrm{HG}(i \to p)$ demonstrates the property of nominals in hybrid logic: under the condition that every pair of points has a lower bound, two distinct points will not share the same name. Tra says that the lower bound is transitive: if a, b have a lower bound, and b, c have a lower bound, then a, c also have a lower bound.

Theorem 2. $\mathbb{HPILc}$ *is sound with respect to the class of lower connected frames* $\mathfrak{F}^{lc}_{bt}$.

Proof. To establish soundness, it is sufficient to demonstrate that all $\mathbb{HLSI}$-axioms are valid in $\mathfrak{F}^{lc}_{bt}$, and this validity is preserved under the rules. These validations are classical and straightforward. Readers may be unfamiliar with the axioms and rules involving hybrid logic, specifically Nom, NAME and PASTE. We will next verifyNom, NAME and $\mathrm{PASTE_{inf}}$ respectively. The case for $\mathrm{PASTE_P}$ is similar.

- Nom:
  - Assume $\mathcal{F}, w, V \vDash \mathrm{PF}(i \wedge p)$. Then there exist uRw, uRv such that $\mathcal{F}, v, V \vDash i \wedge p$. For all $u'Rw$ and for all $u'Rv'$, if $\mathcal{F}, v', V \vDash i$, then by the valuation of i, we have $v' = v$, and thus $\mathcal{F}, v', V \vDash i \wedge p$. Hence, $\mathcal{F}, w, V \vDash \mathrm{PF}(i \wedge p) \to \mathrm{HG}(i \to p)$.
  - Assume $\mathcal{F}, w, V \vDash \mathrm{HG}(i \to p)$. By the valuation of i, there must exist $v \in W$ such that $\mathcal{F}, v, V \vDash i$. Due to the lower connectivity of $\mathcal{F}$, there exists $u \in W$ such that uRw, uRv. Therefore, $\mathcal{F}, w, V \vDash \mathrm{PF}i$. Since $\mathcal{F}, w, V \vDash \mathrm{HG}(i \to p)$, we have $\mathcal{F}, w, V \vDash \mathrm{PF}(i \wedge p)$, which means $\mathcal{F}, w, V \vDash \mathrm{HG}(i \to p) \to \mathrm{PF}(i \wedge p)$.
- NAME: Assume $j \to \varphi$ is valid in $\mathcal{F}$. If φ is not valid in $\mathcal{F}$, then there exists a valuation V and a point $w \in W$ such that $w, V \nvDash \varphi$. Define a new valuation V' as $V' = V \backslash \{(j, V(j))\} \cup \{(j, w)\}$. Since j does not occur in φ, we have $w, V' \vDash \neg\varphi$. By definition, $w, V' \vDash j$ and then $w, V' \vDash k \to \varphi$, which leads to a contradiction. Therefore if $j \to \varphi$ is valid over any frame then φ is also valid over any frame.
- $\mathrm{PASTE_{inf}}$: Assume $\theta(j,k) = \mathrm{PF}(i \wedge \langle \mathrm{inf} \rangle (j,k)) \wedge \mathrm{PF}(j \wedge \gamma) \wedge \mathrm{PF}(k \wedge \varphi) \to \psi$ is valid in $\mathcal{F}$, but $\theta' = \mathrm{PF}(i \wedge \langle \mathrm{sup} \rangle (\gamma, \varphi)) \to \psi$ is not. Then there exists a valuation V and $w \in W$ such that $w, V \vDash \neg\theta'$, which means $w, V \vDash \mathrm{PF}(i \wedge \langle \mathrm{inf} \rangle (\gamma, \varphi))$ and $w, V \vDash \neg\psi$. There are uRw, uRt such that $t, V \vDash i \wedge \langle \mathrm{inf} \rangle (\gamma, \varphi)$. Assuming $R_{\mathrm{inf}}txy$ and $x, V \vDash \gamma$, $y, V \vDash \varphi$, we can construct a new valuation V' such that $V'(j) = x$, $V'(k) = y$, and $V'(m) = V(m)$ for all $m \neq j, m \neq k$. Since j, k are distinct, $j \neq i$, $k \neq i$ and do not occur in γ, φ, ψ, it can be easily checked that $w, V' \vDash \neg\theta$, which leads to a contradiction. □

To prepare for the completeness proof, we present some lemmas about theorems of $\mathbb{HPIIc}$.

Lemma 1. G *is normal modality, i.e., the following formulas are theorems in* $\mathbb{HPIIc}$ *and the rule is derivable:*

$$\mathrm{DUAL_G}: \mathrm{F}p \leftrightarrow \neg\mathrm{G}\neg p$$
$$\mathrm{K_G}: \mathrm{G}(p \to q) \to (\mathrm{G}p \to \mathrm{G}q)$$
$$\mathrm{NEC_G}: \frac{\vdash \varphi}{\vdash \mathrm{G}\varphi}$$

Lemma 2. *The following formulas are theorems in* $\mathbb{HPIIc}$*:*

$\mathrm{Ref_F}$	$p \to \mathrm{F}p$
Eli	$i \wedge \mathrm{PF}(i \wedge p) \to p$
Con	$\mathrm{PF}i$
Uni	$\mathrm{PF}(i \wedge j) \wedge \mathrm{PF}(j \wedge p) \to \mathrm{PF}(i \wedge p)$
Agr	$\mathrm{PF}(j \wedge \mathrm{PF}(i \wedge p)) \leftrightarrow \mathrm{PF}(i \wedge p)$
$\mathrm{Bri_F}$	$\mathrm{F}i \wedge \mathrm{PF}(i \wedge p) \to \mathrm{F}(i \wedge p)$
$\mathrm{Bri_P}$	$\mathrm{P}i \wedge \mathrm{PF}(i \wedge p) \to \mathrm{P}(i \wedge p)$
$\mathrm{Bri_{inf}}$	$\langle\mathrm{inf}\rangle(i, j) \wedge \mathrm{PF}(i \wedge p) \wedge \mathrm{PF}(j \wedge q) \to \langle\mathrm{inf}\rangle(i \wedge p, j \wedge q)$

Proof. Below HS means hypothetical syllogism, ES stands for equivalent substitution while TAUT stands for all propositional and modal tautologies.

