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Model Theoretical Aspects of Normal Polyadic Modal Logic: An Exposition*

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Abstract. In this paper, we give an exposition on the model theoretical aspects of normal polyadic modal logic (PML), which is a modal logic with n-ary modalities generalizing the basic normal modal logic $\mathbb{K}$. Compared to the basic normal modal logic $\mathbb{K}$, PML is much less studied. Basic results about PML scattered in the literature are often stated without proofs, except in certain algebraic setting, as they are considered as straightforward generalizations of the results of $\mathbb{K}$. Besides the missing details, the very limited available expositions are error-prone even in well-known textbooks and papers, since the generalization to the polyadic setting from the monadic one is sometimes non-trivial, which requires different techniques. Therefore, we think there is a need for a detailed exposition of the basic model theoretical results of PML proved in the modal logic setting, to provide a unified reference for further studies of PML, and this is the goal of the paper. In this paper, we review the definition of filtration and ultrafilter extension for polyadic language and give proofs for some basic theorems including the saturation theorem of ultrafilter extension in a purely model theoretical way other than algebraic one. Then we give a clarification on proving van-Benthem characterization theorem of PML in order to exhibit differences in the proof from the monadic cases. Finally, we also give a model theoretical proof for the Craig Interpolation Theorem of PML while the theorem was treated as a corollary of some algebraic results in the literature.

1 Introduction

A polyadic modality is a modality with more than one propositional arguments. In [29] and [30], Jonsson and Tarski first considered the polyadic modal operators in an algebraic context, where they proved a deep theorem about the relation between additive Boolean Algebra and relational structures. As in [42], some views it as the origin of the relational semantics not only for monadic modal logic but also for the polyadic one. So in some sense, the relational semantics is born to be polyadic.

Polyadic modalities are often used in the literature of philosophical logic, such as the since-until operators in temporal logic ([31]), the relativized knowledge operators in epistemic logic ([6, 23]), and the conditional obligation operators in deontic

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logic([13]). Various strict implications studied since the beginning of modal logic can also be viewed as binary modalities. In [42], the author introduced a special kind of PML–rough polyadic modal logic, which is related to rough set theory and has some applications in fuzzy theory and artificial intelligence.

Compared to monadic modal logic, the polyadic modal logic is less studied. However, we do know some deep connection between monadic modal logic and polyadic modal logic. For example, the simulation theorem in [16] shows that a lot of properties can transfer from one to another, but there still remains many differences between the monadic modal language and polyadic modal languages. A significant one is the Sahlqvist Theorem which can be used to decide whether a modal formula has a first order correspondence. In [36], de Rijke showed that we cannot easily generalize Sahlqvist formula for polyadic modal languages. However, in [21], the authors showed that by representing polyadic languages in a combinatorial PDL style, one can still get a generalized Sahlqvist Theorem[1].

In [37], De Rijke gave an unravelling model construction for getting tree-like models of polyadic modal logic, which is useful on finite model theory of PML. In [11], Demri and Gabbay had some model theoretical considerations for polyadic modal logic (under the name of *polymodal* logic[2]) from the algebraic perspective. Moreover, in [19], Goldblatt provided a very good survey for algebraic PML, where he discussed ultrafilter extension and ultraproduct in the polyadic setting. Moreover, Hoogland had a deep observation between the amalgamation on algebras and the interpolation on logic in [26], which includes the Craig Interpolation Theorem for PML as a consequence. Some interesting philosophical applications of polyadic modal logic are discussed in [33].

As we have seen in [9] and other modal logic textbooks, basic results about PML are often stated without proofs, as they are considered as generalizations of the results of the monadic modal logic. Moreover, some results are scattered in the literature of algebraic logic, which are not very accessible to the general audience of modal logic. Moreover, the available expositions are sometimes error-prone, since the generalization to the polyadic setting from the monadic one is sometimes nontrivial, which requires different techniques and some more care. Therefore, we think there is a need for a detailed exposition of the basic model theoretical results of PML proved in the modal logic setting, in order to invite further studies of PML, and this is the goal of the paper.

In this paper, we start with *filtration* and *ultrafilter extension* in the polyadic setting, generalizing the monadic ones. It is worth mentioning that there are some non-trivial differences between polyadic logics and monadic logics in proving the

[1]See [22] for similar results in the hybrid polyadic setting.

[2]Notice that the word "polymodal" in different literatures has two different meanings: One indeed means "polyadic" while the other means "multi-modal", and readers should be careful about that.

saturation theorem for ultrafilter extensions. Next, we give a simple examination on the proof of the van-Benthem Characterization theorem stated in [9] with crucial details in the general case. Finally we use a model theoretical strategy to show the Craig Interpolation Theorem for normal polyadic modal logic. The result itself is not new and can be derived from some algebraic results in [35], but our proof is purely modal as in [3].

All the results can be found in the literature, but some of them just have statements without a complete proof, and some of the proofs only consider the monadic case, while some other proofs used algebraic tools to get a result which can actually be derived by pure modal techniques. For clarification, we summarize the results mentioned above in the following table. We cite those references where the theorems come from as we know, use Ø to mean that the content is omitted, and use $[*](a)$ to mean the proof in * is algebraic.

	FT	UET	SUE	vBC	CIP
ML-statement	[32]	[12],[17]	[12],[17]	[4]	[3]
PML-statement	[9]	[19]	[9]	[9]	[3]
ML-proof	[32]	[12],[17],[5]	[12],[17],[25]	[4],[38]	[3]
PML-proof	Ø	[19](a)	Ø	Ø	[26](a)

- ML: Monadic modal logic PML: Polyadic modal logic
- FT: Filtration Theorem UET: Ultrafilter Extension(UE)[3] Theorem
- SUE: Saturation Theorem of UE Theorem vBC: van-Benthem Characterization
- CIT: Craig Interpolation Theorem

Basically our paper is guided by the following three books [9], [20], and [10], all of which are important for people to learn more about model theory on polyadic modal logic.

2 Normal polyadic modal logic

First we introduce the basic syntax and semantics for PML, which can be found in [9].

Definition 1 (Polyadic Modal Language (PML)) Given a countable set Φ of basic propositional letters and a natural number $n > 1$, the language $\mathrm{ML}^n(\Phi)$ is defined by:

$$\varphi := p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \nabla(\underbrace{\varphi \ldots \varphi}_{n})$$

where $p \in \Phi$.

[3]Some logicians use "canonical extension" instead of "ultrafilter extension", such as Goldblatt in [19].

Intuitively, ∇ is the n-ary version of the $\Box$ in monadic modal logic. We define $\varphi \vee \psi$, $\varphi \to \psi$, and $\triangle(\varphi_1, \ldots, \varphi_n)$ as the abbreviations of $\neg(\neg\varphi \wedge \neg\psi)$, $\neg\varphi \vee \psi$ and $\neg\nabla(\neg\varphi_1, \ldots, \neg\varphi_n)$ respectively.

Definition 2 (Semantics) A frame $\mathcal{F}$ for the modal language $\mathrm{ML}^n(\Phi)$ (call it n-frame) is a pair $\langle W, R_\triangle \rangle$ where W is an nonempty set and $R_\triangle$ is an $(n+1)$-ary relation over W. An n-model $\mathcal{M}$ for $\mathrm{ML}^n(\Phi)$ is a pair $\langle \mathcal{F}, V \rangle$ where the valuation function V assigns each $w \in W$ a subset of Φ. The semantics of ∇ is defined by:

$$\boxed{\begin{array}{lll} \mathcal{M}, w \models \nabla(\varphi_1, \ldots, \varphi_n) & \text{iff} & \text{for each } v_1, \ldots v_n \in W \text{ with } R_\triangle w v_1 \ldots, v_n, \\ & & \mathcal{M}, v_i \models \varphi_i \text{ for some } i \leq n. \end{array}}$$

It is then clear that the semantics for $\triangle$ is as follows:

$$\boxed{\begin{array}{lll} \mathcal{M}, w \models \triangle(\varphi_1, \ldots, \varphi_n) & \text{iff} & \text{there are } v_1, \ldots v_n \in W \text{ with } R_\triangle w v_1 \ldots, v_n, \\ & & \text{such that } \mathcal{M}, v_i \models \varphi_i \text{ for all } i \leq n. \end{array}}$$

Now we give the axioms for normal polyadic modal logics.

