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The Finite Axiomatization of Transitive Pretabular Logics of Finite Depths*

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Abstract. This paper attempts to resolve the problem of how to axiomatize transitive pretabular logics of finite depth. It is the follow-up work on transitive pretabular logics after the crieria for transitive pretabular logics. This paper uses canonical formulas to axiomatize each pretabular logic of finite depth in *NExtK4*.

1 Introduction

This paper attempts to resolve the problem—how to axiomatize transitive pretabular logics of finite depth. It is the follow-up work on transitive pretabular logics after the crieria for transitive pretabular logics in [6] and [5]. We use canonical formulas in [11], [12] and [13] to axiomatize each pretabular logic of finite depth in *NExtK4*.

First let's retrospect the history of the work on pretabular logics in *NExtK4*. In 1970s, [9] and [7] independently proves that there are only 5 pretabular logics in *NExtS4*. Later, [1] establishes that *NExtD4* contains only 10 pretabular logics and that *NExtGL* contains $\aleph_0$ pretabular logics (some discrepancies in his proof were corrected in [3]). [1] also proves that *NExtK4* contains $2^{\aleph_0}$ pretabular logics but only denumerably many pretabular logics of finite depth. [4] proves that every pretabular logic of finite depth in *NExtK4* is finitely axiomatizable and decidable. [2] shows how to axiomatize the pretabular logics in *NExtS4* and *NExtGL* by finite sets of canonical formulas.

The paper consists of three sections. Section 1 is Introduction Part offering some historical notes on pretabular logics in *NExtK4*. Section 2 is Background Part offering some symbols and known results on pretabular logics in *NExtK4* that will be used later in this paper. Section 3 is the main result of the paper, offering the way how to axiomatize the pretabular logics of finite depth in *NExtK4*.

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2 Background

A set of modal formulas is a *normal modal logic* (a *normal logic* or simply a *logic*) if it contains all classical tautologies and $\Box(p \to q) \to (\Box p \to \Box q)$, and is closed under Detachment, Substitution and Necessitation. The smallest normal logic is K, the largest is For, i.e., the set of all modal formulas, and

$$K4 = K \oplus \Box p \to \Box\Box p.$$

Let L be any normal logic. L' is a normal extension of L if L' is a normal modal logic and $L \subseteq L'$. When L' is a normal extension of L and $L \neq L'$, L' is said to be a proper normal extension of L. Let *NExtL* stands for the set of all normal extensions of L, such as *NExtK* for the set of all normal logics and *NExtK4* for the set of all normal extensions of $K4$.

A (Kripke) *frame* and a (Kripke) *model* are pairs $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ repectively, where W is a non-empty set, R is a binary relation on W, and $\mathfrak{V}$ is a valuation of all propositional variables in $\mathfrak{F}$. For each modal formula φ, the validity of φ in $\mathfrak{F}$ (or the truth in $\mathfrak{M}$) is defined in the usual way, written as $\mathfrak{F} \models \varphi$ (or $\mathfrak{M} \models \varphi$).

For each frame $\mathfrak{F}$, $Log\mathfrak{F} = \{\varphi \in For : \mathfrak{F} \models \varphi\}$ is called *the logic characterized by* $\mathfrak{F}$. When $\mathfrak{F}$ is a finite frame, $Log\mathfrak{F}$ is called a *tabular logic*. Similarly, for each class C of frames, $LogC = \cap\{Log\mathfrak{F} : \mathfrak{F} \in C\}$ is called *the logic characterized by* C. When C is a class of finite frames, $LogC$ is said to be *finitely approximable*.

For each normal modal logic L_0 and each $L \in NExtL_0$, L is a *pretabular logic in* $NExtL_0$ if L is not a tabular logic but all of its proper extensions in $NExtL_0$ are tabular logics.

Readers are assumed to be familiar with the following operations and their related theorems: Generation (including Generation by a point), Reduction (also known as p-morphism) and Disjoint Union.