- $\mathrm{Ref_F}$

(1)	$\mathrm{H}p \to p$	$\mathrm{Ref_P}$
(2)	$\mathrm{HF}p \to \mathrm{F}p$	USUB (1)
(3)	$p \to \mathrm{HF}p$	Sym
(4)	$p \to \mathrm{F}p$	HS(3)(2)

- Eli

(1)	$i \wedge \mathrm{PF}(i \wedge p) \to i \wedge \mathrm{HG}(i \to p)$	Nom
(2)	$(i \wedge \mathrm{HG}(i \to p)) \to i \wedge (i \to p)$	Ref
(3)	$i \wedge \mathrm{PF}(i \wedge p) \to p$	HS(1)(2)

- Con

(1)	$\mathrm{HG}(i \to i) \to \mathrm{PF}(i \wedge i)$	Nom
(2)	$\mathrm{PF}i$	TAUT

- Uni

(1)	$\mathrm{PF}(j \wedge i) \to \mathrm{HG}(j \to i)$	Nom
(2)	$\mathrm{HG}(j \to i) \wedge \mathrm{PF}(j \wedge p) \to \mathrm{PF}(j \wedge p \wedge i)$	TAUT
(3)	$\mathrm{PF}(j \wedge p \wedge i) \to \mathrm{PF}(i \wedge p)$	TAUT
(4)	$\mathrm{PF}(i \wedge j) \wedge \mathrm{PF}(j \wedge p) \to \mathrm{PF}(i \wedge p)$	HS(1)(2)(3)

- Agr(→)

(1)	$\mathrm{PF}(j \wedge \mathrm{PF}(i \wedge p)) \to \mathrm{HG}(j \to \mathrm{PF}(i \wedge p)) \wedge \mathrm{PF}j$	Nom and Con
(2)	$\mathrm{HG}(j \to \mathrm{PF}(i \wedge p)) \wedge \mathrm{PF}j \to \mathrm{PFPF}(i \wedge p)$	TAUT
(3)	$\mathrm{PFPF}(i \wedge p) \to \mathrm{PF}(i \wedge p)$	Tra
(4)	$\mathrm{PF}(j \wedge \mathrm{PF}(i \wedge p)) \to \mathrm{PF}(i \wedge p)$	HS(1)(2)(3)

- $\mathrm{Agr}(\leftarrow)$

(1)	$\mathrm{PFPF}(i \wedge \neg p) \rightarrow \mathrm{PF}(i \wedge \neg p)$	Tra
(2)	$\mathrm{HG}(i \rightarrow p) \rightarrow \mathrm{HGHG}(i \rightarrow p)$	TAUT of (1)
(3)	$\mathrm{PF}(i \wedge p) \rightarrow \mathrm{HGPF}(i \wedge p)$	Nom and (2), ES
(4)	$\mathrm{HGPF}(i \wedge p) \rightarrow \mathrm{PF}j \wedge \mathrm{HGPF}(i \wedge p)$	Con
(5)	$\mathrm{PF}j \wedge \mathrm{HGPF}(i \wedge p) \rightarrow \mathrm{PF}(j \wedge \mathrm{PF}(i \wedge p))$	TAUT
(6)	$\mathrm{PF}(i \wedge p) \rightarrow \mathrm{PF}(j \wedge \mathrm{PF}(i \wedge p))$	HS(3)(4)(5)

- $\mathrm{Bri}_{\mathrm{F}}$

(1)	$\mathrm{PF}(i \wedge p) \rightarrow \mathrm{HG}(i \rightarrow p)$	Nom
(2)	$\mathrm{HG}(i \rightarrow p) \rightarrow \mathrm{G}(i \rightarrow p)$	Ref
(3)	$\mathrm{F}i \wedge \mathrm{PF}(i \wedge p) \rightarrow \mathrm{F}i \wedge \mathrm{G}(i \rightarrow p)$	HS(1)(2), TAUT
(4)	$\mathrm{F}i \wedge \mathrm{G}(i \rightarrow p) \rightarrow \mathrm{F}(i \wedge p)$	TAUT
(5)	$\mathrm{F}i \wedge \mathrm{PF}(i \wedge p) \rightarrow \mathrm{F}(i \wedge p)$	HS(3)(4)

- $\mathrm{Bri}_{\mathrm{P}}$

(1)	$\mathrm{PF}(i \wedge p) \rightarrow \mathrm{HG}(i \rightarrow p)$	Nom
(2)	$\mathrm{HG}(i \rightarrow p) \rightarrow \mathrm{H}(i \rightarrow p)$	Ref
(3)	$\mathrm{P}i \wedge \mathrm{PF}(i \wedge p) \rightarrow \mathrm{P}i \wedge \mathrm{H}(i \rightarrow p)$	HS(1)(2), TAUT
(4)	$\mathrm{P}i \wedge \mathrm{H}(i \rightarrow p) \rightarrow \mathrm{P}(i \wedge p)$	TAUT
(5)	$\mathrm{P}i \wedge \mathrm{PF}(i \wedge p) \rightarrow \mathrm{P}(i \wedge p)$	HS(3)(4)

- $\mathrm{Bri}_{\mathrm{inf}}$

(1)	$\mathrm{PF}(i \wedge p) \rightarrow \mathrm{H}(i \rightarrow p)$	Nom and Ref
(2)	$\mathrm{PF}(j \wedge q) \rightarrow \mathrm{H}(j \rightarrow q)$	USUB (1)
(3)	$\mathrm{H}(i \rightarrow p) \rightarrow [\mathrm{inf}](i \rightarrow p, \bot)$	Definition
(4)	$\mathrm{H}(j \rightarrow q) \rightarrow [\mathrm{inf}](j \rightarrow q, \bot)$	USUB (3)
(5)	$[\mathrm{inf}](j \rightarrow q, \bot) \rightarrow [\mathrm{inf}](\bot, j \rightarrow q)$	$\mathrm{Com}_{\mathrm{inf}}$
(6)	$\langle\mathrm{inf}\rangle(i, j) \wedge \mathrm{PF}(i \wedge p) \wedge \mathrm{PF}(j \wedge q) \rightarrow$ $\langle\mathrm{inf}\rangle(i, j) \wedge [\mathrm{inf}](i \rightarrow p, \bot) \wedge [\mathrm{inf}](\bot, j \rightarrow q)$	HS(1)(2)(3)(4)(5)
(7)	$\bot \rightarrow \psi$	TAUT
(8)	$[\mathrm{inf}](\varphi, \bot \rightarrow \psi)$	$\mathrm{NEC}_{\mathrm{inf}}$
(9)	$[\mathrm{inf}](\varphi, \bot) \wedge [\mathrm{inf}](\varphi, \bot \rightarrow \psi) \rightarrow [\mathrm{inf}](\varphi, \psi)$	$\mathrm{K}_{\mathrm{inf}}$
(10)	$[\mathrm{inf}](\varphi, \bot) \rightarrow [\mathrm{inf}](\varphi, \psi)$	HS(8)(9)
(11)	$\langle\mathrm{inf}\rangle(i, j) \wedge [\mathrm{inf}](i \rightarrow p, \neg j) \rightarrow \langle\mathrm{inf}\rangle(i \wedge p, j)$	TAUT of $\mathrm{K}_{\mathrm{inf}}$
(12)	$\langle\mathrm{inf}\rangle(i, j) \wedge [\mathrm{inf}](i \rightarrow p, \bot) \rightarrow \langle\mathrm{inf}\rangle(i \wedge p, j)$	HS(10)(11)
(13)	$\langle\mathrm{inf}\rangle(i \wedge p, j) \wedge [\mathrm{inf}](\bot, j \rightarrow q) \rightarrow \langle\mathrm{inf}\rangle(i \wedge p, j \wedge q)$	Similar to (12)
(14)	$\langle\mathrm{inf}\rangle(i, j) \wedge [\mathrm{inf}](i \rightarrow p, \bot) \wedge [\mathrm{inf}](\bot, j \rightarrow q) \rightarrow$ $\langle\mathrm{inf}\rangle(i \wedge p, j \wedge q)$	HS(12)(13)
(15)	$\langle\mathrm{inf}\rangle(i, j) \wedge \mathrm{PF}(i \wedge p) \wedge \mathrm{PF}(j \wedge q) \rightarrow \langle\mathrm{inf}\rangle(i \wedge p, j \wedge q)$	HS(6)(14) □