Definition 3 (Normal polyadic modal logic) Given a language $\mathrm{ML}^n(\Phi)$, a modal logic Λ is a set of formulas containing all tautologies that is closed under modus ponens and uniform substitution. A modal logic Λ is *normal* if it contains the axiom K^i_∇ and is closed under N^i_∇ for each $i \in [1, n]$.

$$\begin{array}{ll} \mathrm{K}^i_\nabla & \nabla(r_1, \ldots r_{i-1}, p \to q, r_{i+1} \ldots, r_n) \to \\ & \quad (\nabla(r_1, \ldots r_{i-1}, p, r_{i+1} \ldots, r_n) \to \nabla(r_1, \ldots r_{i-1}, q, r_{i+1} \ldots, r_n)) \\ \mathrm{N}^i_\nabla & \quad \text{from } \vdash_\Lambda \varphi \text{ infer } \vdash_\Lambda \nabla(\psi_1 \ldots, \psi_{i-1}, \varphi, \psi_{i+1}, \ldots, \psi_n) \end{array}$$

We call the resulting minimal normal modal logic $\mathbb{K}_n$.

Remark 1 In [28], the author used the following axiom G^i_∇ instead of K^i_∇,[4] and besides N^i_∇, a monotonicity rule RM^i_∇ is also used.

$$\begin{array}{ll} \mathrm{G}^i_\nabla & \nabla(r_1, \ldots r_{i-1}, p, r_{i+1} \ldots, r_n) \to \\ & \quad (\nabla(r_1, \ldots r_{i-1}, q, r_{i+1} \ldots, r_n) \to \nabla(r_1, \ldots r_{i-1}, p \wedge q, r_{i+1} \ldots, r_n)) \\ \mathrm{RM}^i_\nabla & \quad \text{from } \vdash_\Lambda \varphi \to \psi \text{ infer} \\ & \vdash_\Lambda \nabla(\psi_1 \ldots, \psi_{i-1}, \varphi, \psi_{i+1}, \ldots, \psi_n) \to (\psi_1 \ldots, \psi_{i-1}, \psi, \psi_{i+1}, \ldots, \psi_n) \end{array}$$

It is not hard to show the resulting logic is equivalent to our exposition. On the other hand, in [9], the following rule is used instead of N^i_∇:

$$\mathrm{N}^*_\nabla \quad \text{from } \vdash_\Lambda \varphi \text{ infer } \vdash_\Lambda \nabla(\bot \ldots, \bot, \varphi, \bot, \ldots, \bot)$$

[4] The name G^i_∇ is in recognition of the contribution of Goldblatt.

Unfortunately, the resulting logic based on $\mathrm{N}^*_\triangledown$ instead of $\mathrm{N}^i_\triangledown$ is strictly weaker than $\mathbb{K}_n$, and a proof can be found in the following. Also note that the following axiom mentioned in the definition of normal polyadic modal logics from [16] is not valid:[5]

$$\triangledown(p_1 \to q_1, \dots, p_n \to q_n) \to (\triangledown(p_1, \dots, p_k) \to \triangledown(q_1, \dots, q_n))$$

Proposition 4 $\mathrm{N}^i_\triangledown$ is not admissible in the logic $\mathbb{K}^*$ where $\mathrm{N}^i_\triangledown$ is replaced by $\mathrm{N}^*_\triangledown$.

Proof We define a new semantics $\Vdash$ w.r.t. the Kripke model to show the independence. The truth definitions for the propositional letters and Boolean cases are the same as $\models$. For the modal case:

$w \Vdash \triangledown(\varphi_1, \dots, \varphi_n)$ iff one of the followings hold:

- w is a dead end, i.e. there is no $v_1, \dots, v_n$ s.t. $Rwv_1, \dots, v_n$.
- There are some $v_1, \dots, v_n$ s.t. $Rwv_1, \dots, v_n$ and $\exists k \in [1, n]$ $\forall w_1, \dots, w_n(Rww_1, \dots, w_n \to$ $(w_k \Vdash \varphi_k \wedge \forall m \neq k \exists w'_1, \dots, w'_n(Rww'_1, \dots, w'_n \wedge w'_m \Vdash \neg\varphi_m)))$.

The above statement says that there is a unique argument which is true at the corresponding position of every sequence of successors, and we call this argument the unique truth.

Now we verify that $\Vdash$ is valid w.r.t. $\mathbb{K}^*$. Since we don't change any definition of the propositional connectives, each tautology is still valid. The case for dual and US are trivial and it is also easy to show that $\Vdash$ preserves truth under $\mathrm{N}^*_\triangledown$. For the axioms $\mathrm{K}^i_\triangledown$, suppose that $\mathcal{M}, w$ is a pointed model and the two premises are satisfied at $\mathcal{M}, w$. We may assume that w is not a dead end, and otherwise the case is trivial. Then we know that p_i is the unique true argument for some i, and if $i = n+1$, q_{n+1} must be the unique true argument, which means $\triangledown(p_1, \dots q_{n+1}, \dots, p_m)$ is true at w, since other p_j must be wrong at some successors of w. If $i \neq n+1$, it follows that q_{n+1} must be wrong somewhere and hence we also have $\triangledown(p_1, \dots q_{n+1}, \dots, p_m)$ is true at w. As a result, $\Vdash \mathrm{K}^i_\triangledown$.

Let $\mathcal{M}, w$ be a point model where w is not a dead end. Trivially, $\Vdash \top$, but $w \Vdash \neg(\top, \dots, \top, \dots, \top)$, which means $\mathrm{N}^i_\triangledown$ cannot preserve truth. Thus we know that $\mathrm{N}^i_\triangledown$ is independent in the logic $\mathbb{K}^*$. $\square$

3 Filtration

Now we come to consider the filtration construction, which is first from [32], but the name seems to be given by Segerberg in [40], who further developed this method in [41]. Gabbay had some important researches in [14] and [15], too. But all of those

[5] In [16], $\triangle$ is used as the polyadic box.

works are considering the monadic modal language, and there are few references for polyadic modal languages.

The basic content of this section is also from the chapter of filtration in [9], but we add those proofs for polyadic cases that the book omits. The following definition is also from [9].

Definition 5 (Filtration) Let $\mathcal{M} = \langle W, R, V\rangle$ be an $n{+}1$-ary model and Σ be a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma$ be the relation on the states of $\mathcal{M}$ defined by:

$w \leftrightsquigarrow_\Sigma v$ iff for all $\varphi \in \Sigma$: $(\mathcal{M}, w \models \varphi \leftrightarrow \mathcal{M}, v \models \varphi)$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. We denote the equivalence class of a state w of $\mathcal{M}$ with respect to $\leftrightsquigarrow_\Sigma$ by $|w|_\Sigma$, or simply by $|w|$ if the content is clear. The mapping $w \mapsto |w|$ that sends a state to its equivalence class is called the natural map.

Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. If any model $\mathcal{M}^f_\Sigma = \langle W^f, R^f, V^f\rangle$ satisfies the followings:

(i) $W^f = W_\Sigma$;

(ii) $wRw_1, ..., w_n$ implies $|w|R^f|w_1|, ..., |w_n|$;

(iii) If $|w|R^f|w_1|, ..., |w_n|$ then for all $\triangle(\varphi_1, ..., \varphi_n) \in \Sigma$: $(\mathcal{M}, w_i \models \varphi_i$ for all $i \leq n \Longrightarrow \mathcal{M}, w \models \triangle(\varphi_1, ..., \varphi_n))$;

(iv) $V^f(p) = \{|w| \mid \mathcal{M}, w \models p\}$, for all proposition letters $p \in \Sigma$,

then $\mathcal{M}^f_\Sigma$ is called a filtration of $\mathcal{M}$ through Σ.

We also have the following fact like the one for the monadic modal language.

Proposition 6 Let Σ be a finite sub-closed formula set. For any model $\mathcal{M}$, if $\mathcal{M}^f_\Sigma$ is a filtration of $\mathcal{M}$ through Σ, then $\mathcal{M}^f_\Sigma$ contains at most $2^{card(\Sigma)}$ nodes.

Proof The proof for the monadic modal language also works here: the mapping g defined by $g(|w|) = \{\varphi \in \Sigma \mid \mathcal{M}, w \models \varphi\}$ is an injection from W^Σ to $\mathbb{P}(\Sigma)$. $\square$

The proof for the following filtration theorem is also similar to the monadic one in [9].

Proposition 7 (Filtration Theorem) Let $\mathcal{M}^f_\Sigma = \langle W^f, R^f, V^f\rangle$ be a filtration of $\mathcal{M}$ through a subformula closed set Σ. Then for all $\varphi \in \Sigma$, and all $w \in W^f$, we have $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^f, |w| \models \varphi$

There is only a proof for basic modal language in [9], while the authors omit the polyadic case, so here we give a complete proof for polyadic language.

