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On f-generic types in Presburger Arithmetic*

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Abstract. We study the f-generics of ordered additive group G of any model of Presburger Arithmetic, and give a classification for the f-generic types of G^n for any $n \in \mathbb{N}^+$. As an application of this classification theorem, we conclude that every f-generic type of G^n is $\emptyset$-definable. We also consider the multiplicative group H of the p-adic field $\mathbb{Q}_p$, and prove that every f-generic type of H^n is also $\emptyset$-definable.

1 Introduction and Preliminaries

1.1 Introduction

In model theory, we study a group G definable in a structure M and the action of G on its type space $S_G(M)$, which is the collection of all types over M containing the formula defining G. The space of generic types, introduced by Poizat as a generalization of the notation of generic points in an algebraic group, plays a heart role when $Th(M)$ is stable. But, for unstable case, the generic types may not exist. So various of weakenings of the generic were introduced to unstable enviroment to generalize the properties of stables groups to unstable context. The notation of weakly generic types introduced by Newelski in [9], which exists in any context, is a suitable substitution for generic types. In [6], f-generic was introduced, and nice result of [2] shows that f-generic coincides with weakly generic when G is a NIP definably aminable group.

In [11], Marcin Petrykowski gave a nice description of f-generic types in groups $(R, +) \times (R, +)$ with $(R, <, +, \cdot)$ an o-minimal expansion of real closed field. An analogs question is: What are the f-generic types of G^n, the product of n copies of ordered additive groups $(\mathbb{Z}, +, <)$ of integers (or models of Presburger arithmetic)? In [4], Conant and Vojdani gave a couple of nice equivalent characterization for f-generic types of G^n. But they didn't give a classification for the space of f-generic types.

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This paper is inspired by the ideas of [11], and provides a full classification for the space of f-generic types of the product of n copies of ordered groups of any models of Presburger arithmetic. As an application, the main results of [4] spring readily from the classification theorem.

Let M be an elementary extension of $(\mathbb{Z}, +, <, 0)$, $\mathbb{M} \succ M$ a monster model. G denotes the additive group $(\mathbb{M}, +)$, $S_G(M)$ the space of complete types over M extending the formula '$x \in G$'. G^0 is the definable connected component of G. Namely, G^0 the is the intersection of all definable subgroups of G with finite index. Let L_n denote the space of homogeneous n-nary $\mathbb{Q}$-linear functions. For $f, g \in L_n$ and $\alpha, \beta \in \mathbb{M}^n$ such that $\alpha \in dom(f)$ and $\beta \in dom(g)$, by $f(\alpha) \ll_M g(\beta)$ we mean that for all $a, b \in M$ and $k, l \in \mathbb{N}^+$, $kf(\alpha) + a < lg(\beta) + b$. By $f(\alpha) \sim_M g(\beta)$, we mean that neither $f(\alpha) \ll_M g(\beta)$ nor $g(\beta) \ll_M f(\alpha)$. Let $f_0, ..., f_m \in L_n$, we say $0 \ll_M f_1(\alpha) \ll_M ... \ll_M f_m(\alpha)$ is a maximal positive chain of α over M if for any $g \in L_n$ with $g(\alpha) > 0$, neither $f_m(\alpha) \ll_M g(\alpha)$ nor $g(\alpha) \ll_M f_0(\alpha)$.

Now we highlight our main result as follows:

Theorem 1 Let $M \succ \mathbb{Z}$, $\alpha = (\alpha_1, ..., \alpha_n) \in (G^n)^0$. Then there exists a finite subset $\{f_0, ..., f_m\}$ of L_n such that $f_0(\alpha) = 0 \ll_M f_1(\alpha) \ll_M ... \ll_M f_m(\alpha)$ is the maximal positive chain of α over M. If α realizes an f-generic type $p \in S_{G^n}(M)$ then for every $\beta \in G^0$, $p = \text{tp}(\alpha, \beta/M) \in S_{G^{n+1}}(M)$ is an f-generic type if and only if one of the following holds:

- $f_m(\alpha) \ll_M \beta$ or $\beta \ll_M -f_m(\beta)$;
- there is i with $0 \leq i < m$ and $g \in L_n$ such that $f_i(\alpha) \ll_M \epsilon(\beta - g(\alpha)) \ll_M f_{i+1}(\alpha)$, where $\epsilon = \pm 1$;
- there is i with $1 \leq i \leq m$ and $g \in L_n$ such that for all $h \in L_n$ with $h(\alpha) \sim_M f_i(\alpha)$, there is an irrational number $r_h \in \mathbb{R} \backslash \mathbb{Q}$ such that $q_1 h(\alpha) < \beta - g(\alpha) < q_2 h(\alpha)$ for all $q_1, q_2 \in \mathbb{Q}$ with $q_1 < r_h < q_2$.

The paper is organized as follows. In the rest of this introduction we recall some definitions and results , from earlier papers, relevant to our results. In Section 2.1 we will give a characterization for the f-generic types of G^2. In section 2.2, we study the space of n-nary $\mathbb{Q}$-linear function, and conclude that, modulo an equivalence relation, there are at most finitely many $\mathbb{Q}$-linear functions. Section 2.3 contains the main result of the paper, we also discuss the application of our results. In Section 2.4, we consider the multiplicative group $H(\mathbb{Q}_p) = (\mathbb{Q}_p \backslash \{0\}, \times)$ of the p-adic field $\mathbb{Q}_p$, and prove that every f-generic type of H^n is also $\emptyset$-definable.

1.2 Preliminaries

We will assume a basic knowledge of model theory. Good references are [12] and [8]. Let T be a complete theory with infinite models. Its language is L and $\mathbb{M}$

is the monster model, in which every type over a small model $\mathbb{M} \succ M$ is realized. By x, y, z we mean arbitrary n-variables and $a, b, c \in \mathbb{M}$ the n-tuples in $\mathbb{M}^n$ with $n \in \mathbb{N}$. every formula is an $L_{\mathbb{M}}$-formula. For an L_M-formula $\phi(x)$, $\phi(M)$ denote the definable subset of $M^{|x|}$ defined by ϕ, and a set $X \subseteq M$ is definable if there is an L_M-formula $\phi(x)$ such that $X = \phi(M)$. For any subset A of $\mathbb{M}$, by $acl(A)$ we mean the algebraic closure of A. Namely, $b \in acl(A)$ if and only if there is a formula $\phi(x)$ with parameters from A such that $b \in \phi(\mathbb{M})$ and $\phi(\mathbb{M})$ is finite. By $dcl(A)$ we mean the definable closure of A, which is the collection of all $f(a)$ with f an $\emptyset$-definable function and $a \in A^n$. For any n-tuple $(\alpha_1, ..., \alpha_n) \in \mathbb{M}^n$, we denote $acl(A \cup \{\alpha_1, ..., \alpha_n\})$ by $acl(A, \alpha)$. Similarly for $dcl(A, \alpha)$.

Assume that $G \subseteq \mathbb{M}^n$ is a group $\emptyset$-definable in $\mathbb{M}$ defined by the formula $G(x)$. For any $M \prec \mathbb{M}$, $G(M) = \{g \in M^n | g \in G\}$ is a subgroup of G. By $S_G(M)$, we mean the space of all complete types over M concentrating on $G(x)$. From now on, we will, through out this paper, assume that every formula $\phi(x)$, with parameters in $\mathbb{M}$, is contained in $G(x)$, namely, the subset $\phi(\mathbb{M})$ defined by ϕ is contained in G. Suppose that ϕ is an L_M-formula and $g \in G(M)$, then the left translate $g\phi(x)$ is defined to be $\phi(g^{-1}x)$. It is easy to check that $(g\phi)(M) = gX$ with $X = \phi(M)$.

Definition 1.1 Let notations be as above.

- A definable subset $X \subseteq G$ is generic if finitely many left translates of X covers G. Namely, there are $g_1, ..., g_n \in G$ such that $G = \cup_{i \leq i \leq n} g_i X$.
- A definable subset $X \subseteq G$ is weakly generic if there is a non-generic definable subset Y such that $X \cup Y$ is generic
- A definable subset $X \subseteq G$ is f-generic if for some/any model M over which X is defined and any $g \in G$, gX does not divide over M. Namely, for any M-indiscernible sequence $(g_i : i < \omega)$, with $g = g_0$, $\{g_i X : i < \omega\}$ is consistent.
- A formula $\phi(x)$ is generic if the definable set $\phi(\mathbb{M})$ is generic. Similarly for weakly generic and f-generic formulas.
- A type $p \in S_G(M)$ is generic if every formula $\phi(x) \in p$ is generic. Similarly for weakly generic and f-generic types.