In contrast to the canonical model used to prove the completeness of standard modal logic, in $\mathbb{HPII}\mathbb{c}$, we require only *one* maximal consistent set generated from a consistent set that contains sufficient information to prove completeness ([8, 17]).

Lemma 3. *Let* Γ *be an* $\mathbb{HPIIc}$*-MCS. For all nominals* i*, let* $\Delta_i = \{\psi \mid \mathrm{PF}(i \wedge \psi) \in \Gamma\}$*. Then:*

1. *For all* i*,* Δ_i *is an* $\mathbb{HPIIc}$*-MCS that contains* i*.*
2. *For all* i, j*, if* $i \in \Delta_j$ *then* $\Delta_j = \Delta_i$*.*
3. *If* $k \in \Gamma$*, then* $\Gamma = \Delta_k$*.*

Proof. For any $\mathbb{HPIIc}$-MCS Γ:

1. For all nominals i, we have $i \in \Delta_i$ by Con. Next, we show that Δ_i is consistent. If not, then there are $\psi_1, ..., \psi_n \in \Delta_i$ such that $\vdash \neg(\psi_1 \wedge ... \wedge \psi_n)$. Hence $\vdash i \to \neg(\psi_1 \wedge ... \wedge \psi_n)$. By $\mathrm{NEC_G}$ and $\mathrm{NEC_H}$, we have $\mathrm{HG}(i \to \neg(\psi_1 \wedge ... \wedge \psi_n)) \in \Gamma$. Hence $\neg\mathrm{PF}(i \wedge \psi_1 \wedge ... \wedge \psi_n) \in \Gamma$ by $\mathrm{DUAL_u}$. However, since $\psi_1, ..., \psi_n \in \Delta_i$, we have $\mathrm{PF}(i \wedge \psi_1 \wedge ... \wedge \psi_n) \in \Gamma$, which is a contradiction.
   If Δ_i is not maximal, then there is ψ such that $\psi \notin \Delta_i, \neg\psi \notin \Delta_i$. So $\neg\mathrm{PF}(i \wedge \psi) \in \Gamma$, $\neg\mathrm{PF}(i \wedge \neg\psi) \in \Gamma$. However, $\neg\mathrm{PF}(i \wedge \neg\psi) \in \Gamma$ implies $\mathrm{HG}(i \to \psi) \in \Gamma$ due to $\mathrm{DUAL_u}$. Hence by Nom we have $\mathrm{PF}(i \wedge \psi) \in \Gamma$, which is a contradiction.
2. Assume $i \in \Delta_j$, we need to prove $\Delta_j = \Delta_i$. $i \in \Delta_j$ implies $\mathrm{PF}(i \wedge j) \in \Gamma$. If $\psi \in \Delta_j$, then $\mathrm{PF}(j \wedge \psi) \in \Gamma$. By Uni, we get $\mathrm{PF}(i \wedge \psi) \in \Gamma$ and thus $\psi \in \Delta_i$. Hence $\Delta_j \subseteq \Delta_i$. Similarly, we can prove $\Delta_i \subseteq \Delta_j$.
3. Assume $k \in \Gamma$. For all $\psi \in \Gamma$, by Ref we have $k \wedge \psi \in \Gamma$ implies $\mathrm{PF}(k \wedge \psi) \in \Gamma$. Hence $\psi \in \Delta_k$. Conversely, for all $\psi \in \Delta_k$, we have $\mathrm{PF}(k \wedge \psi) \in \Gamma$. So by $k \in \Gamma$ and Eli we have $\psi \in \Gamma$. □

Here we require that the maximal consistent set satisfies additional properties in [8]:

Definition 12. An $\mathbb{HPIIc}$-MCS Γ is *named* if there exists $i \in \mathbf{N}$ such that $i \in \Gamma$. An $\mathbb{HPIIc}$-MCS Γ is *pasted* if it satisfies the following conditions:

- If $\mathrm{PF}(i \wedge \mathrm{P}\psi) \in \Gamma$, then there exists nominal j such that $\mathrm{PF}(i \wedge \mathrm{P}j) \wedge \mathrm{PF}(j \wedge \psi) \in \Gamma$;
- If $\mathrm{PF}(i \wedge \langle \mathrm{inf} \rangle(\psi, \varphi)) \in \Gamma$, then there exist nominals j and k such that $\mathrm{PF}(i \wedge \langle \mathrm{inf} \rangle(j, k)) \wedge \mathrm{PF}(j \wedge \psi) \wedge \mathrm{PF}(k \wedge \varphi) \in \Gamma$.

If Γ is both named and pasted, it is called the *characteristic maximal consistent set* (CMCS).

The CMCS induces all the relevant named MCS, and the named model can be obtained by connecting them with classical canonical relations, which is used to prove the $\mathbb{HPIIc}$ completeness theorem.