Following the notation in [2], for each frame $\mathfrak{F} = \langle W, R\rangle$ and $X \subseteq W$, let

$$X{\uparrow} = \{y \in W : \exists x \in X\ (xRy)\}$$
$$X{\downarrow} = \{y \in W : \exists x \in X\ (yRx)\}$$

and the designations are introduced as follows:

$$X{\uparrow}^- = X{\uparrow} \setminus X, \qquad X{\uparrow}^+ = X{\uparrow} \cup X,$$
$$X{\downarrow}^- = X{\downarrow} \setminus X, \qquad X{\downarrow}^+ = X{\downarrow} \cup X.$$

When X is a singleton $\{x\}$, we will use $x{\uparrow}$ $(x{\downarrow})$, $x{\uparrow}^-$ $(x{\downarrow}^-)$ and $x{\uparrow}^+$ $(x{\downarrow}^+)$ to denote $\{x\}{\uparrow}$ $(\{x\}{\downarrow})$, $\{x\}{\uparrow}^-$ $(\{x\}{\downarrow}^-)$ and $\{x\}{\uparrow}^+$ $(\{x\}{\downarrow}^+)$ respectively. Each point in $x{\uparrow}$ $(x{\downarrow})$ is called a *successor* (*predecessor*) *of* x, each point in $x{\uparrow}^-$ $(x{\downarrow}^-)$ a *proper successor* (*predecessor*) *of* x, and each point $y \in x{\uparrow}^-$ $(x{\downarrow}^-)$ an *immediate successor* (*predecessor*) *of* x if for each $z \in W$, $x\vec{R}z\vec{R}y$ ($y\vec{R}z\vec{R}x$) implies $x = z$ or $y = z$, and $x \notin y\!\uparrow$ ($x \notin y\!\downarrow$), where $u_1\vec{R}u_2$ are designations of $u_1Ru_2 \wedge \neg u_2Ru_1$.

We will repeatedly appeal to Zakharyaschev's results in [11], [12] and [13] concerning "normal modal canonical formulas". Let $\mathfrak{F} = \langle W, R\rangle$ be a finite rooted transitive frame, where $W = \{x_1, \dots, x_n\}$ and x_1 is the root of $\mathfrak{F}$. Let $\mathfrak{D}$ be a set of antichains[1] in $\mathfrak{F}$ that are not reflexive singletons. *The normal modal canonical formula* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ associated with $\mathfrak{F}$ and $\mathfrak{D}$ is as follows:

$$\alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \bigwedge_{x_iRx_j} \varphi_{ij} \wedge \bigwedge_{i=0}^{n} \varphi_i \wedge \bigwedge_{\mathfrak{d}\in\mathfrak{D}} \varphi_{\mathfrak{d}} \wedge \varphi_\bot \to p_0$$

where

$$\varphi_{ij} = \Box^+(\Box p_j \to p_i),$$

$$\varphi_i = \Box^+((\bigwedge_{\neg x_iRx_k} \Box p_k \wedge \bigwedge_{j=0,j\neq i}^{n} p_j \to p_i) \to p_i),$$

$$\varphi_{\mathfrak{d}} = \Box^+(\bigwedge_{x_i\in W\smallsetminus\mathfrak{d}\uparrow^-} \Box p_i \wedge \bigwedge_{i=0}^{n} p_i \to \bigvee_{x_j\in\mathfrak{d}} \Box p_j),$$

$$\varphi_\bot = \Box^+(\bigwedge_{i=0}^{n} \Box^+ p_i \to \bot).$$

Here $\Box^+\varphi$ abbreviates $\Box\varphi \wedge \varphi$ and $i, j \in \{1, \dots, n\}$.

Let $\mathfrak{D}^\sharp$ be the set of all antichains in $\mathfrak{F}$ that are not reflexive singletons. Then the canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ is called the *frame formula* for $\mathfrak{F}$ and is abbreviated by $\alpha^\sharp(\mathfrak{F}, \bot)$. $\alpha^\sharp(\mathfrak{F}, \bot)$ is equivalent to Fine's frame formula in [8]. A refutability criterion for a frame formula in [8] and [2] is formulated as follows:

Theorem 1 (Fine)**.** *Let $\mathfrak{F}$ be a finite rooted transitive frame and $\mathfrak{G}$ be a transitive frame. Then $\mathfrak{G} \not\models \alpha^\sharp(\mathfrak{F}, \bot)$ iff a generated subframe of $\mathfrak{G}$ is reducible to $\mathfrak{F}$.*

Let $\mathfrak{F} = \langle W, R\rangle$ be a transitive frame. A $\vec{R}$-chain of the length n in $\mathfrak{F}$ is a sequence $x_1, x_2, \cdots, x_n$ of n distinct points in W such that

$$x_1\vec{R}x_2\vec{R}x_3\vec{R}\cdots\vec{R}x_n.$$

If the supremum of lengths of $\vec{R}$-chains in $\mathfrak{F}$ is n, the depth of $\mathfrak{F}$ is n, in symbols, $Dep(\mathfrak{F}) = n$. $\mathfrak{F}$ is *of finite depth* if $Dep(\mathfrak{F}) = n$ for some $n \in \omega$, and is *of infinite*