Remark 1.2 It is easy to see that the class of all non-weakly generic formulas forms an ideal. So any weakly generic type $p \in S_G(M)$ has a global extension $\bar{p} \in S_G(\mathbb{M})$ which weakly generic

T is said to be (or have) NIP if for any indiscernible sequence $(b_i : i < \omega)$, formula $\psi(x, y)$, and $a \in \mathbb{M}$, there is an eventual truth value of $\psi(a, b_i)$ as $i \to \infty$.

Recall that a type definable over A subgroup $H \leq G$ has bounded index if $|G/H| < 2^{|T|+|A|}$. For groups definable in NIP structures, the smallest type-definable subgroup G^{00} exist (See [5]). Namely, the intersection of all type-definable subgroup of bounded index still has bounded index. We call G^{00} the type-definable connected component of G. Another model theoretic invariant is G^0, called the definably-connected component of G, which is the intersection all definable subgroup of G of finite index. Clearly, $G^{00} \leq G^0$.

Recall also that the Keisler measure over M on X. with X a definable set over M, is a finitely additive measure on the Boolean algebra of definable, over M, subsets of X. When we take the monster model, $M = \mathbb{M}$, we call it a global Keisler measure. A definable group G is said to be definably amenable if it admits a global (left) G-invariant probability Keisler measure.

Fact 1.3 [2] Assuming NIP. A definable group G is definably amenable if and only if it admits a global type $p \in S_G(\mathbb{M})$ with bounded G-orbit.

Moreover,

Fact 1.4 [2] For a definable amenable NIP group G, we have

- Weakly generic definable subsets, formulas, and types coincide with f-generic definable subsets, formulas, and types, respectively
- $p \in S_G(\mathbb{M})$ is f-generic if and only if it has bounded G-orbit.
- $p \in S_G(\mathbb{M})$ is f-generic if and only if it is G^{00}-invariant.
- A type-definable subgroup H fixing a global f-generic type is exactly G^{00}

Remark 1.5 Assuming that G is a definable amenable NIP group. By Remark 1.2, we see that any f-generic type $p \in S_G(M)$ has an f-generic global extension $\bar{p} \in S_G(\mathbb{M})$.

We now turn to Presburger arithmetic. For now on, we will, throughout this paper, assume that $T = Th(\mathbb{Z}, +, \{D_n\}_{n \in \mathbb{N}^+}, <, 0)$ is the first order theroy of integers in Presburger language $L_{Pres} = (+, \{D_n\}_{n \in \mathbb{N}^+}, <, 0)$, where each D_n is a unary predicate symbol for the set of elements divisible by n, $\mathbb{M}$ is the monster model of T, M is some small elementary submodel of $\mathbb{M}$.

It is well know that T has quantifier elimination [13]. Moreover, a nice result of Cluckers [3] indicated that models of T has cell decomposition, which will mention later. We recall some definitions first.

Definition 1.6 We call a function $f : X \subseteq M^m \longrightarrow M$ linear if there is a constant $\gamma \in M$ and integers a_i, $0 \leq c_i < n_i$ for $i = 1, ..., m$ such that $D_{n_i}(x_i - c_i)$ and

$$f(x) = \sum_{1 \leq i \leq m} a_i(\frac{x_i - c_i}{n_i}) + \gamma$$

for all $x = (x_1, ..., x_m) \in X$. We call f piecewise linear if there is a finite partition $\mathcal{P}$ of X such that all restriction $f|_A$, $A \in \mathcal{P}$ are linear. We speak analogously of linear and piecewise linear maps $g : X \longrightarrow M^n$.

Another important notation is the Presburger cells.

Definition 1.7

- A (0)-cell is a point $\{a\} \subset M$.
- An (1)-cell is a set with infinite cardinality of the form

$$\{x \in M | a\square_1 x\square_2 b, D_n(x - c)\},$$

with $a, b \in M$, integers $0 \leq c < n$ and $\square_i$ either $\leq$ or no condition.
- Let $i_j \in \{0, 1\}$ for $j = 1, ..., m$ and $x = (x_1, ..., x_m)$. A $(i_1, ..., i_m$,1)-cell is a set A of the form

$$\{(x, t) \in M^{m+1} | \ x \in D, \ f(x)\square_1 t\square_2 g(x), \ D_n(t - c)\},$$

with $D = \pi_m(A)$ an $(i_1, ..., i_m)$-cell. $f, g : D \longrightarrow M$ linear functions, $\square_i$ either $\leq$ or no condition and integers $0 \leq c < n$ such that the cardinality of the fibers $A_x = \{t \in M| \ (x, t) \in A\}$ can not be bounded uniformly in $x \in D$ by an integer.
- An $(i_1, ..., i_m$,0)-cell is a set A of the form

$$\{(x, t) \in M^{m+1} | \ x \in D, \ t = g(x)\},$$

with $g : D \longrightarrow M$ a linear function and $D \in M^m$ an $(i_1, ..., i_m)$-cell.

Fact 1.8 [3](Cell Decomposition Theorem). Let $X \subset M^m$ and $f : X \longrightarrow G$ be definable. Then there exists a finite partition $\mathcal{P}$ of X into cells, such that the restriction $f|_A : A \longrightarrow M$ is linear for each cell $A \in \mathcal{P}$. Moreover, if X and f are S-definable, then the parts A can be taken S-definable.

By the Cell Decomposition Theorem, we conclude directly that every definable subset of M^n is a finite union of cells. So every definable subset $X \subseteq M$ is a finite union of points and intervals mod some $n \in \mathbb{N}$. This implies that T has NIP.

From now on, we assume that $G = (\mathbb{M}, +)$ is the additive group of the Presburger arithmetic. Namely, G is defined by the formula "$x = x$", $G = \mathbb{M}$ as a set, and $G(M) = M$ for any $M \prec \mathbb{M}$. For any n-tuple $x = (x_1, .., x_n)$, by $D_m(x)$ we mean $\bigwedge_{1 \leq i \leq n} D_m(x_i)$. For any $\alpha \in \mathbb{M}$, and $A \subseteq \mathbb{M}$, by $\alpha > A$ we mean $\alpha > a$ for all $a \in acl(A)$. It is easy to see that $dcl(A) = acl(A)$ since $\mathbb{M}$ is a linear order structure.

Fact 1.9 For every $n \in \mathbb{N}$,

- G^n is definably amenable;

- The type-definable connected component of G^n is $\bigcap_{m\in\mathbb{N}^+} D_m(\mathbb{M}^n)$.

Proof Let $x = (x_1, ..., x_n)$ be an n-tuple variable. Let $\Pi(x)$ be the partial type of form

$$\{x_1 > \mathbb{M}\} \bigwedge \{x_2 > dcl(\mathbb{M}, x_1)\} \\ \bigwedge ... \{x_n > dcl(\mathbb{M}, x_1, ..., x_{n_1})\} \bigwedge \{D_m(x) : \ m \in \mathbb{N}^+\}.$$

By the Cell Decomposition Theorem, and induction on n, it is easy to see that Π determines a unique type $p \in S_{G^n}(\mathbb{M})$. Moreover, Π is invariant under $\bigcap_{m\in\mathbb{N}^+} D_m$ $(\mathbb{M}^n)$. Since $D_m(\mathbb{M}^n)$ is a definable subgroup of G^n of finite index, we see that $G^{00} \leq \bigcap_{m\in\mathbb{N}^+} D_m(\mathbb{M}^n)$. Thus p is G^{00}-invariant and hence has a bounded orbit. By Fact 1.3, G^n is definably amenable and $G^{00} = \bigcap_{m\in\mathbb{N}^+} D_m(\mathbb{M}^n)$. □

Corollary 1.10 $G^{n0} = G^{n00}$ for all $n \in \mathbb{N}^+$.

Remark 1.11

- G^0 is a densely linear ordered divisible abelian group, hence isomorphic to an ordered vector space over $\mathbb{Q}$.
- For every $n \in \mathbb{N}^+$, $(G^0)^n = (G^n)^0$.

Fact 1.12 Suppose that f is an function M-definable function from $X \subseteq \mathbb{M}^n$ to $Y \subseteq \mathbb{M}$. Then for any $\alpha \in (G^0)^n$ there are $q_1, ..., q_n \in \mathbb{Q}$ and $a \in M$ such that $f(\alpha) = q_1\alpha_1 + ... + q_n\alpha_n + a$.

Proof By Fact 1.8, we may assume that f is linear. Then apply Remark 1.11. □

Definition 1.13 We call the function f of the form $q_1x_1 + ... + q_nx_n + a$ with $q_1, ..., q_n \in \mathbb{Q}$ and $a \in M$ an n-nary $\mathbb{Q}$-linear function over M. If $a = 0$, we call f a homogeneous n-nary $\mathbb{Q}$-linear function. By $L_n(M)$ we mean the space of all n-nary $\mathbb{Q}$-linear functions over M, and L_n the space of all homogeneous n-nary $\mathbb{Q}$-linear functions.