Definition 13. Let Γ be an $\mathbb{HPIIc}$-CMCS, and let i be a nominal, we call the set $\{\psi \mid \mathrm{PF}(i \wedge \psi) \in \Gamma\}$ a *named MCS* (NMCS for short) induced by Γ. We define the characteristic canonical model as $\mathcal{M}^\Gamma = \langle W^\Gamma, R^\Gamma, R^\Gamma_{\mathrm{inf}}, V^\Gamma \rangle$, where:

- W^Γ is the set of all NMCSs induced by Γ, i.e., $W^\Gamma = \{\{\psi \mid \mathrm{PF}(i \wedge \psi) \in \Gamma\} \mid i$ appears in $\Gamma\}$.
- For all MCNSs $w, w_1 \in W^\Gamma$, $wR^\Gamma w_1$ iff for all formula ψ, $\psi \in w$ implies $\mathrm{P}\psi \in w_1$.
- For all MCNSs $w, w_1, w_2 \in W^\Gamma$, $R^\Gamma_{\text{inf}} w w_1 w_2$ iff for all formulas ψ_1, ψ_2: $\psi_1 \in w_1$ and $\psi_2 \in w_2$ implies $\langle\text{inf}\rangle(\psi_1, \psi_2) \in w$.
- $p \in V^\Gamma(w)$ iff $p \in w$.

By Lem. 3, it is easy to verify that $\mathcal{M}^\Gamma$ is indeed a named model. Next we should find a way to construct a CMCS from any consistent set. The following generalization of Lindenbaum's lemma demonstrates that any consistent set can be extended to a CMCS in the language that adds a countably infinite set of new nominals.

Lemma 4 (Characteristic Lindenbaum's Lemma)**.** *Let* **M** *be a countably infinite set of nominals that do not intersect with* **N**. *Let* **HPI**$^+$ *be the extension of* **HPI** *that uses* **M** ∪ **N** *as the nominal set. Then each* $\mathbb{HPIIc}$*-consistent sets in* **HPI** *can be extended to an* $\mathbb{HPIIc}$*-CMCS in* **HPI**$^+$.

Proof. First, we enumerate **M**. For a given $\mathbb{HLSI}$-consistent set Σ, let $\Sigma_k = \Sigma \cup \{k\}$, where k is the minimal element in the enumeration. Σ_k is consistent. If not, then there is a conjunction of finite formulas θ in Σ such that $\vdash k \to \neg\theta$. Since $k \in$ **M** and does not occur in θ, we have $\vdash \neg\theta$ by NAME, leading to a contradiction.

Enumerate all formulas in **HPI**$^+$ as $\{\psi_1, \psi_2, \ldots\}$. Define $\Sigma^0 = \Sigma_k$. If Σ^m is defined, then define Σ^{m+1} as follows: if $\Sigma^m \cup \{\psi_{m+1}\}$ is inconsistent, let $\Sigma^{m+1} = \Sigma^m$. Otherwise:

- Let $\Sigma^{m+1} = \Sigma^m \cup \{\psi_{m+1}\} \cup \{\mathrm{PF}(i \wedge \mathrm{P}j) \wedge \mathrm{PF}(j \wedge \psi)\}$, if ψ_{m+1} has the form $\mathrm{PF}(i \wedge \mathrm{P}\psi)$ (where j is the distinct minimal elements in the enumeration of **M** that does not occur in both Σ^m and $\mathrm{PF}(i \wedge \mathrm{P}\psi)$).
- Let $\Sigma^{m+1} = \Sigma^m \cup \{\psi_{m+1}\} \cup \{\mathrm{PF}(i \wedge \langle\text{inf}\rangle(j,k)) \wedge \mathrm{PF}(j \wedge \gamma) \wedge \mathrm{PF}(k \wedge \varphi)\}$, if ψ_{m+1} has the form $\mathrm{PF}(i \wedge \langle\text{inf}\rangle(\gamma, \varphi))$ (where j and k are the distinct minimal elements in the enumeration of **M** that do not occur in both Σ^m and $\mathrm{PF}(i \wedge \langle\text{inf}\rangle(\gamma, \varphi)))$.
- Otherwise, let $\Sigma^{m+1} = \Sigma^m \cup \{\psi_{m+1}\}$.

Let $\Sigma^+ = \bigcup_{0 \le n} \Sigma^n$. Evidently, Σ^+ contains a nominal k and is maximal. It is also pasted according to the definition of Σ^m. If we can prove that Σ^+ is also consistent, then it is the $\mathbb{HPIIc}$-CMCS in **HPI**$^+$.

The only non-trivial case arises when we add $\mathrm{PF}(i \wedge \mathrm{P}\psi)$ or $\mathrm{PF}(i \wedge \langle\text{inf}\rangle(\psi, \varphi))$ to Σ^m. Consider the $\langle\text{inf}\rangle$-condition; the P-condition is similar. If Σ^{m+1} is inconsistent, then there exists a conjunction of finite formulas θ in Σ^{m+1} such that $\vdash \mathrm{PF}(i \wedge \langle\text{inf}\rangle(j,k)) \wedge \mathrm{PF}(j \wedge \gamma) \wedge \mathrm{PF}(k \wedge \varphi) \to \neg\theta$. By $\mathrm{PASTE}_{\text{inf}}$, we have $\vdash \mathrm{PF}(i \wedge \langle\text{inf}\rangle(\psi, \varphi)) \to \neg\theta$, which contradicts the consistency of $\Sigma^m \cup \{\mathrm{PF}(i \wedge \langle\text{inf}\rangle(\psi, \varphi))\}$. □

Lemma 5 (Characteristic Existence Lemma). *Let* Γ *be an* $\mathbb{HLSI}$*-CMCS, and let* $\mathcal{M}^\Gamma = \langle W, R, R_{\text{inf}}, V\rangle$ *be the induced characteristic canonical model as in Def. 13. Then:*

- *For all* $u \in W$ *and all* $\mathrm{P}\psi \in u$*, there exist* $v \in W$ *such that* vRu *and* $\psi \in v$*.*
- *For all* $u \in W$ *and all* $\langle\text{inf}\rangle(\psi, \varphi) \in u$*, there exist* $v, w \in W$ *such that* $R_{\text{inf}}uvw$ *and* $\psi \in v, \varphi \in w$*.*

Proof. Again we first prove the binary $\langle\text{inf}\rangle$-condition of existence lemma. The unary P-condition is similar.

Let $u \in W$ and $\langle\text{inf}\rangle(\psi, \varphi) \in u$. By definition, there is an i such that $u = \Delta_i$. So, by $\langle\text{inf}\rangle(\psi, \varphi) \in \Delta_i$, we have $\mathrm{PF}(i \wedge \langle\text{inf}\rangle(\psi, \varphi)) \in \Gamma$. Since Γ is pasted, there are nominals j, k such that $\mathrm{PF}(i \wedge \langle\text{inf}\rangle(j, k)) \wedge \mathrm{PF}(j \wedge \psi) \wedge \mathrm{PF}(k \wedge \varphi) \in \Gamma$. Hence, we have $\langle\text{inf}\rangle(j, k) \in \Delta_i$, $\psi \in \Delta_j$, and $\varphi \in \Delta_k$. Now, we need to prove $R_{\text{inf}}\Delta_i\Delta_j\Delta_k$.