[1]A set of points $X \subseteq W$ is an antichain in a frame $\mathfrak{F} = \langle W, R\rangle$ if xRy implies $x = y$ for every $x, y \in X$.

depth otherwise. A transitive frame $\mathfrak{F}$ validates bd_n iff $Dep(\mathfrak{F}) \leq n$, where bd_n is defined inductively as follows:

$$bd_1 = \Diamond\Box p_1 \longrightarrow p_1,$$
$$bd_{n+1} = \Diamond(\Box p_{n+1} \wedge \neg bd_n) \longrightarrow p_{n+1}.$$

A transitive logic $L \in NExtK4$ is *of finite depth* if $bd_n \in L$ for some $n \in \omega$.

We need to introduce a pointwise reduction and its related theorem in [6].

Definition 1 (**Pointwise Reduction**). A reduction f of $\mathfrak{F} = \langle W, R\rangle$ to $\mathfrak{G} = \langle U, S\rangle$ is a *pointwise reduction*, if there are at most two distinct points $a, b \in W$—referred as *chosen points for* f—such that $f(a) = f(b)$, and for other points $x, x' \in W$ at least one of which is not a or b,

$$f(x) = f(x') \text{ only if } x = x'.$$

In the nontrivial case (i.e., $a \neq b$), it is called a *proper pointwise reduction* ("proper" may be omitted if there is no danger of confusion).

The following theorem in [6] shows that the five types P_1–P_5 constitute an exhaustive classification of proper pointwise reductions of transitive frames.

Theorem 2 (Theorem 2.3 in [6]). *There are exactly five types P_1–P_5 of proper pointwise reductions of transitive frames.*

- **Type** P_1 (*Proper-cluster-contracting*): $f \in P_1$ iff aRb and bRa (obviously both a and b are reflexive)(see Figure 1).
- **Type** P_2 (*Reflexive-copy-merging*): $f \in P_2$ iff $a{\uparrow}^- = b{\uparrow}^-$, neither aRb nor bRa, and a, b are both reflexive(see Figure 1).
- **Type** P_3 (*Irreflexive-copy-merging*): $f \in P_3$ iff $a{\uparrow}^- = b{\uparrow}^-$, neither aRb nor bRa, and a, b are both irreflexive(see Figure 1).
- **Type** P_4 (*Reflexive-predecessor-eliminating*): $f \in P_4$ iff $a{\uparrow}^- = b{\uparrow}$ and a, b are both reflexive(see Figure 2).
- **Type** P_5 (*Irreflexive-predecessor-eliminating*): $f \in P_5$ iff $a{\uparrow}^- = b{\uparrow}$, a is irreflexive and b is reflexive(see Figure 2).

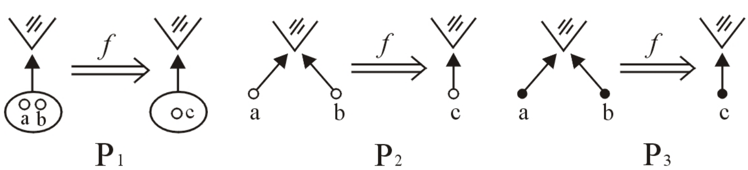

Figure 1

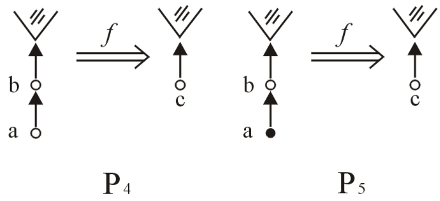

Figure 2

It has been proved in [6] that each pretaublar logic L of finite depth in *NExtK4* is characterized by a pseudo-tack (see Figure 3 (a) and (b)) or a pseudo-top (see Figure 3 (c) and (d)). A pseudo-tack or a pseudo-top is called the *norm form* of L. In Figure 3, $\mathfrak{f}$ represents a finite reduced[2] frame, (b) represents an infinite cluster and S is their common set of successors of those $\triangle$s.[3]