It is easy to see that any $f \in L_n(M)$ is M-definable, and there is a nature number m such that $D_m(\mathbb{M}^n) \subseteq dom(f)$. In particularly, $(G^0)^n \subseteq dom(f)$. By Fact 1.8 and Fact 1.12, we conclude that:

Corollary 1.14 If $\alpha = (\alpha_1, ..., \alpha_n) \in (G^0)^n$. Then for any $\phi(x_1, ...x_n) \in$ tp(α/M), there is a formula $\psi(x_1, ..., x_n) \in$ tp(α/M) of the form

$$\theta(x_1, ..., x_{n-1}) \wedge D_m(x_n) \wedge (f_1(x_1, ..., x_{n-1})\square_1 x_n \square_2 f_2(x_1, ..., x_{n-1})),$$

with $m \in \mathbb{N}^m$, $\theta(M)$ a cell, , $f_i \in L_{n-1}(M)$, and $\square_i$ either $\leq$ or no condition, such that $M \models \forall x(\psi(x) \to \phi(x))$.

Remark 1.15 There are only 2 f-generic types contained in every coset of G^0. More precisely, for any model M,

$$p^+(x) = \{D_n(x) |\ n \in \mathbb{N}^+\} \cup \{x > a |\ a \in M\},$$

and

$$p^-(x) = \{D_n(x) |\ n \in \mathbb{N}^+\} \cup \{x < a |\ a \in M\}.$$

Then every f-generic type over M is one of $G(M)$-translates of p^+ or p^-.

2 Main results

2.1 The f-generics of G^2

Let $\mathbb{M}$ be the saturated model of $Th(\mathbb{Z}, +, D_n, <, 0, 1)_{n \in \mathbb{N}^+}$, T the theory of the Presburger Arithmetic.

Proposition 2.1 For any $M \succ \mathbb{Z}$, the f-generic type tp$(\alpha, \beta/M) \in S_{G^2}(M)$, with α, β in G^0, has one of the following forms:

- $\beta > dcl(M, \alpha)$ (we call it $+\infty$-type);
- $\beta < dcl(M, \alpha)$ (we call it $-\infty$-type);
- there is some $q \in \mathbb{Q}$ such that $q\alpha + m < \beta < (q + \frac{1}{n})\alpha$ for all $m \in M$ and $n \in \mathbb{N}$ (we call it q^+-type);
- there is some $q \in \mathbb{Q}$ such that $(q - \frac{1}{n})\alpha < \beta < q\alpha + m$ for all $m \in M$ and $n \in \mathbb{N}$ (we call it q^--type);
- there is some $r \in \mathbb{R}$ such that $q_1\alpha < \beta < q_2\alpha$ for all $q_1, q_2 \in \mathbb{Q}$ with $q_1 < r < q_2$ (we call it r^0-type).

Proof Let $p = $ tp$(\alpha, \beta/M)$ be a f-generic type which contained in ${G^2}^0$. By the cell decomposition, we may assume that every formula $\phi(x, y)$ in p is of the form

$$D_n(x) \wedge (a < x) \wedge D_n(y) \wedge (f_1(x) \Box_1 y \Box_2 f_2(x)),$$

with $n \in \mathbb{N}$, $a \in M$, $f_i : D_n(M) \longrightarrow M$ linear, and $\Box_i$ either $\leq$ or no condition.

If every formula in p contains a cell of the form $D_n(x) \wedge D_n(y) \wedge f_1(x) \leq y$, then it is easy to see that p is a $+\infty$-type.

Similarly, if every formula in p contains a cell of the form $D_n(x) \wedge D_n(y) \wedge y \leq f_2(x)$, then p is a $-\infty$-type.

Otherwise, there are linear functions $f_1(x) = q_1x + b_1$ and $f_2(x) = q_2x + b_2$, with $q_1, q_2 \in \mathbb{Q}$ and $b_1, b_2 \in M$, such that the cell

$$D_n(x) \wedge (a < x) \wedge D_n(y) \wedge (f_1(x) \leq y \leq f_2(x))$$

is contained in p, where both nq_1 are nq_1 are some integers. We call the above cell a (n, a, q_1, q_2)-cell.

Let

$$Q_1 = \{t \in \mathbb{Q} : \text{there is an } (n, a, t, q_2)\text{-cell which is contained in } p(x, y)\}$$

and

$$Q_2 = \{t \in \mathbb{Q} : \text{there is an } (n, a, q_1, t)\text{-cell which is contained in } p(x, y)\}.$$

Then both Q_1 and Q_2 are nonempty.

Claim (Q_1, Q_2) is a cut of $\mathbb{Q}$

Proof Clearly, $q_1 \leq q_2$ whenever $q_1 \in Q_1$ and $q_2 \in Q_2$. Otherwise, p is inconsistent.

By Remark 1.5, let $\bar{p} \in S_{G^2}(\mathbb{M})$ be any global f-generic type containing p. Now $\bar{p}$ is G^{2^0}-invariant. If there are $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $q_1 = q_2$, take $g \in G^{2^0}$ such that $g > M$, we see that the partial type $(gp) \cup p$ is inconsistent, but $(gp) \cup p \subseteq \bar{p}$. A contradiction. So $q_1 < q_2$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$.

Suppose that there is $q \in \mathbb{Q}$ such that $q_1 < q$ for all $q_1 \in Q_1$. Then for some $n \in \mathbb{N}$ and any $a \in M$, any (n, a, q_1, q)-cell is consistent with p and hence contained in p. So $q \in Q_2$. Similarly, if q, q_2 for all $q_2 \in Q_2$, then $q \in Q_1$. So (Q_1, Q_2) is a cut of $\mathbb{Q}$. □

Let $r \in \mathbb{R}$ be the real number determined by the cut (Q_1, Q_2). By the G^{2^0}-invariance of $\bar{p}$, we have

- If $r = q \in Q_1$, then p is a q^+-type;
- If $r = q \in Q_2$, then p is a q^--type;
- If $r \notin \mathbb{Q}$, then p is a r-type.

This completes the proof. □

Definition 2.2 We say that $\alpha \in \mathbb{M}$ is bounded over M if there are $a, b \in M$ such that $a < \alpha < b$, and unbounded if otherwise.

Remark 2.3 By the above argument, it is easy to conclude that for any $\alpha, \beta \in G^0$, if both tp(α/M) and tp(β/M) are f-generic. Then either tp$(\alpha, \beta/M)$ is f-generic, or there is $q_1, q_2 \in \mathbb{Q}$ such that $q_1\alpha + q_2\beta$ is bouned over M.

Corollary 2.4 Let tp$(\alpha, \beta/M)$ be a f-generic type which contained in G^{2^0}. Then tp$(q_1\alpha, q_2\beta/M)$ is f-generic for all $q_1, q_2 \in \mathbb{Q}\backslash\{0\}$.

By Remark 2.3 and Corollary 2.4, we immediately have:

Corollary 2.5 Let $\alpha, \beta \in G^0$. Then tp$(\alpha, \beta/M)$ is an f-generic type if and only if for all $q_1, q_2 \in \mathbb{Q}$, $q_1\alpha + q_2\beta$ is unbounded over M whenever ${q_1}^2 + {q_2}^2 \neq 0$. In particular, both α and β are unbounded over M, and $\{\alpha, \beta\}$ is algebraic independent over M.

Remark 2.6 As we stated in Remark 1.15, every f-generic type of G^2 over M is one of $G^2(M)$-translate of some f-generic type contained in ${G^2}^0$. So it suffices to study the f-generic types contained in ${G^2}^0$.

Corollary 2.7 Every global f-generic type of G^2 contained in ${G^2}^0$ is $\emptyset$-definable.

Proof Let $\phi(x, y, z)$ be a formula. Then we may assume that ϕ is a finitely many union of the following cells:

$$\begin{aligned} C_i(x,y,z) \ &= D_{n_{1i}}(z-c_{1i}) \wedge D_{n_{2i}}(x-c_{2i}) \wedge D_{n_{3i}}(y-c_{3i}) \wedge (a_{1i} \square_{1i} z \square_{2i} a_{2i}) \\ &\wedge \ (h_{1i}(z) \square_{3i} x \square_{4i} h_{2i}(z)) \ \wedge \ (f_{1i}(x,z) \square_{5i} y \square_{6i} f_{2i}(x,z)), \end{aligned}$$

where $i = 1, ..., m$, integers $c_{1i},\ c_{2i}\ c_{3i}\ a_{1i}\ a_{1i}, \square_{1i}, ..., \square_{6i}$ either $\leq$ or no condition, $h_{li}(x) = b_{li}(\frac{z-c_{1i}}{n_{1i}}) + \gamma_{li}$, and $f_{li}(x,z) = d_{li}(\frac{x-c_{2i}}{n_{2i}}) + e_{li}(\frac{z-c_{1i}}{n_{1i}}) + \xi_{li}$ for $l = 1, 2$ and $b_{li},\ d_{li},\ e_{li},\ \gamma_{li},\ \xi_{li} \in \mathbb{Z}$.