Assume $\varphi \in \Delta_j$ and $\gamma \in \Delta_k$. By definition, we have $\mathrm{PF}(j \wedge \varphi) \wedge \mathrm{PF}(k \wedge \gamma) \in \Gamma$. By Agr and Nom, we have $\mathrm{PF}(i \wedge \mathrm{PF}(j \wedge \varphi) \wedge \mathrm{PF}(k \wedge \gamma)) \in \Gamma$. So, by definition, $\mathrm{PF}(j \wedge \varphi) \wedge \mathrm{PF}(k \wedge \gamma) \in \Delta_i$. By $\langle\text{inf}\rangle(j, k) \in \Delta_i$ and $\mathrm{Bri}_{\text{inf}}$, we get $\langle\text{inf}\rangle(\varphi, \gamma) \in \Delta_i$. Hence, $R_{\text{inf}}\Delta_i\Delta_j\Delta_k$. □

By using the Characteristic Existence Lemma, we can prove the Characteristic Truth Lemma using standard methods.

Lemma 6 (Characteristic Truth Lemma). *Let* Γ *be an* $\mathbb{HLSI}$*-CMCS, and let* $\mathcal{M}^\Gamma = \langle W, R, R_{\text{inf}}, V\rangle$ *be the induced characteristic canonical model as in Def. 13. For all* $u \in W$ *and all* **HLSI**$^+$*-formula* ψ*, we have:*

$$\psi \in w \iff \mathcal{M}^\Gamma, w \vDash \psi.$$

Finally, we can prove the completeness theorem for $\mathbb{HPIlc}$ over lower connected frames.

Theorem 3. *Every* $\mathbb{HPIlc}$*-consistent set in* **HPI** *is satisfiable in a countable characteristic model* $\mathcal{M} = \langle W, R\ , R_{\text{inf}}, V\rangle$*, and the frame* $\mathcal{F} = \langle W, R, R_{\text{inf}}\rangle$ *is a lower connected frame.*

Proof. Given an $\mathbb{HPIlc}$-consistent set Σ in **HPI**, we extend it to the countable CMCS Σ^+ in **HPI**$^+$. Let $\mathcal{M}^{\Sigma^+} = \langle W, R_{\text{sup}}, R_{\text{inf}}, V\rangle$ be the induced characteristic canonical model. By Lem. 3, we have $\Sigma^+ \in W$. According to the Truth Lemma (Lem. 6), $\mathcal{M}^{\Sigma^+}, \Sigma^+ \vDash \Sigma$. Since all worlds in $\mathcal{M}^{\Sigma^+}$ are named by the nominals in **HPI**$^+$ and **HPI**$^+$ has countably many nominals, we conclude that $\mathcal{M}^{\Sigma^+}$ is countable.

Next, we show that the frame of $\mathcal{M}^{\Sigma^+}$ is a lower connected frame (see Def. 11):

- $\forall x(xRx)$: Standard argument by using Ref_P.
- $\forall x\forall y\forall z(xRy \leftrightarrow \exists z R_{\text{inf}}xyz)$: For all $x, y \in W$, if xRy, let us assume $i \in y$. Then $i \to \mathrm{HF}i \in y$ by Sym, and thus $\mathrm{HF}i \in y$. By the definition of canonical

relation, $\mathrm{F}i \in x$, which means $\langle \inf \rangle(i, \top) \in x$. According to Lem. 5, there are y', z such that $R_{\sup}xy'z$ and $i \in y'$. By $i \in y$, we have $y' = y$ by Lem. 3.
On the contrary, suppose there is z s.t. $R_{\inf}xyz$ hold. Let us assume $j \in x$. Then $j \to \mathrm{GP}j \in x$ by Sym, and thus $\mathrm{GP}j \in x$. It implies $\mathrm{F}\neg\mathrm{P}j \notin x$ since x is a MCS. It is equivalent to $\langle \inf \rangle(\neg\mathrm{P}j, \top) \notin x$. By the definition of canonical relation and $R_{\inf}xyz$, we have $\neg\mathrm{P}j \notin y$ and hence $\mathrm{P}j \in y$. According to Lem. 5, there is x' s.t. $x'Ry$ and $j \in x'$. By $j \in x$, we have $x' = x$ by Lem. 3.

- $\forall x \forall y \exists z(zRx \wedge zRy)$: For each $x, y \in W$, assume $j \in y$ as y is named. By Con, $\mathrm{PF}j \in x$. Thus according to Lem. 5, there is z s.t. zRx and $\mathrm{F}j \in z$, i.e., $\langle \inf \rangle(\mathrm{F}j, \top) \in z$. Again by Lem. 5, there are y', t s.t. $R_{\inf}zy't$ and $j \in y'$. Note that $j \in y$, so $y' = y$ and thus $R_{\inf}zyt$ holds. We have just proved $R_{\inf}zyt$ implies zRy.
- $\forall x \forall y \forall z(R_{\inf}xyz \to R_{\inf}xzy)$: Standard argument by using $\mathrm{Com}_{\inf}$. □

3.2 Completeness theorem for $\curlywedge$-semi-lattices

In this section, we introduce the nominal polyadic modal logic over $\curlywedge$-semi-lattices. The key point is to incorporate *pure formulas* for describing semi-lattice properties.

A formula in **HLSI** is referred to as a *pure formula* if it does not contain any propositional variables. Unlike modal formula, each pure formula has the first-order frame correspondence (FOC). If we add pure formulas as extra axioms to the hybrid system, the completeness result can be obtained directly ([8, 17]):

Theorem 4. *Let* Π *be a set of* $\mathbb{HPIIc}$*-consistent pure formulas, and consider the extension* $\mathbb{HPIIc}_\Pi = \mathbb{HPIIc} + \Pi$. *Then, any* $\mathbb{HPIIc}_\Pi$*-consistent set is satisfiable in a characteristic model that satisfies the FOC of* Π.