Proof We prove by induction on φ, and the only non-trivial case is that

$\varphi = \triangle(\psi_1, ..., \psi_n)$. So we first assume that $\varphi \in \Sigma$.

($\Longrightarrow$) Suppose that $\mathcal{M}, w \models \varphi$, from which it follows that there are some $v_1, ...v_n$ s.t. $wR^f v_1, ...v_n$ and for each $i \leq n$, $\mathcal{M}, v_i \models \psi_i$. By the fact that Σ is subformula closed, we have $\psi_i \in \Sigma$, and hence by I.H. we know that $\mathcal{M}, |v_i| \models \psi_i$ for each i. But the condition (ii) in the definition of filtration shows that $|w|R^f|v_1|, ..., |v_n|$, which means $\mathcal{M}, |w| \models \varphi$.

($\Longleftarrow$) Suppose that $\mathcal{M}, |w| \models \varphi$. Thus there are are some $v_1, ...v_n \in W^f$ s.t. $|w|R^f|v_1|, ..., |v_n|$ and $\mathcal{M}, |v_i| \models \psi_i$ for each $i \leq n$. Since each $\psi_i \in \Sigma$, it follows that $\forall i \leq n(\mathcal{M}, v_i \models \psi_i)$ by I.H. So the condition (iii) is applicable, and hence we can conclude that $\mathcal{M}, w \models \varphi$. □

Like the case for basic modal language, We have the smallest and largest filtration due to our definition.

Define R^s and R^l as follows:

- $|w|R^s|v_1|, ..., |v_n|$ iff $\exists w' \in |w| \forall i \leq n \exists v'_i \in |v_i|(w' R v'_1, ..., v'_n)$;
- $|w|R^l|v_1|, ..., |v_n|$ iff for all $\triangle(\varphi_1, ..., \varphi_n) \in \Sigma$: ($\mathcal{M}, v_i \models \varphi_i$ for all $i \leq n \Longrightarrow \mathcal{M}, w \models \triangle(\varphi_1, ..., \varphi_n)$).

The following fact shows that R^s and R^l are indeed the smallest and largest filtration relations, which is a simple generalization of the monadic case in [9].

Proposition 8 Let $\mathcal{M}$ be any model and Σ any sub formula closed set. Then both $\mathcal{M}^s_p = \langle W^f_\Sigma, R^s, V^f \rangle$ and $\mathcal{M}^l_p = \langle W^f_\Sigma, R^l, V^f \rangle$ are filtrations. Furthermore, for any filtration (W^f_Σ, R^f, V^f) of $\mathcal{M}$ though Σ, $R^s \subseteq R^f \subseteq R^l$.

Proof First we show that R^s satisfies the condition (iii), and condition (ii) is trivial.

Suppose that $|w|R^s|w_1|, ..., |w_n|$. By our definition, there must be $w' \in |w|$ s.t. $\forall i \leq n \exists w'_i \in |w_i|(w' R w'_1, ..., w'_n)$. Thus, for any $\triangle(\varphi_1, ..., \varphi_n) \in \Sigma$, if $\mathcal{M}^s_p, w_i \models \varphi_i$ for all $i \leq n$, then $\mathcal{M}^s_p, w'_i \models \varphi_i$ for all $i \leq n$ by $w'_i \in |w_i|$. Therefore $\mathcal{M}^s_p, w' \models \triangle(\varphi_1, ..., \varphi_n)$, which means $\mathcal{M}^s_p, w \models \triangle(\varphi_1, ..., \varphi_n)$ by $w' \in |w|$.

Now we need to show that R^l_p satisfies condition (ii).

Suppose that $wRw_1, ..., w_n$. we need to prove that $|w|R^l|w_1|, ..., |w_n|$:

for each $\triangle(\varphi_1, ..., \varphi_n) \in \Sigma$, if for all $i \leq n$, $(\mathcal{M}^l_p, w_i \models \varphi_i)$, then by our assumption, $\mathcal{M}^l_p, w \models \triangle(\varphi_1, ..., \varphi_n)$. Hence by our definition of R^l, $|w|R^l_p|w_1|, ..., |w_n|$ holds.

Suppose that $|w|R^s|v_1|, ..., |v_n|$, from which it follows that that $\exists w' \in |w| \forall i \leq n \exists v'_i \in |v_i|(w' R v'_1, ..., v'_n)$, and hence for any filtration relation R^f, $|w'|R^f|v'_1|, ...,$ $|v'_n|$ by condition (ii). But $|w'| = |w|$ and $|v_i| = |v'_i|$ for all i, which means $|w|R^f|v_1|, ..., |v_n|$. As a result, $R^s \subseteq R^f$.

Suppose that $\neg|w|R^l|v_1|, ..., |v_n|$, which means there is some $\triangle(\varphi_1, ..., \varphi_n) \in \Sigma$ s.t. ($\mathcal{M}, v_i \models \varphi_i$ for all $i \leq n$ and $\mathcal{M}, w \models \neg\triangle(\varphi_1, ..., \varphi_n)$). So for any R^f, $\neg|w|R^f|v_1|, ..., |v_n|$ by condition (iii), and hence $R^f \subseteq R^l$ as a conclusion.

□

We say a logic L admits filtration iff for any frame $\mathcal{F}$ of L, any subformula closed set Σ, and any model $\mathcal{M}$ on $\mathcal{F}$, there is some filtration $\mathcal{M}^f_\Sigma$ s.t. the frame $\mathcal{F}^f$ of $\mathcal{M}^f_\Sigma$ is still a frame of L.

A standard canonical model method in [9] shows that each $\mathbb{K}_n$ is complete with respect to all n-frame, and by the filtration method we give above, it directly follows that $\mathbb{K}_n$ admits filtration and hence has the finite model property.

4 Ultrafilter Extension

Now we consider a more complex construction–ultrafilter extension. The name is first from [5], but the idea is earlier in [17], which is about modal duality theory, and independently in [12], where Fine proved the Canonicity theorem for first-order definable classes. Fine and Van Benthem's works are basicly on the monadic modal language. Goldblatt's original work is also monadic but he has a further research from the algebraic view, which is polyadic, in [19], and he talks about ultraproduct in that paper too.

In this section we will prove two important theorems for polyadic language: the ultrafilter extension (UE) theorem and the saturation theorem of UE. Both proofs are in pure modal method without using algebra.

We use the notation in [9] as follows.

Definition 9 Let $\mathfrak{F} = (W, R_\triangle)$ be an n-frame. For each $n{+}1$-ary $R_\triangle$, we define the following two operations $m_\triangle$ and $m^\delta_\triangle$ on the power set $\mathbb{P}(W)$ of W.

- $m_\triangle(X_1, ..., X_n) := \{w \in W \mid$ there are $w_1, ..., w_n$ s.t. $wR_\triangle w_1, ..., w_n$ and $w_i \in X_i$ for all $i\}$
- $m^\delta_\triangle(X_1, ..., X_n) := \{w \in W \mid$ for all $w_1, ..., w_n$: if $wR_\triangle w_1, ..., w_n$, then there is an i with $w_i \in X_i\}$

There is a duality for $m_\triangle$ and $m^\delta_\triangle$.

Proposition 10 $m^\delta_\triangle(X_1, ..., X_n) = W - m_\triangle(W - X_1, ..., W - X_n)$.

Proof $x \in m^\delta_\triangle(X_1, ..., X_n)$ iff $\forall w_1, ..., w_n(xR_\triangle w_1, ..., w_n \to \exists i(w_i \in X_i))$. $x \in m_\triangle(W - X_1, ..., W - X_n)$ iff $\exists w'_1, ..., w'_n(xR_\triangle w'_1, ..., w'_n \wedge \forall i(w'_i \in W - X_i))$. So we have $x \in m^\delta_\triangle(X_1, ..., X_n)$ iff $x \in W - m_\triangle(W - X_1, ..., W - X_n)$. □

Now we can give the definition of ultrafilter extension for polyadic modal languages, which is also from [9].

Definition 11 (ultrafilter extension) Let $\mathfrak{F} = (W, R_\triangle)$ be an n-frame. The $ultrafilter$ $extension$ $ue\mathcal{F}$ of $\mathcal{F}$ is defined as the frame $(Uf(W), R^{ue}_\triangle)$:

- $Uf(W)$ is the set of all ultrafilters over W;

- $u_0 R^{ue}_{\triangle} u_1, ..., u_n$ iff $m_{\triangle}(X_1, ..., X_n) \in u_0$ whenever $X_i \in u_i$ for all $i \leq n$.