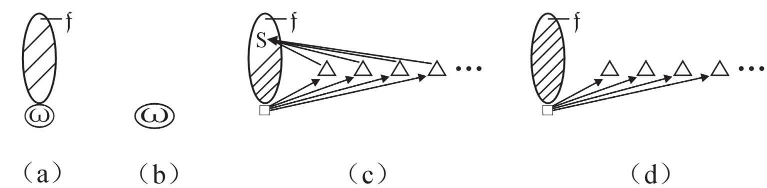

Figure 3

[6] proves that each pseudo-tack or a pseudo-top satisfies Condition $C1$, i.e.,

for each frame $\mathfrak{F}' \in R(G(\mathfrak{F}))$, there is a frame $\mathfrak{G} \in E(\mathfrak{F})$ such that $\mathfrak{F}' \cong \mathfrak{G}$.[4]

Here

$R(C) = \{\mathfrak{F}' : \mathfrak{F}' \text{ is a reduct of } \mathfrak{G} \in C\}$.
$R(\mathfrak{F}) = R(\{\mathfrak{F}\})$.
$G(\mathfrak{F}) = \{\mathfrak{F}' : \mathfrak{F}' \text{ is a generated subframe of } \mathfrak{F}\}$.
$Su(\mathfrak{F}) = \{\mathfrak{F}' : \mathfrak{F}' \text{ is a subframe of } \mathfrak{F}\}$.
$R^{Su}(\mathfrak{F}) = R(\mathfrak{F}) \cap Su(\mathfrak{F})$.
$E(\mathfrak{F}) = G(\mathfrak{F}) \cup R^{Su}(\mathfrak{F})$.

Theorem 4.10 in [6] says that each pseudo-tack or a pseudo-top satisfies Condition $C1$. It is listed as the following Proposition 1.

[2]A frame $\mathfrak{F}$ is *invariant under reductions* iff each reduct of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}$ under any reduction. A frame *invariant under reductions* is called a *reduced* frame.

[3]Each $\triangle$ or $\square$ represents a reflexive or an irreflexive point. (b) is the special case of (a) as its $\mathfrak{f}$ doesn't exist and (d) is the special case of (c) as its $S = \varnothing$.

[4]Condition $C1$ says that if a frame $\mathfrak{F}'$ is a reduct of some generated subframe of a frame $\mathfrak{F}$, then there is a generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ or a subframe $\mathfrak{G}$ of $\mathfrak{F}$ to which $\mathfrak{F}$ can be reduced satisfying that $\mathfrak{F}' \cong \mathfrak{G}$.

Proposition 1 (Theorem 4.10 in [6]). *If a frame $\mathfrak{F}$ is a pseudo-tack or a pseudo-top, then $\mathfrak{F}$ satisfies Condition $C1$.*

3 The Finite Axiomatization of Transitive Pretabular Logics of Finite Depth

Let $L \in \mathit{NExtK4}$ be a pretabular logic of finite depth. Proposition 2 shows that each finite transitive frame $\mathfrak{G}$ with $\mathfrak{G} \models L$ share some common characteristics with the norm form of L.

Proposition 2. *Let $L = Log\mathfrak{F}^* \in \mathit{NExtK4}$ be a transitive pretabular logic of finite depth and $\mathfrak{F}^*$ be the norm form of L, i.e., a pseudo-tack or a pseudo-top. Then for each finite transitive frame $\mathfrak{G}$, $\mathfrak{G} \models L$ iff each subframe of $\mathfrak{G}$ generated by a point is isomorphic to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated by a point.*

Proof. Let $L = Log\mathfrak{F}^* \in \mathit{NExtK4}$ be a transitive pretabular logic of finite depth and $\mathfrak{F}^*$ be the norm form of L. ($\Leftarrow$) Assume that $\mathfrak{G}$ is a finite transitive frame such that $\mathfrak{G} \not\models L$. Then there exists a formula $\varphi \in L$ and $\mathfrak{G} \not\models \varphi$. So there is a subframe $\mathfrak{G}'$ of $\mathfrak{G}$ generated by a point such that $\mathfrak{G}' \not\models \varphi$. By $L = Log\mathfrak{F}^*$, $\mathfrak{F}^* \models \varphi$. Thus $\mathfrak{G}'$ is not isomorphic to any reduct of $\mathfrak{F}^*$ or any subframe of $\mathfrak{F}^*$ generated by point. Otherwise, $\mathfrak{F}^* \not\models \varphi$ holds. ($\Rightarrow$) Let $\mathfrak{H}$ be a finite transitive frame and there exists a subframe $\mathfrak{H}'$ of $\mathfrak{H}$ generated by a point such that it is not isomorphic to any reduct of $\mathfrak{F}^*$ or any subframe of $\mathfrak{F}^*$ generated by a point. By Proposition 1, $\mathfrak{F}^*$ satisfies the following condition:

$C1$ for each $\mathfrak{F}' \in R(G(\mathfrak{F}^*))$, there is a $\mathfrak{G} \in E(\mathfrak{F}^*)$ such that $\mathfrak{F}' \cong \mathfrak{G}$.