It is easty to check all $\curlywedge$-semi-lattices (see Def. 10) are lower connected. Based on lower connected frames, we give the extra axioms and their FOCs for meet semi-lattices:

Axiom:	**FOC**
4_F: $\mathrm{FF}i \to \mathrm{F}i$	$\forall x \forall y \forall z(xRy \wedge yRz \to xRz)$
Asym: $i \to \mathrm{G}(\mathrm{F}i \to i)$	$\forall x \forall y(xRy \wedge yRx \to x = y)$
$\inf_\mathrm{E}$: $\mathrm{F}i \wedge \mathrm{F}j \to \mathrm{F}\langle \inf \rangle(i, j)$	$\forall x \forall y \forall z(xRy \wedge xRz) \to \exists t(xRt \wedge R_{\inf}tyz)$
$\inf_\mathrm{U}$: $k \wedge \langle \inf \rangle(i, j) \to \mathrm{HG}(\langle \inf \rangle(i, j) \to k)$	$\forall x \forall y \forall z(R_{\inf}xyz \to \forall t(R_{\inf}tyz \to t = x))$

4_F, Asym and Ref in $\mathbb{HPIIc}$ shows that R is a partial order; $\inf_\mathrm{E}$ says that if x is the lower bound of y, z, then there is t s.t. xRt and $R_{\inf}tyz$ holds; $\inf_\mathrm{U}$ means that the uniqueness of $R_{\inf}$ relation: $R_{\inf}xyz$ and $R_{\inf}x'yz$ implies $x = x'$. It is easy to varify:

Lemma 7. *For any* **HLSI**-*lower connected frame* $\mathcal{F}^{lc}_{bt}$, *if* $\{4_\mathrm{F}, \mathrm{Asym}, \inf_\mathrm{E}, \inf_\mathrm{U}\}$ *is valid in* $\mathcal{F}_{bt}$ *then* $\mathcal{F}^{lc}_{t}$ *is a* $\curlywedge$*-semi-lattice in Def.10.*

Let $\mathbb{HPI}_{\wedge} = \mathbb{HPIIc} + \{4_{\mathrm{F}} + \mathrm{Asym} + \mathrm{inf}_{\mathrm{E}} + \mathrm{inf}_{\mathrm{U}}\}$. Based on Thm. 3, since all axioms we add are pure, by Thm. 4 and Lem. 7, we can get: Since all axioms we add are pure, we have:

Theorem 5. $\mathbb{HPI}_{\wedge}$ *is sound and strongly complete with respect to the class of* $\wedge$*-semi-lattices* $\mathfrak{L}_{bt}^{\wedge}$.

4 Hybrid Polyadic Modal Logic over Functional Lattices

In Section 2, we introduced how to use polyadic hybrid logic $\mathbb{HLSI}_L$ to characterize the class of lattice frames $\mathfrak{L}_t$ (see Def. 3) , and presented another more algebraic definition of lattices as relational structures, $\mathfrak{L}_f$ (see Def. 6). In this section, we will give the polyadic hybrid logic for $\mathfrak{L}_f$.

In constructing $\mathbb{HLSI}_L$, the key idea is to use $\langle\sup\rangle$ and $\langle\inf\rangle$ to define the unary modalities P and F, and then use the bundled modality PF as the global modality E in lower connected frames, see [17]. The same approach was also applied in Section 3 for constructing $\mathbb{HPIIc}$ to characterize semi-lattices, as shown in Prop. 2. From the definition of $\mathcal{L}_f$, it is evident that the binary partial order relation is not present, so we no longer introduce the defined unary modalities P and F into the language. As a consequence, we need to add the global modality E into our language.

Definition 14. Given a countable set of proposition letters **P**, a countable set of nominals **N**, an unary modality E and binary modalities $\langle\sup\rangle$, $\langle\inf\rangle$, the language of hybrid logic with sup, inf and E (**HLSIE**) is defined by the following BNF grammar:

$$\varphi ::= p \in \mathbf{P} \mid i \in \mathbf{N} \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathrm{E}\varphi \mid \langle\sup\rangle(\varphi,\varphi) \mid \langle\inf\rangle(\varphi,\varphi).$$

Define the following modalities:

$$[\inf](\psi,\varphi) := \neg\langle\inf\rangle(\neg\psi,\neg\varphi)$$
$$[\sup](\psi,\varphi) := \neg\langle\sup\rangle(\neg\psi,\neg\varphi)$$
$$\mathrm{A}\psi := \neg\mathrm{E}\neg\psi$$

The Kripke model $\mathcal{M}_t = \langle W, R_{\sup}, R_{\inf}, V\rangle$ and Kripke semantics of **HLSIE** are standard ([12]):

$\mathcal{M}, s \vDash \mathrm{E}\varphi$	$\iff$	there is $t \in W$ such that $\mathcal{M}, t \vDash \varphi$
$\mathcal{M}, s \vDash \langle\sup\rangle(\varphi,\psi)$	$\iff$	there are $t, u \in W$ such that $R_{\sup}stu$, $\mathcal{M}, t \vDash \varphi$ and $\mathcal{M}, u \vDash \psi$
$\mathcal{M}, s \vDash \langle\inf\rangle(\varphi,\psi)$	$\iff$	there are $t, u \in W$ such that $R_{\inf}stu$, $\mathcal{M}, t \vDash \varphi$ and $\mathcal{M}, u \vDash \psi$

Definition 15. The Hybrid polyadic logic $\mathbb{HSIE}$ has the following axioms $\{1\text{–}10\}$ and all following rules; the Hybrid polyadic logic for lattices $\mathbb{HSIE}_L = \mathbb{HSIE} + \{11\text{–}21\}$.