The $ultrafilter\ extension$ of an n-model $\mathcal{M} = (\mathcal{F}, V)$ is the model $ue\mathcal{M} = (ue\mathcal{F}, V^{ue})$ where $V^{ue}(p_i) = \{u \text{ is an ultrafilter on } W \mid V(p_i) \in u\}$.

The following lemma is important in dealing with the ultrafilter extension theorem for PML, which is very different from the situation for the monadic modal language. Since [9] omits proofs for polyadic languages, we cannot find a similar lemma in that book. But actually it is a corollary of a classical model theory result, and Guozhen Shen helps us on finding the power of it here.

Lemma 1 Suppose u is an ultrafilter on W^n. Let $\prod_i : W^n \to W$ be the i-th coordinate projection and $b_i = \{\prod_i(x) \mid x \in b\}$ be the projection of $b \in \mathbb{P}(W^n)$. Then $u_i = \{b_i \mid b \in u\}$ is an ultrafilter on W.

Proof First we define a function $' : \mathbb{P}(W) \to \mathbb{P}(W^n)$ as follows:

for each $a \in \mathbb{P}(W)$, $a' = \{(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \mid x \in a \text{ and } a_j \in W \text{ for each } j \leq n\} = \{x \in W^n \mid \prod_i(x) \in a\}$. Obviously, $a \subseteq b$ only if $a' \subseteq b'$ and one can check that $(a')_i = a$.

Now we verify the four conditions for ultrafilter.

1. First since u is an ultrafilter on W^n, $W^n \in u$, which means $W = (W^n)_i \in u_i$.
2. If $a \supseteq b \in u_i$, then $\exists c \in u$ s.t. $b = c_i$. To get $a \in u_i$, we only need to show $a' \supseteq c$. If $x \in c$, then $\prod_i(x) \in b$ and hence $\prod_i(x) \in a$, which means $x \in a'$.
3. If $a, b \in u_i$, then $a = x_i$ and $b = y_i$ for $x, y \in u$. We know that $a \cap b = x_i \cap y_i \supseteq (x \cap y)_i$, which means $a \cap b \in u_i$ since $x \cap y \in u$ and we already showed that 2 holds.
4. if $a \notin u_i$, then $a' \notin u$, which means $W^n - a' \in u$. It follows that $(W^n - a')_i \in u_i$, but $(W^n - a')_i = \{\prod_i(x) \mid x \in W^n - a'\}$. Assume that $y \in \{\prod_i(x) \mid x \in W^n - a'\}$, then $y = \prod_i(x)$ for some $x \in W^n - a'$. If $y \in a$, then $\prod_i(x) \in a$ which means $x \in a'$, a contradiction. So $(W^n - a')_i \subseteq W - a$. By 2 again, $W - a \in u_i$. □

Before showing the ultrafilter extension theorem, we will first prove the saturation theorem because that proof is more general and the key point for both is the above lemma. Recall that we do have a modal saturation notion for PML in [9], as follows.

Definition 12 ($\mathfrak{m}$-saturated) Let $\mathcal{M} = (W, R_{\triangle}, V)$ be an model where $R_{\triangle}$ is n+1-ary. $\mathcal{M}$ is called $\mathfrak{m}$-saturated if for every state $w \in W$ and every sequence $\Sigma_1, \ldots, \Sigma_n$ of sets of PML formulas we have the following.

If for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \ldots, v_n$ s.t. $R_{\triangle} w v_1, \ldots, v_n$ and for each i, $v_i \models \Delta_i$. then there are $w_1, \ldots, w_n$ s.t. $R_{\triangle} w w_1, \ldots, w_n$ and for each i, $w_i \models \Sigma_i$.

The name ‘$\mathfrak{m}$-saturation’ stems from [43], but actually the notion is older: its first occurrence is in [12]. In those original papers, the notion is monadic, while the polyadic case is a direct generalization.

The following theorem is introduced in several books, like [9], without being given a complete proof for polyadic cases. Here we will give a model theoretical proof with the help of the above lemma.

Proposition 13 Let $\mathcal{M}$ be an n-model. Then $ue\mathcal{M}$ is $\mathfrak{m}$-saturated.

Proof Let $\mathcal{M} = \langle W, R, V\rangle$; we will show that its ultrafilter extension $ue\mathcal{M}$ is $\mathfrak{m}$-saturated.

Let $w \in W^{ue}$ and $\Delta_1, \dots, \Delta_n$ be a sequence of sets of formulas. Assume that for every sequence of finite subsets $\Sigma_1 \subseteq \Delta_1, \dots, \Sigma_n \subseteq \Delta_n$, there are states $v_1, \dots, v_n$ s.t. $wR^{ue}v_1, \dots, v_n$ and for each i, $v_i \models \Sigma_i$. We need to construct ultrafilters $u_1, \dots, u_n$ s.t. $wR^{ue}u_1, \dots, u_n$ and $u_i \models \Sigma_i$.

Let $A_i = \{W_1 \times \cdots \times W_{i-1} \times V(\varphi) \times W_{i+1} \cdots \times W_n \mid \varphi \in \Delta_i \text{ and } W_j = W \text{ for all } j\}$, $A = \cup_{1\le i\le n} A_i$ and $B = \{\cup_{1\le i\le n}(W_1 \times \cdots \times W_{i-1} \times Y_i \times W_{i+1} \cdots \times W_n) \mid m^{\delta}_{\triangle}(Y_1, \dots, Y_n) \in w \text{ and } W_j = W \text{ for all } j\}$.

Let $\Delta = A \cup B$ and it is easy to see that $\Delta \subseteq \mathbb{P}(W^n)$.

Claim: Δ has the finite intersection property.

If the claim is true, let u be the ultrafilter on W^n extended Δ, and by the above lemma we know that u_i is an ultrafilter on W. By the definition of u_i, we know that $V(\varphi) \in u_i$ for each $\varphi \in \Delta_i$, that is $u_i \models \Delta_i$. For any $Y_1, \dots, Y_n \subseteq W$, if $m^{\delta}_{\triangle}(Y_1, \dots, Y_n) \in w$, then $\cup_{1\le i\le n}(W_1 \times \cdots \times W_{i-1} \times Y_i \times W_{i+1} \cdots \times W_n) \in u$ and hence $(W_1 \times \cdots \times W_{i-1} \times Y_i \times W_{i+1} \cdots \times W_n) \in u$ for some i since u is an ultrafilter. It follows that $Y_i \in u_i$ for some $i \le n$, which means $wR^{ue}u_1, \dots, u_n$.

Now we come to prove the claim.

First we pick finitely many members of Δ, as $a^i_1, \dots, a^i_{k_i} \in A_i$ for each $i \le n$ and $b_1, \dots, b_m \in B$. By definition, $\cap_{j\le k_i} a^i_j = W_1 \times \cdots \times W_{i-1} \times (V(\varphi^i_1) \cap \cdots \cap V(\varphi^i_{k_i})) \times W_{i+1} \cdots \times W_n$ for $\varphi^i_j \in \Delta_i$, and $\cap_{i\le n} \cap_{j\le k_i} a^i_j = (V(\varphi^1_1) \cap \cdots \cap V(\varphi^1_{k_1})) \times \cdots \times (V(\varphi^n_1) \cap \cdots \cap V(\varphi^n_{k_n}))$ for $\varphi^i_j \in \Delta_i$. Let $\Sigma_i = \{\varphi^i_j \mid 1 \le j \le k_i\}$, and we know that Σ_i is a finite subset of Δ_i. Using the assumption, it follows that there are $v_1, \dots, v_n$ s.t. $wR^{ue}v_1, \dots, v_n$ and for each i, $v_i \models \Sigma_i$, which means if $Y_i \in v_i$ for all i, then $m_{\triangle}(Y_1, \dots, Y_n) \in w$. Since $\cap_{\varphi\in\Sigma_i} V(\varphi) \in v_i$, we know that $m_{\triangle}(\cap_{\varphi\in\Sigma_1} V(\varphi), \dots, \cap_{\varphi\in\Sigma_n} V(\varphi)) \in w$.