Let $p = \mathrm{tp}(\alpha, \beta/\mathbb{M})$be a global f-generic type of G^2 contained in ${G^2}^0$. We assume that, for example, $\alpha > \mathbb{M}$, and p is a q^+-type for some $q \in \mathbb{Q}$. Then $\phi(x, y, b) \in p$ iff there is some $i \leq m$ such that

(i) $\mathbb{M} \models D_{n_{2i}}(c_{2i}) \wedge D_{n_{2i}}(c_{3i})$;
(ii) $\square_{4i}$ is no condition;
(iii) $\frac{d_{1i}}{n_{2i}} \leq q$ if $\square_{5i}$ is $\leq$, and $\frac{d_{2i}}{n_{2i}} > q$ if $\square_{6i}$ is $\leq$.
(iv) $\mathbb{M} \models D_{n_{1i}}(b - c_{1i}) \wedge (a_{1i} \square_{1i} b \square_{2i} a_{2i})$.

Let $E \subseteq \{1, ..., m\}$ be the set of all i such that (i), (ii), and (iii) hold. Then

$$\phi(x,y,b) \in p \iff \mathbb{M} \models \bigvee_{i \in E} D_{n_{1i}}(b-c_{1i}) \wedge (a_{1i} \square_{1i} b \square_{2i} a_{2i}).$$

This implies that p is $\emptyset$-definable. □

Actually, a slight modification of the above proof would conclude that

Corollary 2.8 Every global f-generic type of G^2 is $\emptyset$-definable.

Proof Let $\phi(x, y, z)$ and $C_i(x, y, z)$ be as stated above. Let p' be a global f-generic type of G^2. Then there is $g = (g_1, g_2) \in G$ such that $gp' \in {G^2}^0$. Similarly, as mentioned above, we assume that gp' is a q^+-type for some $q \in \mathbb{Q}$. Now

$$\phi(x, y, b) \in p' \iff g\phi(x, y, b) \in gp'.$$

It is easy to see that $g\phi(x,y,z)$ is the union the

$$\begin{aligned} gC_i(x,y.z) \ &= D_{n_{1i}}(z-c_{1i}) \wedge D_{n_{2i}}(x-g_1-c_{2i}) \wedge D_{n_{3i}}(y-g_2-c_{3i}) \\ &\wedge (a_{1i}\,\square_{1i}\,z\,\square_{2i}\,a_{2i}) \wedge \ (h_{1i}(z)\,\square_{3i}\,x-g_1\,\square_{4i}\,h_{2i}(z)) \\ &\wedge \ (f_{1i}(x-g_1,z)\,\square_{5i}\,y-g_2\,\square_{6i}\,f_{2i}(x-g_1,z)). \end{aligned}$$

For each $i \leq m$ and $b \in \mathbb{M}$, $gC_i(x,y,z) \in gp'$ iff

(i) $\mathbb{M} \models D_{n_{2i}}(g_1+c_{2i}) \wedge D_{n_{2i}}(g_2+c_{3i})$;
(ii) $\square_{4i}$ is no condition;
(iii) $\frac{d_{1i}}{n_{2i}} \leq q$ if $\square_{5i}$ is $\leq$, and $\frac{d_{2i}}{n_{2i}} > q$ if $\square_{6i}$ is $\leq$.
(iv) $\mathbb{M} \models D_{n_{1i}}(b-c_{1i}) \wedge (a_{1i}\,\square_{1i}\,b\,\square_{2i}\,a_{2i})$.

Let $\epsilon_i = (\epsilon_{1i},\epsilon_{2i}) \in \mathbb{Z}^2$ such that $\mathbb{M} \models D_{n_{2i}}(g_1-\epsilon_{1i}) \wedge D_{n_{2i}}(g_2-\epsilon_{2i})$. Then $gC_i(x,y,b) \in gp'$ iff $\epsilon_i C_i(x,y,b) \in gp'$ iff

(i') $\mathbb{M} \models D_{n_{2i}}(\epsilon_{1i}+c_{2i}) \wedge D_{n_{2i}}(\epsilon_{2i}+c_{3i})$;
(ii') $\square_{4i}$ is no condition;
(iii') $\frac{d_{1i}}{n_{2i}} \leq q$ if $\square_{5i}$ is $\leq$, and $\frac{d_{2i}}{n_{2i}} > q$ if $\square_{6i}$ is $\leq$.
(iv) $\mathbb{M} \models D_{n_{1i}}(b-c_{1i}) \wedge (a_{1i}\,\square_{1i}\,b\,\square_{2i}\,a_{2i})$.

Let $E' \subseteq \{1,...,m\}$ be the set of all i such that (i'), (ii'), and (iii') hold. Then $\phi(x,y,b) \in p$ if and only if $\mathbb{M} \models \bigwedge_{i\in E'} D_{n_{1i}}(b-c_{1i}) \wedge (a_{1i}\,\square_{1i}\,b\,\square_{2i}\,a_{2i})$, and thus p is $\emptyset$-definable. □

Definition 2.9 Let $p = \text{tp}(\alpha.\beta/M) \in S_{G^2}(M)$ be any f-generic type.

- We call p an ∞-TYPE if p is a $-\infty$-type or a $+\infty$-type;
- Let $r \in \mathbb{Q}$, We call p an r-TYPE if r p is an r^+-type or an r^+-type;
- Let $r \in \mathbb{R}\backslash\mathbb{Q}$, We call p an r-TYPE if r p is an r^0-type;

We call p a *rational-TYPE* if p is an r-TYPE and $r \in \mathbb{Q}$, and *irrational-TYPE* if p is an r-TYPE and $r \in \mathbb{R}-\mathbb{Q}$.

Lemma 2.10 Suppose that $\alpha = (\alpha_1,...\alpha_n) \in G^{n^0}$ and $\text{tp}(\alpha_1,...,\alpha_n/M)$ is f-generic. Then for all n-nary definable f over M, either $f(\alpha) > M$ or $f(\alpha) < M$, whenever f is nonconstant.

Proof By Remark 1.5, let $p = \text{tp}(\alpha'/\mathbb{M})$ be an f-generic global extension of $\text{tp}(\alpha/M)$. Then p is G^{n^0}-invariant. By cell decomposition, we may assume that f is of form $f(x) = c + q_1x_1 + ...q_nx_n$ with $q_1,...,q_n \in \mathbb{Q}$ and $c \in M$. If $q_i \neq 0$ for some $1 \leq i \leq n$, then for any $a \in G^0$,

$$c + q_1\alpha_1' + ... + q_i\alpha_i' + ... + q_n\alpha_n' + a = c + q_1\alpha_1' + ... + q_i(\frac{a}{q_i} + \alpha_i') + ... + q_n\alpha_n'.$$

So $f(\alpha')+a = f(\alpha'_1, ..., \frac{a}{q_i}+\alpha'_i, ...\alpha'_n)$. Since p is G^{n^0}-invariant and $(0, ..., \frac{a}{q_i}, ...0) \in G^{n^0}$, we have

$$\text{tp}(\alpha'/\mathbb{M}) = \text{tp}(\alpha'_1, ..., (\alpha'_i + \frac{a}{q_i}), ..., \alpha'_n/\mathbb{M}).$$

Now we see that

$$\text{tp}(f(\alpha') + a/\mathbb{M}) = \text{tp}(f(\alpha'_1, ..., \frac{a}{q_i} + \alpha'_i, ...\alpha'_n)/\mathbb{M}) = \text{tp}(f(\alpha')/\mathbb{M}).$$

This conclude that $\text{tp}(f(\alpha')/\mathbb{M})$ is G^0-invariant, thus f-generic, so either $f(\alpha') > \mathbb{M}$ or $f(\alpha') < \mathbb{M}$, and hence $f(\alpha) > M$ or $f(\alpha) < M$. □

2.2 An equivalence relation on homogeneous linear functions

Recall that $L_n = \{q_1x_1 + ... + q_nx_n|\ q_1, ...q_n \in \mathbb{Q}\}$ is the space of all homogeneous n-nary $\mathbb{Q}$-linear functions, and by $L_n(M) = \{f + a|\ f \in L_n,\ a \in M\}$ the space of all n-nary $\mathbb{Q}$-linear functions definable over M for any $M \prec \mathbb{M}$. For each $f \in L_n(M)$, there is $m \in \mathbb{N}^+$ such that f is a $\emptyset$-definable function from $D_m(G^n)$ to G.

Definition 2.11 Let $M \prec \mathbb{M}$, $f, g \in L_n(M)$ and $\alpha \in (G^n)^0$.