1. TAUT
2. $\mathrm{Dual}_{\mathrm{sup}}, \mathrm{Dual}_{\mathrm{inf}}, \mathrm{Dual}_{\mathrm{E}}$
3. $\mathrm{K}_{\mathrm{sup}}, \mathrm{K}_{\mathrm{inf}}$
4. $\mathrm{Ref}_{\mathrm{E}}$: $p \to \mathrm{E}p$
5. $\mathrm{Tra}_{\mathrm{E}}$: $\mathrm{EE}p \to \mathrm{E}p$
6. $\mathrm{Sym}_{\mathrm{E}}$: $p \to \mathrm{AE}p$
7. $\mathrm{Inc}_{\mathrm{sup}}$: $\langle \mathrm{sup} \rangle (p, q) \to \mathrm{E}p$
8. $\mathrm{Inc}_{\mathrm{inf}}$:$\langle \mathrm{inf} \rangle (p, q) \to \mathrm{E}p$
9. $\mathrm{Inc}_{\mathrm{i}}$: $\mathrm{E}i$
10. Nom: $\mathrm{E}(i \wedge p) \to \mathrm{A}(i \to p)$
11. FunE: $\mathrm{E}\langle \mathrm{sup} \rangle (i, j) \wedge \mathrm{E}\langle \mathrm{inf} \rangle (i, j)$
12. $\mathrm{FunU}_{\mathrm{sup}}$: $i \wedge \langle \mathrm{sup} \rangle (j, k) \to \mathrm{A}(\langle \mathrm{sup} \rangle (j, k) \to i)$
13. $\mathrm{FunU}_{\mathrm{inf}}$: $i \wedge \langle \mathrm{inf} \rangle (j, k) \to \mathrm{A}(\langle \mathrm{inf} \rangle (j, k) \to i)$
14. $\mathrm{Ide}_{\mathrm{sup}}$: $i \to \langle \mathrm{sup} \rangle (i, i)$
15. $\mathrm{Ide}_{\mathrm{inf}}$: $i \to \langle \mathrm{inf} \rangle (i, i)$
16. $\mathrm{Ass}_{\mathrm{sup}}$: $\langle \mathrm{sup} \rangle (i, \langle \mathrm{sup} \rangle (j, k)) \to \langle \mathrm{sup} \rangle (\langle \mathrm{sup} \rangle (i, j), k)$
17. $\mathrm{Ass}_{\mathrm{inf}}$: $\langle \mathrm{inf} \rangle (i, \langle \mathrm{inf} \rangle (j, k)) \to \langle \mathrm{inf} \rangle (\langle \mathrm{inf} \rangle (i, j), k)$
18. $\mathrm{Com}_{\mathrm{sup}}$: $\langle \mathrm{sup} \rangle (i, j) \to \langle \mathrm{sup} \rangle (j, i)$
19. $\mathrm{Com}_{\mathrm{inf}}$: $\langle \mathrm{inf} \rangle (i, j) \to \langle \mathrm{inf} \rangle (j, i)$
20. $\mathrm{Abs}_{\mathrm{sup}}$: $i \to \langle \mathrm{sup} \rangle (\langle \mathrm{inf} \rangle (j, i), j)$
21. $\mathrm{Abs}_{\mathrm{inf}}$: $i \to \langle \mathrm{inf} \rangle (\langle \mathrm{sup} \rangle (j, i), j)$

Rules

$$\mathrm{MP}\ \frac{\psi, \psi \to \varphi}{\varphi} \qquad \mathrm{NEC}_{\mathrm{sup}}\ \frac{\vdash \varphi}{\vdash [\mathrm{sup}](\varphi, \psi)} \qquad \mathrm{NEC}_{\mathrm{inf}}\ \frac{\vdash \varphi}{\vdash [\mathrm{inf}](\varphi, \psi)} \qquad \mathrm{NEC}_{\mathrm{A}}\ \frac{\vdash \varphi}{\vdash \mathrm{A}\varphi} \qquad \mathrm{USUB}\ \frac{\vdash \varphi(p, i)}{\vdash \varphi[\psi/p, j/i]}$$

Nominal-Rules

$$\mathrm{NAME}\ \frac{\vdash j \to \varphi}{\vdash \varphi} \qquad \mathrm{PASTE}_{\mathrm{E}}\ \frac{\vdash \mathrm{E}(j \wedge \varphi) \to \psi}{\vdash \mathrm{E}\varphi \to \psi}$$

$$\mathrm{PASTE}_{\mathrm{sup}}\ \frac{\vdash (\langle \mathrm{sup} \rangle (j, k) \wedge \mathrm{E}(j \wedge \varphi) \wedge \mathrm{E}(k \wedge \theta)) \to \psi}{\vdash \langle \mathrm{sup} \rangle (\varphi, \theta) \to \psi}$$

$$\mathrm{PASTE}_{\mathrm{inf}}\ \frac{\vdash (\langle \mathrm{inf} \rangle (j, k) \wedge \mathrm{E}(j \wedge \varphi) \wedge \mathrm{E}(k \wedge \theta)) \to \psi}{\vdash \langle \mathrm{inf} \rangle (\varphi, \theta) \to \psi}$$

In Nominal-Rules, j, k are distinct and do not occur in φ, θ, ψ.

Formulas 1–10 are classical axioms of polyadic hybrid logic with E ([4, 12]). Below, we provide some intuitions for these axioms: E, as an unary modality, describes a binary relation R_{E} in $\mathcal{M}_t$. $4, 5, 6$ says R_{E} is an equivalence relation. $7, 8, 9$

implies R_{E} is a total relation in $\mathcal{M}_t$ and each moninal i is true in some points in $\mathcal{M}_t$. 10 says i is true at most one point in $\mathcal{M}_t$. Formulas $11, 12, 13$ implies $R_{\sup}, R_{\inf}$ are essentially binary functions in $\mathcal{M}_t$. 14–21 are pure correspondences of the first order formulas listed in Def. 2.[3]

Here is the strategy to prove completeness theorem for $\mathbb{HSIE}_L$ with respect to lattices $\mathfrak{L}_f$. At first we prove the completeness theorem for $\mathbb{HSIE}$ with respect to all **HLSIE**-frames. The method is similar to prove completeness theorem for $\mathbb{HPII}\mathbb{c}$ with respect to lower connected frames. We list the key definitions and lemmas for the proof and point out the difference from Section 3. Then we show the FOCs of 11–21 is a lattice $\mathcal{L}_f$. Since 11–21 are all pure, we get the completeness theorem for lattices $\mathfrak{L}_f$ directly.

Theorem 6. $\mathbb{HSIE}$ *is sound with respect to all* **HLSIE**-*frames.*

Lemma 8. *Let* Γ *be an* $\mathbb{HSIE}$*-MCS. For all nominals* i*, let* $\Delta_i = \{\psi \mid \mathrm{E}(i \wedge \psi) \in \Gamma\}$. *Then:*

1. *For all* i*,* Δ_i *is an* $\mathbb{HSIE}$*-MCS that contains* i.
2. *For all* i, j*, if* $i \in \Delta_j$ *then* $\Delta_j = \Delta_i$.
3. *If* $k \in \Gamma$*, then* $\Gamma = \Delta_k$.

Definition 16. An $\mathbb{HSIE}$-MCS Γ is *named* if there exists a unique $i \in \mathbf{N}$ such that $i \in \Gamma$. An $\mathbb{HSIE}$-MCS Γ is *pasted* if it satisfies the following conditions:

- If $\mathrm{E}(i \wedge \mathrm{E}\psi) \in \Gamma$, then there exists nominal j not occurring in $i \wedge \psi$ such that $\mathrm{E}i \wedge \mathrm{E}(j \wedge \varphi) \in \Gamma$;
- If $\mathrm{E}(i \wedge \langle \sup \rangle(\psi, \varphi)) \in \Gamma$, then there exist nominals j and k such that $\mathrm{E}(i \wedge \langle \sup \rangle(j, k)) \wedge \mathrm{E}(j \wedge \psi) \wedge \mathrm{E}(k \wedge \varphi) \in \Gamma$;
- If $\mathrm{E}(i \wedge \langle \inf \rangle(\psi, \varphi)) \in \Gamma$, then there exist nominals j and k such that $\mathrm{E}(i \wedge \langle \inf \rangle(j, k)) \wedge \mathrm{E}(j \wedge \psi) \wedge \mathrm{E}(k \wedge \varphi) \in \Gamma$.