Consider $b_1, \dots, b_m$. For each $j \le m$, $b_j = \cup_{1\le i\le n}(W_1 \times \cdots \times W_{i-1} \times Y_i \times W_{i+1} \cdots \times W_n)$ for some $Y^j_1, \dots, Y^j_n \subseteq W$ s.t. $m^{\delta}_{\triangle}(Y^j_1, \dots, Y^j_n) \in w$. Thus we have

$$m_{\triangle}(\cap_{\varphi\in\Sigma_1} V(\varphi), \dots, \cap_{\varphi\in\Sigma_n} V(\varphi)) \cap (\cap_{j\le m} m^{\delta}_{\triangle}(Y^j_1, \dots, Y^j_n)) \in w$$

since w is an ultrafilter. It follows that there is

$$x \in m_{\triangle}(\cap_{\varphi\in\Sigma_1}V(\varphi), \dots, \cap_{\varphi\in\Sigma_n}V(\varphi)) \cap (\cap_{j\leq m} m^{\delta}_{\triangle}(Y^j_1, \dots, Y^j_n))$$

which means the followings hold for x by the definition of $m^{\delta}_{\triangle}$ and $m_{\triangle}$:

1. There are $w_1, \dots, w_n$ s.t. $Rxw_1, \dots, w_n$ s.t. $w_i \in \cap_{\varphi\in\Sigma_i}V(\varphi)$ for each $i \leq n$.

2. For each $j \leq m$, for all $t_1, \dots, t_n \in W$, if $Rxt_1, \dots, t_n$, then $\exists i$ s.t. $t_i \in Y^j_i$.

As a consequence, for $\forall j \leq m$ $\exists i$ s.t. $w_i \in Y^j_i \cap \cup_{\varphi\in\Sigma_i}V(\varphi)$, which means $(w_1, \dots, w_n) \in \cap_{i\leq n} \cap_{j\leq k_i} a^i_j \cap \cap_{j\leq m} b_j$. As a result, Δ has the finite intersection property. □

Now we come to deal with the ultrafilter extension theorem, which can be found in [19], where Goldblatt had already considered the polyadic languages, but the proof there is algebraic. However, the saturation theorem is in some sense a special version of the UE-theorem, so we can use a same strategy to give a modal proof here by using the above lemma about ultrafilters.

Proposition 14 (ultrafilter extension theorem) Let $\mathcal{M}$ be an n-model. Then, for any formula φ and any ultrafilter u over W, $V(\varphi) \in u$ iff $ue\mathcal{M}, u \models \varphi$.

Hence, for each state w in $\mathcal{M}$ we have $w \leftrightsquigarrow \prod_w$, where $\prod_w$ is the principal ultrafilter generated by $\{w\}$.

Proof We prove by induction on φ, and the only nontrivial case is that of the polyadic modal operator. So suppose $\varphi = \triangle(\varphi_1, ..., \varphi_n)$.

($\Leftarrow$) Assume $ue\mathcal{M}, u \models \varphi$. It follows that there are $u_1, ..., u_n$ s.t. $uR^{ue}_{\triangle}u_1, ..., u_n$ and $ue\mathcal{M}, u_i \models \varphi_i$. By induction hypothesis, $V(\varphi_i) \in u_i$ for all i, and hence by the definition of $R^{ue}_{\triangle}$, $m_{\triangle}(V(\varphi_1), ..., V(\varphi_n)) \in u$. Now the conclusion follows directly from the fact that $m_{\triangle}(V(\varphi_1), ..., V(\varphi_n)) = V(\varphi)$, which can be easily checked.

($\Rightarrow$) Assume $V(\varphi) \in u$. we need to find ultrafilters $u_1, ..., u_n$ s.t. $uR^{ue}_{\triangle}u_1, ..., u_n$ and $ue\mathcal{M}, u_i \models \varphi_i$ for all i, but this is just a special case of the Δ construction in the above proof, where each Σ_i is a singleton. The only thing we need to check again is the claim:

Δ has the finite intersection property.

For finitely many arbitrary members of Δ, as $a_i \in A_i$ for each $i \leq n$ and $b_1, \dots, b_m \in B$. By definition, $\cap_{i\leq n} a_i = V(\varphi_1) \times \cdots \times V(\varphi_n)$. Using the assumption, we also know that $m_{\triangle}(V(\varphi_1), ..., V(\varphi_n)) \in u$.

Consider $b_1, \dots, b_m$. For each $j \leq m$, $b_j = \cup_{1\leq i\leq n}(W_1 \times \cdots \times W_{i-1} \times Y^j_i \times W_{i+1} \cdots \times W_n)$ for some $Y^j_1, \dots, Y^j_n \subseteq W$ s.t. $m^{\delta}_{\triangle}(Y^j_1, \dots, Y^j_n) \in u$. Thus we have

$$m_{\triangle}(V(\varphi_1), \dots, V(\varphi_n)) \cap (\cap_{j\leq m} m^{\delta}_{\triangle}(Y^j_1, \dots, Y^j_n)) \in u$$

since u is an ultrafilter. It follows that there is

$$x \in m_{\triangle}(V(\varphi_1), \ldots, V(\varphi_n)) \cap (\cap_{j \leq m} m^{\delta}_{\triangle}(Y^j_1, \ldots, Y^j_n))$$

which means the followings hold for x by the definition of $m^{\delta}_{\triangle}$ and $m_{\triangle}$:

1. There are $w_1, \ldots, w_n$ s.t. $Rxw_1, \ldots, w_n$ s.t. $w_i \in V(\varphi_i)$ for each $i \leq n$.
2. For each $j \leq m$, for all $t_1, \ldots, t_n \in W$, if $Rxt_1, \ldots, t_n$, then $\exists i$ s.t. $t_i \in Y^j_i$.

As a consequence, for $\forall j \leq m$ $\exists i$ s.t. $w_i \in Y^j_i \cap V(\varphi_i)$, which means $(w_1, \ldots, w_n) \in \cap_{i \leq n} a_i \cap \cap_{j \leq m} b_j$. As a result, Δ has the finite intersection property. □

5 Characterization via bisimulation

In this section, we give a proof for the van-Benthem Characteristic Theorem on PML by using a bisimulation invariant method. The bisimulation notion for modal logic is first from [4], where van Benthem proved the characterization theorem. But he used different names in [7], where the zigzag relation came from. But the saturation-based strategy we will use is due to [38]. Note, though, that their works are all dealing with only the monadic modal language. The basic structure of our proof in this section is guided by [9], where the authors only gave a proof for the basic modal language, and omit the polyadic cases, so we will focus on the polyadic cases here. Notice that in [9], all the statements in this section has already been mentioned, but if we give a proof here, it means that the book omits such a detailed proof.

First we introduce polyadic bisimulation notions as follows.([9])

Definition 15 ((P-)pm-bisimulation) Let $\mathcal{M} = (W, R_{\triangle}, V)$ and $\mathcal{M}' = (W', R'_{\triangle}, V')$ be two models. A non-empty binary relation $Z \subseteq W \times W'$ is called a *pm-bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ if the following conditions are satisfied:

- i If wZw', then w and w' satisfy the same propositional letters (in P).
- ii If wZw' and $R_{\triangle}wv_1, \ldots, v_n$ then there are $v'_1, \ldots, v'_n$ in W' s.t. $R'_{\triangle}w'v'_1, \ldots, v'_n$ and $v_iZv'_i$ for all $i \leq n$; (the forth condition).
- iii If wZw' and $R'_{\triangle}w'v'_1, \ldots, v'_n$ then there are $v_1, \ldots, v_n$ in W s.t. $R_{\triangle}wv_1, \ldots, v_n$ and $v_iZv'_i$ for all $i \leq n$. (the back condition)

When Z is a bisimulation linking two states w in $\mathcal{M}$ and w' in M' we say that w and w' are bisimilar, and we write $Z : \mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$. If there is a bisimulation Z such that $Z : \mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$, we sometimes write $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$; likewise, if there is some bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, we write $\mathcal{M} \leftrightarroweq \mathcal{M}'$,saying $\mathcal{M}$ and $\mathcal{M}'$are bisimilar.

We already know that PML is a fragment of first order logic, and hence we will show that it is exactly the fragment closed under the bisimulation above. The standard translation also works for PML as in [9], and for instance we can translate

$\triangle(p_1, ..., p_n)$ as $\exists y_1, ..., y_n(xR_\triangle y_1, ..., y_n \wedge \bigwedge_{i\leq n} Py_i)$. In the following we show that this bisimulation is indeed sound w.r.t. the PML-equivalence.