- We say that $f(\alpha) \ll_M g(\alpha)$ (or $g(\alpha) \gg_M f(\alpha)$) if
$$nf(\alpha) + a < mg(\alpha) + b$$
for all $n, m \in \mathbb{N}^+$ and $a, b \in M$.
- We say that $f \underset{M\alpha}{\sim} g$ (or $f(\alpha) \sim_M g(\alpha)$) if neither $f(\alpha) \ll_M g(\alpha)$ nor $g(\alpha) \ll_M f(\alpha)$.

Clearly, for any $M \prec \mathbb{M}$ and $\alpha \in (G^n)^0$, $\underset{M\alpha}{\sim}$ is an equivalence relation on $L_n(M)$. We denote the equivalence class of f by $[f]_{M\alpha}$. For any $f \in L_n(M)$, there is $g \in L_n$ such that $f \in [g]_{M\alpha}$. For $f, g \in L_n$, by $[f]_{M\alpha} < [g]_{M\alpha}$ we mean that there exist (or for all) $f' \in [f]_{M\alpha}$ and $g' \in [f]_{M\alpha}$ such that $f'(\alpha) \ll_M g'(\alpha)$.

Remark 2.12 If both $f(\alpha)$ and $g(\alpha)$ are positive (or negative), then $f(\alpha) \ll_M g(\alpha)$ if and only if $dcl(M, f(\alpha)) < g(\alpha)$ (or $f(\alpha) < dcl(M, g(\alpha))$, respectively).

Lemma 2.13 Suppose that $\alpha_1, \alpha_2 \in G^0$. Then $\{[f]_{M\alpha}|\ f \in L_2(M)\}$ has at most 5 elements.

Proof Let $p = \text{tp}(\alpha_1, \alpha_2/M)$. We first suppose that p is not f-generic. Then, by Corollary 2.5, $q_1\alpha_1 + q_2\alpha_2$ is bounded over M for some $q_1, q_2 \in \mathbb{Q}$. If $q_1 \neq 0$, then for each $f \in L_2$, there is $g \in L_1(M)$ such that $f(\alpha_1, \alpha_2) \sim_M g(\alpha_2)$. Assume that $\alpha_2 > 0$. It is easy to see that

$$\{[g]_{M\alpha_2}|\ g \in L_1(M)\} = \{[0]_{M\alpha_2}\}$$

if α_2 is bounded over M, and

$$\{[g]_{M\alpha_2} \mid g \in L_1(M)\} = \{[-x_2]_{M\alpha_2},\ [0]_{M\alpha_2},\ [x_2]_{M\alpha_2}\}$$

if α_2 is unbounded over M.

Now suppose that p is an f-generic type. Without lose of generality, we assume that $\alpha_1 > 0$.

- Suppose that p is a q-TYPE with $q \in \mathbb{Q}$, say a q^+-type.
Let $h(x_1, x_2) = ax_1 + bx_2 \in L_2$ and $g(x_1, x_2) = a'x_1 + b'x_2 \in L_2$ with $a, b, a', b' \in \mathbb{Q}$, such that $h(\alpha_1, \alpha_2) > 0$, $g(\alpha_1, \alpha_2) > 0$, and $h(\alpha) \gg_M g(\alpha)$. Then we have
$$a\alpha_1 + b\alpha_2 > n(a'\alpha_1 + b'\alpha_2)$$
for all $n \in \mathbb{N}^+$.
If $b' = 0$, we conclude that either $\alpha_2 < dcl(M, \alpha_1)$ or $\alpha_2 > dcl(M, \alpha_1)$, and hence p should be an ∞-TYPE. A contradiction.
If $b' < 0$, then $a' > -b'q$ as $a'\alpha_1 + b'\alpha_2 > 0$. For any sufficiently large $n \in \mathbb{N}^+$ we have
$$(a - na')\alpha_1 + (b - nb')\alpha_2 > 0.$$
We now assume that $b - nb' > 0$. Since $\alpha_2 < (q + \frac{1}{m})\alpha_1$ for all $m \in \mathbb{N}^+$, we have
$$(a - na')\alpha_1 + (b - nb')(q + \frac{1}{m})\alpha_1 > 0,$$
which implies that for all sufficiently large $m, n \in \mathbb{N}^+$,
$$(a + b(q + \frac{1}{m})) - n(a' + b'(q + \frac{1}{m})) > 0.$$
So $a' + b'(q + \frac{1}{m}) \leq 0$ for all $0 < m \in \mathbb{N}$. But $a' > -b'q$, so for sufficiently large m, $a' > -b'(q + \frac{1}{m})$. A contradiction. We conclude that $b' > 0$. For sufficiently large n, $(b - nb') < 0$ and hence $(b - nb')q\alpha_1 > (b - nb')\alpha_2$. So we have
$$(a - na')\alpha_1 + (b - nb')q\alpha_1 > 0,$$
which implies that
$$(a + bq) - n(a' + b'q) > 0$$
for all sufficiently large $n \in \mathbb{N}$, and hence $a' + b'q \leq 0$. Since $a'\alpha_1 + b'\alpha_2 > 0$, we have $a' + b'q \geq 0$. So $a' + b'q = 0$. For any $h(x_1, x_2) = a''x_1 + b''x_2$ with $b'' > 0$ and $a'' + b''q = 0$, there is some $n \in \mathbb{N}$ such that $h = ng$ or $g = nh$. So in this case,
$$\{[f]_{M\alpha} \mid f \in L_2\} = \{[-h]_{M\alpha}, [-g]_{M\alpha}, [0]_{M\alpha}, [g]_{M\alpha}, [h]_{M\alpha}\}.$$

- Suppose that p is an ∞-TYPE.
  Clearly, p is an ∞-TYPE if and only if tp$(\alpha_2, \alpha_1/M)$ is a 0-TYPE.
- Suppose that p is an r-TYPE with $r \in \mathbb{R}\backslash\mathbb{Q}$, and let $h(x_1, x_2) = ax_1 + bx_2$ and $g(x_1, x_2) = a'x_1 + b'x_2$ as above.
  If $b' < 0$, then a' greater than $-b'r$. Let $r < q \in \mathbb{Q}$ such that $a' > -b'q$. For all sufficiently large $n \in \mathbb{N}$, we have
  $$(a - na')\alpha_1 + (b - nb')q\alpha_1 > (a - na')\alpha_1 + (b - nb')\alpha_2 > 0,$$
  which implies that
  $$(a + bq) - n(a' + b'q) > 0.$$
  This is a contradiction as $a' + b'q > 0$.
  If $b' > 0$, then $a'\alpha_1 + b'\alpha_2 > 0$ implies that there is some $q \in \mathbb{Q}$ such that $a' + b'q > 0$ and $q < r$.
  For all sufficiently large $n \in \mathbb{N}$, we have
  $$(a - na')\alpha_1 + (b - nb')q\alpha_1 > (a - na')\alpha_1 + (b - nb')\alpha_2 > 0,$$
  which implies that
  $$(a + bq) - n(a' + b'q) > 0.$$
  This is a contradiction since $a' + b'q > 0$. So $[h]_{M\alpha} = [g]_{M\alpha}$ whenever $h(\alpha) > 0$, $g(\alpha) > 0$, and $f, g \in L_2$, and hance
  $$\{[f]_{M\alpha} \mid f \in L_2(M)\} = \{[-h]_{M\alpha}, [0]_{M\alpha}, [h]_{M\alpha}\}.$$

This completes our proof. □

The proof of Lemma 2.13 also concludes that

Corollary 2.14 Suppose that $p = $ tp$(\alpha_1, \alpha_2/M)$ is a f-generic with $\alpha_1, \alpha_2 \in G^0$. Let $f_1(x_1, x_2) = ax_1 + bx_2$ and $f_2(x_1, x_2) = a'x_1 + b'x_2$ be linear functions such that $f_i(\alpha_1, \alpha_2) > 0$. If $f_1(\alpha_1, \alpha_2) \ll_M f_2(\alpha_1, \alpha_2)$ then p is a q-TYPE with $q \in \mathbb{Q}$ and $a + bq = 0$.

Lemma 2.15 For any $\alpha = (\alpha_1, ..., \alpha_n) \in {G^0}^n$ and $\beta \in G^0$, $\{[f]_{M\alpha\beta} | f \in L_{n+1}\}$ is finite.