If Γ is both named and pasted, it is called the *characteristic maximal consistent set* (CMCS).

Definition 17. Let Γ be an $\mathbb{HSIE}$-CMCS, and let i be a nominal, we call the set $\{\psi \mid \mathrm{E}(i \wedge \psi) \in \Gamma\}$ a *named MCS* (NMCS for short) induced by Γ. We define the characteristic canonical model as $\mathcal{M}^\Gamma = \langle W^\Gamma, R^\Gamma_{\sup}, R^\Gamma_{\inf}, V^\Gamma \rangle$, where:

- W^Γ is the set of all NMCSs induced by Γ, i.e., $W^\Gamma = \{\{\psi \mid \mathrm{E}(i \wedge \psi) \in \Gamma\} \mid i \text{ appears in } \Gamma\}$;
- For all MCNSs $w, w_1, w_2 \in W^\Gamma$, $R^\Gamma_{\sup} w w_1 w_2$ iff for all formulas ψ_1, ψ_2: $\psi_1 \in w_1$ and $\psi_2 \in w_2$ implies $\langle \sup \rangle(\psi_1, \psi_2) \in w$. Similar for $R^\Gamma_{\inf}$;

[3][17] shows that formulas 14–21 are theorems in the relational-based system $\mathbb{HLSI}_L$. Here we pick them as the axioms of the functional-based system $\mathbb{HSIE}_L$. This is one of the starting points of this paper: whether it is possible to construct a modal system for the class of lattices based on the formulas 14–21 as axioms.

- $p \in V^{\Gamma}(w)$ iff $p \in w$.

Lemma 9 (Characteristic Lindenbaum's Lemma). *Let* **M** *be a countably infinite set of nominals that do not intersect with* **N**. *Let* **HLSIE**$^+$ *be the extension of* **HLSIE** *that uses* **M** $\cup$ **N** *as the nominal set. Then each* $\mathbb{HSIE}$*-consistent sets in* **HLSIE** *can be extended to an* $\mathbb{HSIE}$*-CMCS in* **HLSIE**$^+$.

Lemma 10 (Characteristic Existence Lemma). *Let* Γ *be an* $\mathbb{HSIE}$*-CMCS, and let* $\mathcal{M}^{\Gamma} = \langle W, R_{\sup}, R_{\inf}, V\rangle$ *be the induced characteristic canonical model as in Def. 17. Then:*

- *For all* $u \in W$ *and all* $\mathrm{E}\psi \in u$*, there exists* $v \in W$ *such that* $\psi \in v$.
- *For all* $u \in W$ *and all* $\langle \sup\rangle(\psi, \varphi) \in u$*, there exist* $v, w \in W$ *such that* $R_{\sup}uvw$ *and* $\psi \in v, \varphi \in w$.
- *For all* $u \in W$ *and all* $\langle \inf\rangle(\psi, \varphi) \in u$*, there exist* $v, w \in W$ *such that* $R_{\inf}uvw$ *and* $\psi \in v, \varphi \in w$.

Lemma 11 (Characteristic Truth Lemma). *Let* Γ *be an* $\mathbb{HSIE}$*-CMCS, and let* $\mathcal{M}^{\Gamma} = \langle W, R_{\sup}, R_{\inf}, V\rangle$ *be the induced characteristic canonical model as in Def. 17. For all* $w \in W$ *and all* **HLSIE**$^+$*-formula* ψ*, we have:*

$$\psi \in w \iff \mathcal{M}^{\Gamma}, w \vDash \psi.$$

Theorem 7. *Every* $\mathbb{HSIE}$*-consistent set in* **HLSIE** *is satisfiable in a countable characteristic* **HLSIE***-model* $\mathcal{M}_t = \langle W, R_{\sup}, R_{\inf}, V\rangle$.

Theorem 8. *Let* Π *be a set of* $\mathbb{HSIE}$*-consistent pure formulas, and consider the extension* $\mathbb{HSIE}_{\Pi} = \mathbb{HSIE} + \Pi$. *Then, any* $\mathbb{HSIE}_{\Pi}$*-consistent set is satisfiable in a characteristic model that satisfies the FOC of* Π.

Lemma 12. *For any* **HLSIE***- frame* $\mathcal{F}_t$*, if formulas* 11–21 *are valid in* $\mathcal{F}_t$ *then* $\mathcal{F}_t$ *is a lattice* $\mathcal{L}_f$ *as in Def. 6.*

Theorem 9. $\mathbb{HSIE}_L$ *is sound and strongly complete with respect to the class of lattices* $\mathfrak{L}_f$.

5 Conclusion and Future Work

In conclusion, the study significantly builds upon previous explorations of modal logics over lattices by X. Wang and Y. Wang ([16, 17]), delving deeper into the complex interplay between modal logic and lattice theory. Initially, our research employed polyadic hybrid logic, utilizing binary modalities $\langle \sup\rangle$ and $\langle \inf\rangle$ to describe lattice structures.

In this paper, by employing similar techniques, we provide modal characterizations of semi-lattice structures as well as functional lattice structures. These results are beneficial for the proof theory of modal logic and for understanding lattice theory from different perspectives.

As for future work, we will consider other relatives of lattices such as bounded lattices, complemented lattices, Heyting algebra, quantum algebra, and so on, which may also involve other binary operators, unary operators, and constants. Corresponding modalities are to be introduced. More generally, we can look at the modal logic over other algebraic structures. We believe the method presented in this paper can be generalized to a wider class of algebras. Finally, it is an interesting question to go from these modal logic back to algebraic semantics.

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半格模态逻辑以及格的模态公理化新方法

王潇扬

摘 要

本文在《Modal Logic over Lattices》的基础工作之上，进一步探索了模态逻辑与格理论之间的关系。在之前的研究中，使用带二元模态词 ⟨sup⟩, ⟨inf⟩ 的多元混合逻辑通过标准克里普克语义讨论格结构。本文将讨论如何使用模态逻辑刻画下半格结构。为了刻画下半格，本文使用了带有一元模态词 P 和二元模态词 ⟨inf⟩ 的多元混合逻辑语言并给出了半格上的多元混合逻辑的完整公理化。在已有的相关结果中，格的定义主要基于偏序关系。在本文的后半部分，提出了一种更符合代数视角的格的替代定义，并给出了相应的模态公理化结果。

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