Proposition 16 Let $\mathcal{M} = (W, R_\triangle, V)$ and $\mathcal{M}' = (W', R'_\triangle, V')$ be two models. Then for every $w \in W$ and $w' \in W'$, $w \leftrightarrows w'$ implies $w \leftrightsquigarrow w'$. In words, PML formulas are invariant under pm-bisimulation.

Proof We use induction on formulas, and we focus on the modality case, since others are trivial. Suppose that $w \leftrightarrows w'$ and $w \models \triangle(\varphi_1, ..., \varphi_n)$. Then we know that there are $v_1, \ldots, v_n$ s.t. $R_\triangle wv_1, \ldots, v_n$, and each $v_i \models \varphi_i$. By the forth condition, there are $v'_1, \ldots, v'_n$ in W' s.t. $R_\triangle w'v'_1, \ldots, v'_n$ and $v_i \leftrightarrows v'_i$ for each i. From the I.H. we have each $v'_i \models \varphi_i$. As a result, $w' \models \triangle(\varphi_1, ..., \varphi_n)$. For the converse direction just use the back condition. $\square$

The bisimulation notion for PML is similar with that of the monadic modal language, and it is possible to prove a restricted converse to the above theorem, namely the Hennessy-Milner Theorem forPML as follows, which can be found in [9]. However, The key reference for this theorem is [24].

Theorem 17 (Hennessy-Milner Theorem)**.** *Let* $\mathcal{M} = (W, R_\triangle, V)$ *and* $\mathcal{M}' = (W', R'_\triangle, V')$ *be two image-finite models. Then for every* $w \in W$ *and* $w' \in W'$*,* $w \leftrightarrows w'$ *iff* $w \leftrightsquigarrow w'$.

Proof A direct generalization of the monadic proof will work here. $\square$

So there remains a question: is there any generalization of the above theorem, or in which situation can we treat modal equivalence and bisimularity as the same thing. This question leads to the following definition.

Definition 18 (Hennessy-Milner Classes) K is a Hennessy-Milner class, if for every two pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ in K, $w \leftrightarrows w'$ iff $w \leftrightsquigarrow w'$.

The concept of a Hennessy-Milner class is first from [18] and [25], but those works are mostly on monadic cases. In [25], Hollenberg proved that equivalence of models implies bisimilarity between their ultrafilter extensions, and the following theorem is a generalization, which can be found in [9]. Even though they only dealt with the monadic operator, the polyadic cases are similar.

Proposition 19 The class of $\mathfrak{m}$-saturated models has the Hennessy-Milner property.

Proof Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be two $\mathfrak{m}$-saturated models. It is sufficient to show that the modal equivalence relation $\leftrightsquigarrow$ is indeed a bisimulation. We focus on the forth condition, and the back condition is similar.

Suppose that $w \leftrightsquigarrow w'$ for $w \in W$ and $w' \in W'$, and for some $v_1, ..., v_n \in W$, $wR_1v_1, ..., v_n$. Let Σ_i be the truth set at v_i. It is clear that for each sequence

$(A_1, ..., A_n)$ s.t. each A_i is a finite subset of Σ_i, we know that $\mathcal{M}, v_i \models \wedge A_i$, hence $\mathcal{M}, w \models \triangle(\wedge A_1, ..., \wedge A_n)$. Since $w \leftrightsquigarrow w'$, we have $\mathcal{M}', w' \models \triangle(\wedge A_1, ..., \wedge A_n)$. Thus there are $v_1', ..., v_n'$ s.t. $w'R'v_1', ..., v_n'$ and $\mathcal{M}', v_i' \models \wedge A_i$. Therefore by m-saturation, there are $u_1, ..., u_n$ s.t.$w'R'u_1, ..., u_n$ and $\mathcal{M}, u_i \models \Sigma_i$ for each i, that is, $w_i \leftrightsquigarrow u_i$. As a consequence, the forth condition holds for $\leftrightsquigarrow$. □

Actually $\mathfrak{m}$-saturation is a special case of the saturation property in classical model theory, as the following theorem says, and For more on saturated models, see [10].

Proposition 20 Any countably saturated model is $\mathfrak{m}$-saturated. It follows that the class of countably saturated models has the Hennessy-Milner property.

Proof Suppose that $\mathcal{M} = (W, R_\triangle, V)$ is a countably saturated model. Let $w \in W$ and $\Sigma_1, \dots, \Sigma_n$ be a sequence of sets of PML formulas s.t. for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \dots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \dots, v_n$ s.t. $R_\triangle w v_1, \dots, v_n$ and for each i $v_i \models \Delta_i$.

Define $\Sigma = \{Rwx_1, \dots, x_n\} \cup \bigcup_{i \leq n}\{ST_{x_i}(\varphi) \mid \varphi \in \Sigma_i\}$.

Claim: Σ is consistent with $Th((\mathcal{M}, w))$, the first-order theory of $(\mathcal{M}, w)$.

If we prove the Claim, we know that Σ itself is realized in some $v_1, \dots, v_n \in W$, since Σ is a n-type with just one parameter and $\mathcal{M}$ is countably saturated. By $(\mathcal{M}, w) \models Rwx_1, \dots, x_n[v_1, \dots, v_n]$ it follows that $Rwv_1, \dots, v_n$ and by $(\mathcal{M}, w) \models \bigcup_{i \leq n}\{ST_{x_i}(\varphi) \mid \varphi \in \Sigma_i\}[v_1, \dots, v_n]$, $v_i \models \{ST_{x_i}(\varphi) \mid \varphi_i \in \Sigma_i\}$. Thus, $v_i \models \Sigma_i$ for each i.

Now we prove the Claim: Suppose that Σ is not consistent with the first-order theory of $(\mathcal{M}, w)$. Hence there is a sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \dots, \Delta_n \subseteq \Sigma_n$ s.t. $\bigcup_{i \leq n}\{ST_{x_i}(\varphi) \mid \varphi \in \Delta_i\} \cup \{Rwx_1, \dots, x_n\} \cup Th((\mathcal{M}, w))$ is inconsistent. But that is impossible since we already know that for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \dots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \dots, v_n$ s.t. $R_\triangle w v_1, \dots, v_n$ and $v_i \models \Delta_i$. □

Notice that in the above proof, we only need to assume that the model is 2-saturated instead of ω-saturated, even though we deal with polyadic language, because the parameter set is still a singleton. In [9], the author said that when dealing with polyadic case we need a stronger saturation property, but it is a misunderstanding.

Here we also need to use the finite k-bisimulation notion on PML, which can be found in [9].

Definition 21 ((P)-k-pm-bisimulation) Let $\mathcal{M} = (W, R_\triangle, V)$ and $\mathcal{M}' = (W', R_\triangle', V')$ be two models. We say that w and w' are k-bisimular (notation: $w \leftrightarroweq_k w'$) if there is a sequence of binary relations $Z_k \subseteq ... \subseteq Z_0$ with the following properties (for $i + 1 \leq k$ and any $v \in W$,$v' \in W'$):

i wZ_kw';
ii If vZ_0v', then v and v' agree on all propositional letters (in P);
iii If $vZ_{i+1}v'$ and $R_\triangle vv_1, \dots, v_n$ then there are $v'_1, \dots, v'_n$ in W' s.t. $R'_\triangle v'v'_1, \dots, v'_n$ and $v_iZ_iv'_i$ for all $i \leq n$; (the forth condition).
iv If $vZ_{i+1}v'$ and $R'_\triangle v'v'_1, \dots, v'_n$ then there are $v_1, \dots, v_n$ in W s.t. $R_\triangle vv_1, \dots, v_n$ and $v_iZ_iv'_i$ for all $i \leq n$. (the back condition)

The following proposition is just like the one in the monadic modal language.

Proposition 22 Let Φ be a finite set of proposition letters, $\mathcal{M} = (W, R_\triangle, V)$ and $\mathcal{M}' = (W', R'_\triangle, V')$ be two models of the correspondent language under Φ. Then the following are equivalent.

i $w \leftrightarroweq_k w'$
ii w and w' agree on all PML formulas of degree at most k.

Proof By an induction on k with a similar strategy for proving the Hennessy-Milner theorem. □

Like the van-Benthem Characterization Theorem for monadic modal logics, the proof for PML is based on a Detour Lemma.