Proof Induction on $n \in \mathbb{N}$. By Lemma 2.13, the Lemma holds for $n = 1$. By induction hypothesis, there are finitely many n-nary linear functions $h_1, ..., h_k \in L_n$ such that $0 \ll_M h_1(\alpha) \ll_M ... \ll_M h_k(\alpha)$ and

$$\{[h]_{M\alpha} \mid 0 < h(\alpha) \in L_n\} = \{[h_1]_{M\alpha}, ..., [h_k]_{M\alpha}\}$$

Claim1 For each $\epsilon \in \{1, ..., k\}$, there do not exist $u_i \in [h_\epsilon]_{M\alpha}$, $c_i \in \mathbb{Q}$, and $\gamma \in G^0$, with $i \in \mathbb{N}^+$, such that

$$u_1(\alpha_1, ..., \alpha_n) + c_1\gamma \ll_M u_2(\alpha_1, ..., \alpha_n) + c_2\gamma \ll_M ...$$

is an infinite chain.

Proof We prove the following Claim first.

Claim2 If there are $\epsilon \in \{1, ..., k\}$, $u_i \in [h_\epsilon]_{M\alpha}$, $c_i \in \mathbb{Q}$, and $\gamma \in G^0$, with $i \in \mathbb{N}^+$, such that

$$u_1(\alpha_1, ..., \alpha_n) + c_1\gamma \ll_M u_2(\alpha_1, ..., \alpha_n) + c_2\gamma \ll_M ...$$

is an infinite chain. Then tp$(u_i(\alpha), \gamma/M)$ is a q_i-TYPE with $q_i \in \mathbb{Q}\backslash\{0\}$ for all $i \in \mathbb{N}^+$.

Proof If there are $j \in \mathbb{N}^+$, $d_1, d_2 \in \mathbb{Q}$ such that $d_1 u_j(\alpha_1, ..., \alpha_n) + d_2\gamma$ is bounded over M, then

$$-\frac{d_1}{d_2}u_j(\alpha_1, ..., \alpha_n) + a < \gamma < -\frac{d_1}{d_2}u_j(\alpha_1, ..., \alpha_n) + b$$

for some $a, b \in M$. So we conclude that

$$(u_i(\alpha_1, ..., \alpha_n) + c_i\gamma) \ \sim_M \ (u_i(\alpha_1, ..., \alpha_n) + -c_i\frac{d_1}{d_2}u_j(\alpha_1, ..., \alpha_n)).$$

Let

$$v_i(\alpha_1, ..., \alpha_n) = u_i(\alpha_1, ..., \alpha_n) + -c_i\frac{d_1}{d_2}u_j(\alpha_1, ..., \alpha_n),$$

then we have an infinite chain of

$$v_1(\alpha_1, ..., \alpha_n) \ll_M v_2(\alpha_1, ..., \alpha_n) \ll_M ... ,$$

which contradicts our induction hypothesis.

We now assume that $d_1 u_i(\alpha_1, ..., \alpha_n) + d_2\gamma$ is unbounded over M for all $i \in \mathbb{N}^+$, and $d_1, d_2 \in \mathbb{Q}$ such that $d_1^2 + d_2^2 \neq 0$. By Remark 2.3 and Lemma 2.10, tp$(u_i(\alpha), \gamma/M)$ is f-generic for each $i \in \mathbb{N}^+$. As $u_{i+1} \sim_{M\alpha} u_i$, there exists $q \in \mathbb{Q}$ such that for all $m \in \mathbb{N}^+$,

$$qu_i(\alpha) + c_{i+1}\gamma > u_{i+1}(\alpha) + c_{i+1}\gamma > m(u_i(\alpha) + c_i\gamma).$$

By Corollary 2.14, tp$(u_i(\alpha), \gamma/M)$ is either non-f-generic, or a $-c_i^{-1}$-TYPE. □

We now turn to Claim 1. For a contradiction, let $1 \leq t \leq k$ be the least number such that there exist $u_i \in [h_t]_{M\alpha}$, $c_i \in \mathbb{Q}$, and $\gamma \in G^0$, with $i \in \mathbb{N}^+$, such that

$$u_1(\alpha_1, ..., \alpha_n) + c_1\gamma \ll_M u_2(\alpha_1, ..., \alpha_n) + c_2\gamma \ll_M ...$$

is an infinite chain. By the Claim 2, we may assume that tp$(u_i(\alpha), \gamma/M)$ is a q_i-TYPE for each $i \in \mathbb{N}^+$, with $q_i \in \mathbb{Q}^+$. Moreover, we assume that each $u_i \in L_n$. It is easy to see that tp$(u_1(\alpha), u_i(\alpha)/M)$ is either a $\frac{q_1}{q_i}$-TYPE, or $u_i = \frac{q_1}{q_i} u_1$.

- If $t = 1$, then for all $i \in \mathbb{N}^+$, $u_i = \frac{q_1}{q_i} u_1$. Otherwise, tp$(u_1(\alpha), u_i(\alpha)/M)$ will be a $\frac{q_1}{q_i}$-TYPE, and this implies that either $0 \ll_M u_1(\alpha) - \frac{q_1}{q_i} u_i(\alpha) \ll_M u_1(\alpha)$ or $0 \ll_M \frac{q_1}{q_i} u_i(\alpha) - u_1(\alpha) \ll_M u_1(\alpha)$. But $u_1 \sim_{M\alpha} h_1$ is the least one, a contradiction. So $u_i = \frac{q_1}{q_i} u_1$ for all $i \in \mathbb{N}^+$. Let $u_1(\alpha) = \theta$. Then we have the infinity chain of

$$\theta + c_1\gamma \ll_M ... \ll_M \frac{q_1}{q_i}\theta + c_i\gamma \ll_M \frac{q_1}{q_{i+1}}\theta + c_{i+1}\gamma \ll_M ...,$$

  and which contradicts to the Lemma 2.13.
- If $t > 1$, then for each $i \in \mathbb{N}^+$, there is $d_i \in \cup_{\eta<t}[h_\eta]_{M\alpha}$ such that $u_i(\alpha) = \frac{q_1}{q_i} u_1(\alpha) + d_i(\alpha)$. So for $i < j \in \mathbb{N}^+$, we have

$$\frac{q_1}{q_j} u_1 g_1(\alpha) + d_j(\alpha) + c_j\gamma > n(\frac{q_1}{q_i} u_1(\alpha) + d_i(\alpha) + c_i\gamma)$$

  for all sufficiently large $n \in \mathbb{N}$. By Lemma 2.14, we see that $\frac{q_1}{q_i} + c_i = 0$ for all $i \in \mathbb{N}^+$. Let $\gamma' = u_1(\alpha) + c_1\gamma$. Then $\frac{q_1}{q_i} u_1(\alpha) + c_i\gamma = \frac{q_1}{q_i}\gamma'$ and

$$d_j(\alpha) + \frac{q_1}{q_j}\gamma' > n(d_i(\alpha) + \frac{q_1}{q_i}\gamma')$$

  for all $i < j \in \mathbb{N}$ and $n \in \mathbb{N}^+$. So we have an infinity chain of

$$d_1(\alpha) + \gamma' \ll_M ... \ll_M d_i(\alpha) + \frac{q_1}{q_i}\gamma' \ll_M d_{i+1}(\alpha) + \frac{q_1}{q_{i+1}}\gamma' \ll_M ...$$

  It's no harm to assume that $d_j \sim_\alpha d_{1j}$ for all $i, j \in \mathbb{N}^+$. Since $d_i \in \cup_{\eta<t}[h_\eta]_{M\alpha}$, there is $1 \leq \eta_0 < t$ such that $d_j \sim_{M\alpha} h_{\eta_0}$ for all $j \in \mathbb{N}^+$, and this contradicts to the minimality of t.

□

Suppose for a contradiction that

$$0 \ll_M f_1(\alpha\beta) \ll_M f_2(\alpha\beta) \ll_M ...$$

is an infinity chain, where

$$f_i(x_1, ..., x_n, x_{n+1}) = g_i(x_1, ..., x_n) + b_i x_{n+1},$$

with $g_i \in L_n$ for each $1 \leq i \leq m$. Then there is $t \in \{1, ..., k\}$ such that infinitely many g_i's are in $[h_t]_{M\alpha}$. But this contradicts Claim 1. □

Definition 2.16 Let $p = \text{tp}(\alpha_1, ..., \alpha_n/M)$ with $\alpha_1, ..., \alpha_n \in G^0$. By ***maximal positive chain*** of p we mean a finite ascending chain of form $0 \ll_M f_1(\alpha) \ll_M ... \ll_M f_m(\alpha)$, with each $f_i \in L_n$, such that

$$\{[f]_{M\alpha} | h(\alpha) > 0 \in L_n\} = \{[f_1]_{M\alpha}, ..., [f_m]_{M\alpha}\}.$$

Remark 2.17

- Let $p = \text{tp}(\alpha_1, ..., \alpha_n/M)$ with $\alpha_1, ..., \alpha_n \in G^0$. Suppose that $0 \ll_M f_1(\alpha) \ll_M ... \ll_M f_m(\alpha)$ is a maximal positive chain of p. Then for every $h \in L_n(M)$, either there is $i \leq m$ such that $h \sim_{M\alpha} f_i$ or $h \sim_{M\alpha} (-f_i)$, or $h \sim_{M\alpha} 0$.
- The maximal positive chain of p is independent of the choice of the realizations of p. Namely, for any $\alpha' \in \mathbb{M}^n$ realizes p, $0 \ll_M f_1(\alpha') \ll_M ... \ll_M f_m(\alpha')$ is also a maximal positive chain of p.