Since $\mathfrak{H}'$ is rooted, our assumption shows that no frame in $E(\mathfrak{F}^*)$ is isomorphic to $\mathfrak{H}'$. Since $\mathfrak{F}^*$ satisfies $C1$, it means that $\mathfrak{H}' \notin R(G(\mathfrak{F}^*))$, i.e., no generated subframe of $\mathfrak{F}^*$ can be reducible to $\mathfrak{H}'$. So by Theorem 1, $\mathfrak{F}^* \models \alpha^\sharp(\mathfrak{H}', \bot)$, i.e., $\alpha^\sharp(\mathfrak{H}', \bot) \in Log\mathfrak{F}^* = L$. Obviously, according to Theorem 1, $\mathfrak{H}' \not\models \alpha^\sharp(\mathfrak{H}', \bot)$, i.e., $\mathfrak{H}' \not\models L$. Then we have that $\mathfrak{H} \not\models L$. $\square$

Proposition 3 offers the first step of the finite axiomatization of a transitive pretabular logic of finite depth.

Proposition 3. *Let $L = Log\mathfrak{F}^* \in \mathit{NExtK4}$ be a pretabular logic of finite depth and $\mathfrak{F}^*$ be the norm form of L. Let $L^* = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta\}$, where Δ is a set of finite, rooted transitive frames which are not isomorphic to any reduct of $\mathfrak{F}^*$ or any subframe of $\mathfrak{F}^*$ generated by a point. Then for each finite transitive frame $\mathfrak{H}$,*

$$\mathfrak{H} \models L \textit{ iff } \mathfrak{H} \models L^*.$$

Proof. Let $L = Log\mathfrak{F}^* \in NExtK4$ be a pretabular logic of finite depth and $\mathfrak{F}^*$ be the norm form of L. Let $L^* = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta\}$, where Δ is defined as the one in Proposition 3. Let $\mathfrak{H}$ be a finite transitive frame. $(\Leftarrow)$ Assume that $\mathfrak{H} \models L^*$. From the definition of L^*, any subframe of $\mathfrak{H}$ generated by a point is isomorphic to a reduct or $\mathfrak{F}^*$or a subframe of $\mathfrak{F}^*$ generated by a point. Otherwise, there exists a subframe $\mathfrak{H}'$ of $\mathfrak{H}$ generated by a point that is not isomorphic to any reduct of $\mathfrak{F}^*$ or any subframe of $\mathfrak{F}^*$ generated by a point. Now from the definition of L^*,

$$\alpha^\sharp(\mathfrak{H}', \bot) \in L^*.$$

By Theorem 1, $\mathfrak{H} \not\models \alpha^\sharp(\mathfrak{H}', \bot)$. So we have that

$$\mathfrak{H} \not\models L^*.$$

This is contrary to our assumption. Therefore, any subframe of $\mathfrak{H}$ generated by a point is isomorphic to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated by a point. Thus by Proposition 2,

$$\mathfrak{H} \models L.$$

$(\Rightarrow)$ Assume that $\mathfrak{H} \models L$ and $\mathfrak{H} \not\models L^*$. Since $\mathfrak{H}$ is a finite transitive frame, by Proposition 2, from $\mathfrak{H} \models L$ we have that

(1) any subframe of $\mathfrak{H}$ generated by a point is isomorphic to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated by a point .