Theorem 23 (Characterization Theorem). *Let φ be a first order formula. φ is invariant for* PML*-bisimulations iff it is equivalent to the standard translation of a* PML *formula.*

Proof A standard detour strategy for the van Benthem theorem suffices.

$$\begin{array}{ccc} \mathcal{M}, w & & \mathcal{N}, v \\ \preceq\downarrow & & \downarrow\preceq \\ \mathcal{M}^*, w^* & \leftrightarroweq & \mathcal{N}^*, v^* \end{array}$$

Where the $*$ model construction need to preserve first order truth and get saturated models. So we cannot just use the ultrafilter extension. As in [9], the construction could be taking the ultrapower of $\mathcal{M}$ under a ω-incomplete ultrafilter u. So now we need to use some properties about ultrapower to complete our proof. □

[10] is a classic reference for the ultraproduct construction on the first order language, which can directly apply to polyadic modal models, while Doets and Van Benthem [8] gave an intuitive explanation of the ultraproduct construction. We Recall the definition of Ultraproduct and Ultrapower for modal models in [9] as the following.

Let $C = \prod_{i \in I} W_i$ be the Cartesian product of $\{W\}_{i \in I}$ and u be an ultrafilter on the index set I. For two functions $f, g \in C$ we say that f and g are u-equivalent ($f \sim_u g$) if $\{i \in I \mid f(i) = g(i)\} \in u$. One can easily check this is indeed an equivalence relation.

Let $f_u = \{g \in C \mid g \sim_u f\}$. The ultraproduct of $\{W\}_{i \in I}$ modulo u is define as follows:

$$\prod_u W_i = \{f_u \mid f \in \prod_{i \in I} W_i\}$$

Definition 24 (ultraproduct) Let $\mathcal{M}_i = (W_i, R_{\triangle i}, V_i)(i \in I)$ be n-models. The ultraproduct $\prod_u \mathcal{M}$ modulo u is describled as follows.

- (i) The universe W_u is the set $\prod_u W_i = \{f_u \mid f \in \prod_{i \in I} W_i\}$.
- (ii) The valuation V_u is defined by

$$f_u \in V_u(p) \text{ iff } \{i \in I \mid f(i) \in V_i(p)\} \in u.$$

- (iii) The n-ary relation $R_{\triangle u}$ is given by

$$f_u^0 R_{\triangle u} f_u^1 \ldots f_u^n \text{ iff } \{i \in I \mid f^0(i) R_{\triangle i} f^1(i) \ldots f^n(i)\} \in u.$$

If all the $\mathcal{M}_i$ are the same model $\mathcal{M}$, we say $\prod_u \mathcal{M}$ the ultrapower of $\mathcal{M}$ modulo u.

Now there are just two things we need to show. The first one is a classical fact in model theory as follows, and on can find proofs in model theory textbooks like [10]. The book [9] claims that there is a proof in its appendix, but actually it only state the following proposition without giving a proof.

Lemma 2 (cf. Thm 6.1.1 in [10]) Let $\mathcal{L}$ be a countable first-order language, u a countably incomplete unltrafilter over a non-empty set I, and $\mathcal{M}$ an $\mathcal{L}$-model. Then the ultrapower $\prod_u \mathcal{M}$ is countably saturated.

The second one is the fact that ultrapower can preserve local modal truth, which is actually a special case of Łös's theorem. One can find the following theorem in [9], but the authors leaves the proof as an exercise, so we give a proof here.

Lemma 3 Let $\prod_u \mathcal{M}$ be an ultrapower of $\mathcal{M}$. Then for all PML formulas φ, we have $\mathcal{M}, w \models \varphi$ iff $\prod_u \mathcal{M}, (f_w)_u \models \varphi$, where f_w is the constant function s.t. $f_w(i) = w$ for all $i \in I$.

Proof We do an induction on φ to show a strong result: for any $f \in \prod W$,

$$f_u \models \varphi \text{ iff } \{i \in I \mid \mathcal{M}, f(i) \models \varphi\} \in u$$

So first we fix an f. The basic case is directly followed from our definition for $V_u(p)$ by $I \in u$ since u is an ultrafilter on I, and the Boolean connective cases are easy to check. We give the negation case here:

$$\begin{aligned} f_u \models \neg\psi \text{ iff } & \text{ iff } \{i \in I \mid \mathcal{M}, f(i) \models \psi\} \notin u \\ & \text{ iff } I - \{i \in I \mid \mathcal{M}, f(i) \models \psi\} \in u \end{aligned}$$

iff $\{i \in I \mid \mathcal{M}, f(i) \not\models \psi\} \in u$
iff $\{i \in I \mid \mathcal{M}, f(i) \models \neg\psi\} \in u$

Here one can see why we need u to be an ultrafilter, and it is not sufficient to just assume that u is a filter.

Now we focus on the modal operator and assume that $\varphi = \triangle(\varphi_1, ..., \varphi_n)$.

($\Rightarrow$)Suppose that $f_u \models \varphi$ and hence there are $f_1, ..., f_n \in \prod W$ s.t. $(f_j)_u \models \varphi_j$ for each j and $f_u R_{\triangle u}(f_1)_u, ..., (f_n)_u$, which means the following two conditions hold:

1. $A_j = \{i \in I \mid \mathcal{M}, f_j(i) \models \varphi_j\} \in u$ for each $j \leq n$; (by I.H.)
2. $B = \{i \in I \mid f(i) R_{\triangle u} f_1(i), ..., f_n(i)\} \in u$. (by definition of $R_{\triangle u}$)

Therefore we know that $\bigcap_{j \leq n} A_j \cap B \in u$, which means

$\{i \in I \mid$ for each $j \leq n : \mathcal{M}, f_j(i) \models \varphi_j$ and $f(i) R_{\triangle u} f_1(i), ..., f_n(i)\}$ is in u and hence its superset $\{i \in I \mid f(i) \models \triangle(\varphi_1, ..., \varphi_n)\}$ belong to u by u is an ultrafilter.

($\Leftarrow$) Suppose that $\{i \in I \mid \mathcal{M}, f(i) \models \varphi\} \in u$, which means

$C = \{i \in I \mid \exists x_1, ..., x_n \in W(\forall_{j \leq n} \mathcal{M}, x_j \models \varphi_j \bigwedge f(i) R_{\triangle u} x_1, ..., x_n)\} \in u$

It is sufficient to find $f_1, ..., f_n$ s.t.

$\{i \in I \mid \forall_{j \leq n} \mathcal{M}, f_j(i) \models \varphi_j$ and $f(i) R_{\triangle u} f_1(i), ..., f_n(i)\} \in u$, because then we can use the fact that u is closed under taking supersets to get the result. (just like a converse procedure of the $\Rightarrow$ part)

To find such $f_1, ..., f_n$, first we need to use the Axiom of Choice as follows:

1. For each $i \in C$, select $(a^i_1, ..., a^i_n)$ as a sequence of witnesses.
2. For each $i \notin C$, select $(a^i_1, ..., a^i_n)$ as a sequence of arbitrary members of W.

For each $j \leq n$, define f_j as $f_j(i) = a^i_j$, and then one can check that each f_j satisfies our requirement. $\Box$

One should be careful about the using of AC in our proof and see how strong we need the “choice” to be, compared with the case in proving Łös’s theorem.

6 Craig Interpolation Theorem

There is a standard model theoretical method of proving the Craig Interpoaltion Theorem (CIT) for monadic normal modal logics as in [2] and [3]. Rosen gave another proof which can work within finite models in [39]. In [3], the author also says that the general version of this interpolation theorem for minimal modal logics with an arbitrary number of polyadic modalities follows from the results in [35], which is about some amalgamation properties for Boolean Algebra, and actually it’s a more general result than the one we will deal with. Moreover, in [26], there is a deep connection between the amalgamation on algebras and the interpolation on logic, which includes the theorem we will talk about in this section—CIT for PML.

Even though there are already algebraic proofs of CIT for PML, we find that both the proofs in [39] and [3] can direcly apply to PML, and those proofs are purely model theoretical on modal logic, which is much more easier than the algebraic one in [26]. We choose to give a proof using the method in [3], since Rosen's proof requires an "unravelling" model construction to get "tree-like" models, which is more complicated and for our purpose, we don't need to limit us on finite models. But one should notice that Rosen's method is very important in proving CIT for some other special logics, where the finitary method is essential. The result here is not new, but we use a new simple proof by using a similar method in [34].