2.3 The f-generics of G^n

Theorem 2.18 Let $M \succ \mathbb{Z}$, $\alpha = (\alpha_1, ..., \alpha_n) \in (G^n)^0$ realizes an f-generic in $S_{G^n}(M)$. Let $f_0(\alpha) = 0 \ll_M f_1(\alpha) \ll_M ... \ll_M f_m(\alpha)$ be the maximal positive chain of $\text{tp}(\alpha/M)$. Then for every $\beta \in G^0$, $p = \text{tp}(\alpha, \beta/M) \in S_{G^{n+1}}(M)$ is an f-generic type if and only one of the following cases holds:

Case I $\text{tp}(f_m(\alpha), \beta/M)$ is an ∞-TYPE. In this case p is determined by the partial type

$$\text{tp}(\alpha/M) \cup \text{tp}(f_m(\alpha), \beta/M);$$

Case II There are i with $0 \leq i < m$ and $g \in L_n$ such that $\text{tp}(f_i(\alpha), \beta - g(\alpha)/M)$ is an ∞-TYPE and $\text{tp}(f_{i+1}(\alpha), \beta - g(\alpha)/M)$ is a 0-TYPE. In this case p is determined by the partial type

$$\text{tp}(\alpha/M) \cup \text{tp}(f_i(\alpha), \beta - g(\alpha)/M) \cup \text{tp}(f_{i+1}(\alpha), \beta - g(\alpha)/M);$$

Case III There are i with $1 \leq i \leq m$ and $g \in L_n$ such that for all $h \in [f_i]_{M\alpha}$, $\text{tp}(h(\alpha), \beta - g(\alpha)/M)$ is an irrational-TYPE. In this case p is determined by the partial type

$$\text{tp}(\alpha/M) \cup \text{tp}(f_i(\alpha), \beta - g(\alpha)/M).$$

Proof If $\beta > M_\alpha$ or $\beta < M_\alpha$, then **Case I** holds.

Now Suppose that neither $\beta > M_\alpha$ nor $\beta < M_\alpha$. For each $h \in [f_m]_{M\alpha}$, suppose that $\text{tp}(h(\alpha), \beta)$ is an r_h-TYPE, with $r_h \in \mathbb{R}$.

If $r_h \in \mathbb{R}\backslash\mathbb{Q}$ for all $h \in [f_m]_{M\alpha}$. Let $x = (x_1, ...x_n)$. By cell decomposition, we may assume that every formula in $\text{tp}(\alpha, \beta)$ is of form

$$\psi(x, y) = D_N(x, y) \wedge \phi(x) \wedge g_1(x) \leq y \leq g_2(x)$$

where $\phi(x) \in \text{tp}(\alpha/M)$, g_1 and g_2 are n-nary linear functions, and $N \in \mathbb{N}$. We claim that

Claim For any $h \in [f_m]_{M\alpha}$, and any n-nary linear functions h_1, h_2 such that $h_1(x) < y < h_2(x) \in$ tp(α, β), we have

$$\text{tp}(\alpha/M) \cup \text{tp}(h(\alpha), \beta/M) \models h_1(x) < y < h_2(x).$$

Proof There are $q_1, q_2 \in \mathbb{Q}$, with $q_1 < r_h < q_2$, such that

$$h_1(\alpha) < q_1 h(\alpha) + M < q_2 h(\alpha) + M < h_2(\alpha).$$

Otherwise, we have either tp$(h_1(\alpha), \beta/M)$ is an 1-TYPE or tp$(h_2(\alpha), \beta/M)$ is an 1-TYPE, which contradicts our assumption. Now

$$\text{tp}(\alpha/M) \models h_1(x) < q_1 h(x) < q_2 h(x) < h_2(x),$$

and

$$p'(x', y) = \text{tp}(h(\alpha), \beta/M) \models q_1 x' < y < q_2 x' .$$

So

$$\text{tp}(\alpha/M) \cup \text{tp}(h(\alpha), \beta/M) \models h_1(x) < y < h_2(x).$$

□

Since tp$(\alpha/M) \models D_N(x) \wedge \phi(x)$ and tp$(h(\alpha), \beta/M) \models D_N(y)$, we conclude that tp$(\alpha, \beta/M)$ is determined by the partial type

$$\text{tp}(\alpha/M) \cup \text{tp}(h(\alpha), \beta/M)$$

by our Claim.

Now suppose that there is $h' \in [f_m]_{M\alpha}$ such that tp$(h'(\alpha), \beta/M)$ is a rational-TYPE, say, a ${q_m}^{e_m}$-type, with $q_m \in \mathbb{Q}$ and $e_m \in \{+, -\}$.

Let $g(x) = q_m h'(x)$. Then $M < e_m(\beta - g(\alpha)) < h'(\alpha)/k$ for all $k \in \mathbb{N}^+$ and thus tp$(f_m(\alpha), \beta - g(\alpha)/M)$ is a 0-TYPE.

- If for all $h \in [f_{m-1}]_{M\alpha}$, tp$(h(\alpha), \beta - g(\alpha)/M)$ is an irrational-TYPE, then the above argument concludes that tp$(\alpha, \beta/M)$ is determined by the partial type

  $$\text{tp}(\alpha/M) \cup \text{tp}(h(\alpha), \beta - g(\alpha)/M).$$

- If tp$(f_{m-1}(\alpha), \beta - g(\alpha)/M)$ is an ∞-TYPE. As we claimed above, it is easy to show that for any n-nary linear functions h_1, h_2 such that $h_1(x) < y < h_2(x) \in$ tp(α, β), we have

  $$\text{tp}(\alpha) \cup \text{tp}(f_m(\alpha), \beta - g(\alpha)/M) \cup \text{tp}(f_m(\alpha), \beta - g(\alpha)/M) \models h_1(x) < y < h_2(x).$$

  So p is determined by

  $$\text{tp}(\alpha) \cup \text{tp}(f_m(\alpha), \beta - g(\alpha)/M) \cup \text{tp}(f_m(\alpha), \beta - g(\alpha)/M).$$

- If there is $h \in [f_{m-1}]_{M\alpha}$ such that tp$(h(\alpha), \beta - g(\alpha)/M)$ is a rational-TYPE, we could iterate the above steps until we meet a $0 < j \leq m$ such that there is $g \in \bigcup_{j \leq i \leq m} [f_i]_{M\alpha}$ such that

$$\text{tp}(f_j(\alpha), (\beta - g(\alpha))/M)$$

is a 0-TYPE, and for each $g_{j-1} \in [f_{j-1}]_{M\alpha}$,

$$\text{tp}(g_{j-1}(\alpha), (\beta - g(\alpha))/M)$$

is NOT a rational-TYPE. So either tp$(f_{j-1}(\alpha), \beta - g(\alpha)/M)$ is an ∞-TYPE and p is determined by

$$\text{tp}(\alpha/M) \cup \text{tp}(f_{j-1}(\alpha), \beta - g(\alpha)/M) \cup \text{tp}(f_j(\alpha), \beta - g(\alpha)/M),$$

and **Case II** holds, or tp$(f_{j-1}(\alpha), \beta - g(\alpha)/M)$ is an irrational-TYPE and p is determined by the partial type

$$\text{tp}(\alpha/M) \cup \text{tp}(f_{j-1}(\alpha), \beta - g(\alpha)/M),$$

and **Case III** holds.

□

The following two Corollaries are main results of [4]. By Theorem 2.18, we could prove them directly by induction on $n \in \mathbb{N}^+$.

Corollary 2.19 Every global f-generic type of G^n is $\emptyset$-definable for all $n \in \mathbb{N}^+$.

Proof Induction on $n \in \mathbb{N}^+$, and applying Corollary 2.8. □

Corollary 2.20 A global type tp$(\alpha_0, ..., \alpha_{n-1}/\mathbb{M}) \in S_{G^n}(\mathbb{M})$ is f-generic iff

$$q_0\alpha_0 + ... + q_{n-1}\alpha_{n-1}$$

is unbounded over $\mathbb{M}$ for all $q_0, ..., q_{n-1} \in \mathbb{Q}$ with $\sum_{i=0}^{n-1} q_i^2 \neq 0$.