According to the definition of L^*, from $\mathfrak{H} \not\models L^*$ we have that there is a frame formula $\alpha^\sharp(\mathfrak{G}, \bot)$ such that

$$\alpha^\sharp(\mathfrak{G}, \bot) \in L^*, \mathfrak{G} \in \Delta \text{ and } \mathfrak{H} \not\models \alpha^\sharp(\mathfrak{G}, \bot).$$

From $\mathfrak{H} \not\models \alpha^\sharp(\mathfrak{G}, \bot)$ we have that there is a subframe $\mathfrak{H}'$ of $\mathfrak{H}$ generated by a point such that

$$\mathfrak{H}' \not\models \alpha^\sharp(\mathfrak{G}, \bot).$$

Since $\mathfrak{H}'$ is a subframe of $\mathfrak{H}$ generated by a point, by (1), $\mathfrak{H}'$ is isomorphic to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated a point. Therefore, from $\mathfrak{H}' \not\models \alpha^\sharp(\mathfrak{G}, \bot)$ we have that

$$\mathfrak{F}^* \not\models \alpha^\sharp(\mathfrak{G}, \bot).$$

So by Theorem 1, a generated subframe of $\mathfrak{F}^*$ is reducible to $\mathfrak{G}$. Since $\mathfrak{F}^*$ is the norm form of L, $\mathfrak{F}^*$ satisfies Condition $C1$, i.e.,

for each $\mathfrak{F}' \in R(G(\mathfrak{F}^*))$, there exists a $\mathfrak{G}' \in E(\mathfrak{F}^*)$ satisfying that $\mathfrak{F}' \cong \mathfrak{G}'$. It means that $\mathfrak{G}$ is isomorphic to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated by a point. So from the definition of Δ, we have that $\mathfrak{G} \notin \Delta$. This is contrary to our assumption that $\mathfrak{G} \in \Delta$. Therefore, $\mathfrak{H} \models L$ implies $\mathfrak{H} \models L^*$. □

By Theorem 12.6 in [2][5] and Exercise 12.10 in [2][6], we have that each transitive pretabular logic of finite depth is a finite union-splitting[7]. Then by Theorem 10.51 in [2][8] and Corollary 12.12 in [2][9], we have Theorem 3.

Theorem 3. *Each transitive pretabular logic of finite depth is strictly Kripke complete and strictly finitely approximable.*[10]

By Proposition 3 and Theorem 3, we can prove that $L = L^*$, which is the second step of the finite axiomatization of a transitive pretabular logic of finite depth.

Proposition 4. *Let* $L = Log\mathfrak{F}^* \in$ *NExtK4 be a pretabular logic of finite depth and* $\mathfrak{F}^*$ *be the norm form of L. Let* $L^* = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta\}$*, where* Δ *is defined as the one in Proposition 3. Then* $L = L^*$.

Proof. Let $L = Log\mathfrak{F}^* \in$ *NExtK4* be a pretabular logic of finite depth and $\mathfrak{F}^*$ is the norm form of L. Let $L^* = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta\}$, where Δ is defined as the one in Proposition 3. From Theorem 3 we have that L is strictly finitely approximable. By Proposition 3, for each finite transitive frame $\mathfrak{H}$,

$$\mathfrak{H} \models L \text{ iff } \mathfrak{H} \models L^*.$$

Therefore, $L = L^*$. □

Let $L = Log\mathfrak{F}^*$ be a pretabular logic of finite depth, $\mathfrak{F}^*$ be the norm form of L and L^* be defined as the one in Proposition 4. Since $L = L^*$, by Segberg's Theorem in [10], L^* is characterized by a class of finite transitive frames with their depths $\leq Dep(\mathfrak{F}^*)$.

Let Δ be a set of frames defined in Proposition 3. Let $\Delta_1 \subset \Delta$ and each $\mathfrak{G} \in \Delta_1$ is reduced with its depth $\leq Dep(\mathfrak{F}^*)$. Let $\Delta_2 \subset \Delta$ and each $\mathfrak{G} \in \Delta_2$ is not reduced but can be completely reducible to the reduced reduct of $\mathfrak{F}^*$ or a reduced subframe of $\mathfrak{F}^*$ generated by a point by using a proper pointwise reduction only once.

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[5]Theorem 12.6 in [2] says that each finitely axiomatizable logic $L \in$ *NExtK4* of finite depth is a finite union-splitting.

[6]Exercise 12.10 in [2] says that the set of pretabular logics in *(N)ExtK4BD*$_n$ is finite for every $n < \omega$ and that all of them are finitely axiomatizable.

[7]For reference, see the definition of a union-splitting in Page 360 in [2].

[8]Theorem 10.51 says that every Kripke complete (finitely approximable) union-splitting $L = L_0/F$ is strictly Kripke complete (respectively, strictly finitely approximable) in *NExtL*$_0$.

[9]Corollary 12.12 says that every pretabular logic in *NExtK4* is finitely approximable, i.e., the finite frame (model) property.