Theorem 25. *Each normal polyadic modal logic $\mathbb{K}_n$ has the Craig Interpolation Theorem. More precisely, Let $atom(\alpha) = \{p \mid p \text{ occurs in } \alpha\}$, and $\varphi \vdash_{\mathbb{K}_n} \psi$, then there is a formula α s.t. $\varphi \vdash_{\mathbb{K}_n} \alpha \vdash_{\mathbb{K}_n} \psi$ and $atom(\alpha) \subseteq atom(\varphi) \cap atom(\psi)$.*

Proof First we fix an n and just use $\vdash$ without a subscript. Since we already know that $\mathbb{K}_n$ is strongly complete w.r.t to all n-frames, we could freely switch between $\models$ and $\vdash$. For convenience, let $P = atom(\varphi)$, $Q = atom(\psi)$ and $R = atom(\alpha)$. We show that the set $cons_R(\varphi)$ of all consequences of φ in R language satisfies the following claim:

$$cons_R(\varphi) \models \psi.$$

By a standard compactness argument, we can find the interpolant. To prove the claim, let $\mathcal{M} = (W, R_\triangle, V)$ be an n-model s.t. $(\mathcal{M}, a) \models cons_R(\varphi)$ for some $a \in W$, and we need to show that $(\mathcal{M}, a) \models \psi$. By a routine argument, the R-theory $Th_R(\mathcal{M}, a)$ is consistent with $\{\varphi\}$, and by compactness again, there is a P-model $(\mathcal{N}, b) \models \varphi$ s.t. $(\mathcal{M}, a) \equiv_R (\mathcal{N}, b)$. Suppose that $(\mathcal{N}, b) = (W', R'_\triangle, V')$. We have already shown that there are $\mathfrak{m}$-saturated models which can preserve modal truth in this paper before, so without loss of generality we assume that both $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are $\mathfrak{m}$-saturated. It follows that the $\equiv_R$ is indeed an R-bisimulation. Next we construct a product model $\mathcal{MN}, (a, b)$ s.t. $(\mathcal{M}, a) \leftrightarroweq_Q \mathcal{MN}, (a, b)$ and $(\mathcal{N}, b) \leftrightarroweq_P \mathcal{MN}, (a, b)$, which is sufficient for our proof:

$$(\mathcal{N}, b) \models \varphi \Rightarrow \mathcal{MN}, (a, b) \models \varphi \Rightarrow \mathcal{MN}, (a, b) \models \psi \Rightarrow (\mathcal{M}, a) \models \psi$$

Now we come to the construction. Let $Z = \{(x, y) \in W \times W' \mid x \leftrightarroweq_R y\}$, and define $\mathcal{MN} = (Z, R^*_\triangle, V^*)$ as follows:

$$(x, y)R^*_\triangle(x_1, y_1), ..., (x_n, y_n) \text{ iff } xR_\triangle x_1, ..., x_n \text{ and } yR'_\triangle y_1, ..., y_n$$

For each $(x, y) \in Z$,

$$(x, y) \in V^*(p) \iff \begin{cases} x \in V(p) & \text{if} \quad p \in Q \\ y \in V'(p) & \text{if} \quad p \in P \\ \text{never} & \text{if} \quad \text{otherwise} \end{cases}$$

Notice that V^* is well-defined since every $(x, y) \in Z$ satisfies $x \leftrightarroweq_R y$. Now it is sufficient to check that our construction satisfies the requirement.

Let $B_1 = \{(x, (z_1, z_2)) \mid x \leftrightarroweq_Q z_1 \text{ and } z_2 \in W'\}$ be a relation on $W \times Z$ and $B_2 = \{(y, (z_1, z_2)) \mid y \leftrightarroweq_P z_2 \text{ and } z_1 \in W\}$ be a relation on $W' \times Z$. Obviously, $aB_1(a, b)$ and $bB_2(a, b)$. We show that B_1 is a Q-bisimulation and B_2 is a P-bisimulation. The propositional letters case is directly from the definition of V^*. We give a verification for B_1, and the B_2 case is similar.

Forth condition:

Suppose that $xB_1(w, v)$ and there are $x_1, ..., x_n$ s.t. $xR_\triangle x_1, ..., x_n$. By $x \leftrightarroweq_Q w$, there are $w_1, ..., w_n$ s.t. $wR_\triangle w_1, ..., w_n$ and $x_i \leftrightarroweq_Q w_i$ for each i. Again, by $w \leftrightarroweq_R v$, there are $v_1, ..., v_n$ s.t. $vR'_\triangle v_1, ..., v_n$ and $w_i \leftrightarroweq_R v_i$ for each i, and hence $(w_i, v_i) \in Z$. But then by the definition of B_1 and $R^*_\triangle$, we know that $x_iB_1(w_i, v_i)$ for each i and $(w, v)R^*_\triangle(w_1, v_1), ..., (w_n, v_n)$.

Back condition:

Suppose that $xB_1(w, v)$ and there are $(w_1, v_1), ..., (w_n, v_n)$ such that $(w, v)R^*_\triangle(w_1, v_1), ..., (w_n, v_n)$. By the definition of $R^*_\triangle$, we have $wR_\triangle w_1, ..., w_n$. Thus, by $x \leftrightarroweq_Q w$, there are $x_1, ..., x_n$ s.t. $xR_\triangle x_1, ..., x_n$ and $x_i \leftrightarroweq_Q w_i$ for each i. It follows that $x_iB_1(w_i, v_i)$ for each i by the definition of B_1. □

7 Conclusion and Further Work

This paper considered some basic model theoretical properties for PML. Even though we can see all of those results in existing literatures, they always only gave proof details for the monadic modal language and omitted the polyadic cases. Moreover, we can also see some mistakes in existing literatures when dealing with normal polyadic modal logics. So we tried to fix and make up those lacks in our paper by giving complete proofs for polyadic cases. Mainly, we did a model theoretical exposition work for PML in this paper.

First we clarified some notable mistakes in axiomatizing normal polyadic modal logics, where one could also see some differences between the monadic modal logic and polyadic modal logics. Then we considered two important model constructions: filtration and ultrafilter extension. For filtration, the basic theory is similar to the monadic case, but for ultrafilter extension, we need to use a much more complicate strategy to get the results. The key point there is that we used a "big" ultrafilter on a product to get "small" unltrafilters by projections.

In the second part we first stated the van-Benthem Characterization theorem for polyadic modal logics, and gave detailed proofs for those we only found monadic ones in textbooks. Even though the method is standard, we tried to exhibit the differences between monadic case and polyadic cases. Next we gave a proof for the Craig Interpolation Theorem of normal polyadic modal logics $\mathbb{K}_n$, by a standard modal model theoretical method which was used in proving the case for normal modal logic $\mathbb{K}$.

For further works, first we know that there is a special kind of modal logic called the weakly aggregative modal logic(WAML), which introduced in [27], and we can see that the modal operator there is unary but the relational models are n-ary. So we may consider whether WAML has some basic properties which both monadic and polyadic modal logic have—such as those we discussed in this paper: van-Benthem Characterization theorem and the Craig Interpolation Theorem.

Another point is about a general semantics for PML. As we know, modal logics have some alternative semantics, such as the neighborhood semantics and the possibility semantics, and both are more general than the relational one. But those semantics can only work for monadic modal language. So it's natural to think about a general version of those semantics for polyadic modal languages. The work in [1] may give us some hints, where the author uses topological-like models as a variant of some given n-ary relational models. But the problem is that we cannot treat those models as a generalization of neighborhood models, and the semantics there is not "standard". Hence there still remains some work to do—finding a neighborhood semantics for polyadic modal logics.

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模型论视角下对正规多元模态逻辑的阐述

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摘　要

本文对正规多元模态逻辑做了模型论视角的整体阐述。正规多元模态逻辑（PML）是对一元模态逻辑系统 K，在 n 元算子上的推广。而多元模态逻辑的研究相对于一元逻辑较为匮乏。文献中的一系列有关 PML 的结果也被看作是有关 K 的结论的直接推广，而缺少部分完整证明，且已有证明多为代数证明。PML 对于 K 的推广在某些方面是非平凡的，忽略这一点导致了一些教材及文章中甚至存在各种错误。从证明的角度上讲，对于 PML 的证明有时也需要不同的方法。基于以上几点考虑，我们认为有必要从模型论视角对 PML 做一个细致的考察，并给出一些模型论版本的证明，来简化以往的代数证明，从而给研究者提供一个统一的参考。本文从两个模态逻辑常用的模型构造方法（滤子和超滤扩张）出发，以经典教材中的定义为准，补全一些重要定理在多元语言下的模型论方法的详细证明。然后我们对 van-Benthem 刻画定理的多元版本证明做了一个澄清，考察了多元语言和一元语言下证明的具体区别。最后我们用模型论方法证明了 PML 具有插值性，而该定理在文献中往往是被当作一些代数事实的推论。

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