[10]A Kripke complete (finitely approximable) logic L is strictly Kripke complete (respectively, strictly finitely approximable) in a lattice of logics $\mathfrak{L}$ if no other logic in $\mathfrak{L}$ has the same Kripke (finite) frames as L. For reference, see Page 361 in [2].

By Theorem 2 and the fact that each reduction from a finite frame to a finite frame is a composition of several pointwise reductions[11], there are only finitely many frames in Δ_1 and Δ_2 up to isomorphism.

Theorem 4 is the final step of the finite axiomatization of a transitive pretabular logic of finite depth.

Theorem 4. *Let $L = Log\mathfrak{F}^* \in NExtK4$ be a pretabular logic of finite depth and $\mathfrak{F}^*$ be the norm form of L. Then $L = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta_1 \cup \Delta_2\}$.*

Proof. Let $L = Log\mathfrak{F}^* \in NExtK4$ be a pretabular logic of finite depth and $\mathfrak{F}^*$ is the norm form of L. Let $L^{\Delta_1 \cup \Delta_2} = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta_1 \cup \Delta_2\}$. Let $L^* = K4 \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \mathfrak{G} \in \Delta\}$, where Δ is defined as the one in Proposition 3. By Proposition 4, we have that $L = L^*$. By the fact that $\Delta_1 \subset \Delta$ and $\Delta_2 \subset \Delta$,

$$L^{\Delta_1 \cup \Delta_2} \subseteq L^*.$$

Now we prove that for each finite rooted transitive frame $\mathfrak{H}$ with its depth $\leq Dep(\mathfrak{F}^*)$,

$$\mathfrak{H} \models L^{\Delta_1 \cup \Delta_2} \text{ iff } \mathfrak{H} \models L^*.$$

Assume the contrary, i.e., there is a finite rooted transitive frame $\mathfrak{H}'$ with $Dep(\mathfrak{H}') \leq Dep(\mathfrak{F}^*)$ such that $\mathfrak{H}' \models L^{\Delta_1 \cup \Delta_2}$ but $\mathfrak{H}' \not\models L^*$. Then there exists a finite rooted transitive frame $\mathfrak{G}'$ such that $\alpha^\sharp(\mathfrak{G}', \bot) \in L^*$ with $\mathfrak{G}' \in \Delta$ but $\mathfrak{H}' \not\models \alpha^\sharp(\mathfrak{G}', \bot)$. So by Theorem 1, we have that

(1) there is a generated subframe of $\mathfrak{H}'$ that is reducible to $\mathfrak{G}'$.

By $Dep(\mathfrak{H}') \leq Dep(\mathfrak{F}^*)$ and (1), $Dep(\mathfrak{G}') \leq Dep(\mathfrak{F}^*)$. From $\mathfrak{G}' \in \Delta$, $\mathfrak{G}'$ is a finite rooted transitive frame and is not isomorphic to any reduct of $\mathfrak{F}^*$ or any subframe of $\mathfrak{F}^*$ generated by a point. There are two cases for such a frame $\mathfrak{G}'$.

1) There is a generated subframe of $\mathfrak{G}'$ that can be completely reduced[12] to a reduced frame in Δ_1.
2) There is no generated subframe of $\mathfrak{G}'$ that can be completely reduced to any frame in Δ_1. Since each frame can be reduced to a reduced frame, by the definition of Δ_1, we have that each generated subframe of $\mathfrak{G}'$ is completely reducible to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated by a point. Otherwise, there will be a generated subframe of $\mathfrak{G}'$ that can be completely reduced to a frame belonging to Δ_1. Now We prove that $\mathfrak{G}'$ is not reduced. Assume the conytrary, i.e., $\mathfrak{G}'$ is reduced. Then $\mathfrak{G}'$ itself, as one of generated subframes of $\mathfrak{G}'$, is completely reducible to a reduct of $\mathfrak{F}^*$ or a subframe of $\mathfrak{F}^*$ generated by a point. Since a reduced frame can only be reducible to a frame that is isomorphic to itself, it means that $\mathfrak{G}'$ is isomorphic to a reduct of $\mathfrak{F}^*$ or a subframe of

[11]For reference, see Example 2.4 in [6].

[12]A complete reduction is a reduction from a frame to a reduced frame.

$\mathfrak{F}^*$ generated by a point, which is contrary to the assumption that $\mathfrak{G}' \in \Delta$. Therefore, $\mathfrak{G}'$ is not reduced. Since every reduction from a finite frame to a finite frame can be obtained by a composition of several pointwise reductions, by Theorem 2, there is a generated subframe of $\mathfrak{G}'$ that can be reducible to a frame in Δ_2.

Therefore, by (1), we have that

(2) one of generated subframes of $\mathfrak{H}'$ is reducible to a frame in $\Delta_1 \cup \Delta_2$.

Therefore, by Theorem 1, $\mathfrak{H}' \not\models L^{\Delta_1 \cup \Delta_2}$, which is contrary to our assumption that $\mathfrak{H}' \models L^{\Delta_1 \cup \Delta_2}$. Thus we have that for each finite rooted transitive frame $\mathfrak{H}$ with its depth $\leq Dep(\mathfrak{F}^*)$,

$$\text{if } \mathfrak{H} \models L^{\Delta_1 \cup \Delta_2}, \text{ then } \mathfrak{H} \models L^*.$$

Now we have completed the proof of the proposition that for each finite rooted transitive frame $\mathfrak{H}$ with its depth $\leq Dep(\mathfrak{F}^*)$,

$$(3) \;\; \mathfrak{H} \models L^{\Delta_1 \cup \Delta_2} \text{ iff } \mathfrak{H} \models L^*.$$

From $L = L^*$ we have that L^* is a transitive pretabular logic of finite depth which is characterized by a class of finite transitive frames with their depths $\leq Dep(\mathfrak{F}^*)$. From Theorem 3 we have that L^* is strictly finitely approximable. Since $L^{\Delta_1 \cup \Delta_2} \subseteq L^*$ and (3) holds, by the fact that L^* is strictly finitely approximable, we have that

$$L^{\Delta_1 \cup \Delta_2} = L^*.$$

Therefore, $L = L^{\Delta_1 \cup \Delta_2}$. □

$L^{\Delta_1 \cup \Delta_2}$ is finitely axiomatizable since there are only finitely many frames in $\Delta_1 \cup \Delta_2$ up to isomorphism and $\alpha^\sharp(\mathfrak{G}_1, \bot) = \alpha^\sharp(\mathfrak{G}_2, \bot)$ whenever $\mathfrak{G}_1 \cong \mathfrak{G}_2$. So by Theorem 4, each transitive pretabular logic of finite depth can be finitely axiomatized as $L^{\Delta_1 \cup \Delta_2}$.

Now we use the following example to show how to construct $L^{\Delta_1 \cup \Delta_2}$ for axiomatizing each transitive pretabular logic L of finite depth by canonical formulas. What we need to do is to determine the norm form of the transitive pretabular logic L of finite depth and its related Δ_1 and Δ_2 according to the norm form of L.

Example 1. Let $L = Log\mathfrak{F}^* \in$ *NExtK4* be a transitive pretabular logic of finite depth and $\mathfrak{F}^*$ be its norm form given in Figure 4. Now by Theorem 4 we have that

$$L = K4 \oplus \alpha^\sharp(\circ, \bot) \oplus \alpha^\sharp(\mathfrak{F}_1, \bot) \oplus \alpha^\sharp(\mathfrak{F}_2, \bot) \oplus \alpha^\sharp(\mathfrak{F}_3, \bot) \oplus \alpha^\sharp(\mathfrak{F}_4, \bot)$$

while $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ and $\mathfrak{F}_4$ are given in Figure 5. Each of the frames $\circ(= \langle x, \{\langle x, x\rangle\}\rangle)$, $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ belongs to Δ_1 and $\mathfrak{F}_4 \in \Delta_2$ according to the definitions of Δ_1 and Δ_2.

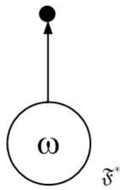

Figure 4

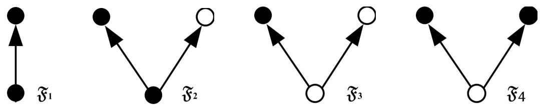

Figure 5

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有穷深度的传递濒表格逻辑的有穷公理化问题

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摘　要

本文试图解决有穷深度的传递的濒表格逻辑的公理化问题。这是作者之前所解决的传递的濒表格逻辑判据工作的后继。本文使用了模态逻辑的先进技术典范公式来解决 *NExtK4* 格（即传递逻辑格）中的每一个有穷深度濒表格逻辑的公理化问题。我们所得到的结论不仅是它们的有穷可公理化，更是如何可公理化的可操作性方法。这种可操作性方法是和模型密切相关的。

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