Proof Induction on $n \in \mathbb{N}^+$, and applying Corollary 2.5. □

2.4 The f-generics of $({\mathbb{Q}_p}^{*n}, \times)$

We now consider the structure of p-adic field $\mathbb{Q}_p$ in the language of rings $\mathcal{L}_{ring} = \{+, \times, 0, 1\}$. By [1], the valuation ring $\mathbb{Z}_p$ is a definable subset of $\mathbb{Q}_p$ in the language of rings. Let ${\mathbb{Z}_p}^\times = \{a \in \mathbb{Z}_p | a^{-1} \in \mathbb{Z}_p\}$ be the definable subgroup of ${\mathbb{Q}_p}^* = \mathbb{Q}_p \backslash \{0\}$, then the map $\bar{\nu} : ({\mathbb{Q}_p}^*/{\mathbb{Z}_p}^\times, \times) \longrightarrow (\mathbb{Z}, +)$ is a group isomorphism. Let $\pi : {\mathbb{Q}_p}^* \longrightarrow {\mathbb{Q}_p}^*/{\mathbb{Z}_p}^\times$ be the nature projection. The valuation map $\nu = \bar{\nu} \circ \pi$ is group homomorphism from $(\mathbb{Q}_p{}^*, \times)$ onto $(\mathbb{Z}, +)$. Moreover, we see that $\nu(x) \leq \nu(y)$ if and only if $\nu(\frac{y}{x}) \in \mathbb{Z}_p$ for all $x, y \in \mathbb{Q}_p{}^*$. So the structure $(\mathbb{Z}, +, <, 0)$ is interpretable in the structure $(\mathbb{Q}_p, \times, +, 0, 1)$.

Fact 2.21 Let $\nu : \mathbb{Q}_p^* \longrightarrow \mathbb{Z}$ as above. Then we have

1. $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in \mathbb{Q}_p$;
2. $\nu(x + y) \geq min\{\nu(x), \nu(y)\}$, and $\nu(x + y) = min\{\nu(x), \nu(y)\}$ if $\nu(x) \neq \nu(y)$;
3. $d(x, y) = p^{-\nu(x-y)}$ defines a metric on $\mathbb{Q}_p$.

For each $k \in \mathbb{N}^+$, let P_k be a unary predicate symbol for the set of k-th power. Then $P_k(\mathbb{Q}_p^*) \leq \mathbb{Q}_P^*$ is definable subgroup of finite index. Moreover, each $P_k(\mathbb{Q}_p^*)$ is an open subgroup of $\mathbb{Q}_p^*$ and every coset $P_k(\mathbb{Q}_p^*)$ is an open subset of $\mathbb{Q}^*$. By [7], the structure $\mathbb{Q}_p$ has quantifier elimination in the language $\mathcal{L}_{ring} \cup \{P_k | \ k \in \mathbb{N}^+\}$. It is easy to see from quantifier elimination that

Fact 2.22 For any $M \models$ Th$(\mathbb{Q})$, $A \subseteq M$, any $a_1, ..., a_n, b_1, ..., b_n \in M$, we have

$$\text{tp}(a_1, ..., a_n/A) = \text{tp}(b_1, ..., b_n/A)$$

if and only if

$$M \models P_k(f(a_1, ..., a_n)) \iff M \models P_k(f(b_1, ..., b_n))$$

for all $k \in \mathbb{N}^+$ and $f \in \bar{A}[x_1, ..., x_n]$, where $\bar{A}$ is the subfield generted by A and $\bar{A}[x_1, ..., x_n]$ is the n-nary polynominal ring over $\bar{A}$.

Let $\mathbb{K}$ be a very saturated model of Th$(\mathbb{Q}_p)$. The valuation map ν can be naturally extended to a homomorphism from $\mathbb{K}\backslash\{0\}$ onto the saturated model $\Gamma \models$ Th$(\mathbb{Z})$. Let H denote the multiplication group $(\mathbb{K}^*, \times)$, where $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$. Then we have

Fact 2.23 ([10])H is definably amenable, and

- A global 1-type tp$(\alpha/\mathbb{K}) \in S_1(\mathbb{K})$ is an f-generic type of H if and only if either $\nu(\alpha) < \gamma$ for all $\gamma \in \Gamma$ or $\nu(\alpha) > \gamma$ for all $\gamma \in \Gamma$.
- $H^{00} = H(\mathbb{K})^0 = \bigcap_{k \in \mathbb{N}^+} P_k(\mathbb{K}^*)$.

Let $\bar{\mathbb{K}} \succ \mathbb{K}$ and $\bar{\Gamma} = \nu(\bar{\mathbb{K}}) \succ \Gamma$. Let G be the additive group of the Presburger arithmetic $(\Gamma, +)$.

Proposition 2.24 Let $\alpha = (\alpha_0, ..., \alpha_{n-1}) \in \bar{\mathbb{K}}^n$. Then tp$(\alpha_0, ..., \alpha_{n-1}/\mathbb{K})$ is an f-generic type of H^n if and only if tp$(\nu(\alpha_0), ..., \nu(\alpha_{n-1})/\Gamma)$ is an f-generic type of G^n. Moreover, every f-generic type of H^n is $\emptyset$-definable.

Proof Since H is definably amenable, we see that if $p(\bar{x}) = \mathrm{tp}(\alpha_0, ..., \alpha_{n-1}/\mathbb{K})$ is f-generic, then for every $g \in H^{00} = \bigcap_{k\in\mathbb{N}^+} P_k(\mathbb{K}^*)$, $gp = p$. So $\mathrm{tp}(\nu(\alpha_0), ..., \nu(\alpha_{n-1})/\Gamma)$ is invariant under G^{00} and hence f-generic, as $\nu(\bigcap_{k\in\mathbb{N}^+} P_k(\mathbb{K}^*)) = \bigcap_{k\in\mathbb{N}^+} D_k(\mathbb{K}^*) = G^{00}$..

Conversely, suppose that $\mathrm{tp}(\nu(\alpha_0), ..., \nu(\alpha_{n-1})/\Gamma)$ is an f-generic type of G^n. By quantifier elimination, we may assume that each formula in $p(\bar{x}) = \mathrm{tp}(\alpha_0, ..., \alpha_{n-1}/\mathbb{K})$ is of form $P_k(f(x_0, ..., x_{n-1}))$ with $k \in \mathbb{N}^+$ and $f \in \mathbb{K}[x_0, ..., x_{n-1}]$. Let $f \in \mathbb{K}[x_0, ..., x_{n-1}]$ be a polynomial of form $\Sigma_{\tau\in D_0} d_\tau \bar{x}^\tau$, where $0 \neq d_\tau \in \mathbb{K}$, D_0 is a finite subset of $\mathbb{N}^n$, and $\bar{x}^\tau = x_0^{\tau_0} ... x_{n-1}^{\tau_{n-1}}$ for $\tau \in D_0$. By Corollary 2.20, every nontrivial $\mathbb{Q}$-linear combination of $\{\nu(\alpha_0), ..., \nu(\alpha_{n-1})\}$ is unbounded over Γ. We see that there is $\tau^* \in D_0$ such that

$$\nu(\alpha^{\tau^*}) = \Sigma_{0\leq i\leq n-1} \tau_i^* \nu(\alpha_i) < \nu(\alpha^\tau) + \Gamma = \Sigma_{0\leq i\leq n-1} \tau_i \nu(\alpha_i) + \Gamma$$

for all $\tau \in D_0 \backslash \{\tau^*\}$. So $\nu(\frac{f(\alpha)}{\alpha^{\tau^*}} - d_{\tau^*}) > \Gamma$, and hence $\frac{f(\alpha)}{\alpha^{\tau^*}}$ is infinitesimal close to d_{τ^*} over $\mathbb{K}$. Since $P_k(\mathbb{K}^*)$ is a definable subgroup of H of finite index, there exists $\lambda \in \mathbb{Q}_p$ such that $\alpha^{\tau^*} \in \lambda P_k(\bar{\mathbb{K}}^*)$, then

$$f(\alpha) \in P_k(\bar{\mathbb{K}}^*) \iff \frac{f(\alpha)}{\alpha^{\tau^*}} \in \lambda^{-1} P_k(\bar{\mathbb{K}}^*).$$

Since each coset of $P_k(\mathbb{K}^*)$ is a clopen subset of H, and $\frac{f(\alpha)}{\alpha^{\tau^*}}$ is infinitesimal close to d_{τ^*} over $\mathbb{K}$, we see that

$$\frac{f(\alpha)}{\alpha^{\tau^*}} \in \lambda^{-1} P_k(\bar{\mathbb{K}}^*) \iff d_{\tau^*} \in \lambda^{-1} P_k(\bar{\mathbb{K}}^*).$$

Thus, we conclude that

$$\bar{\mathbb{K}} \models P_k(f(\alpha)) \iff \bar{\mathbb{K}} \models \lambda^{-1} P_k(d_{\tau^*}) \qquad (*)$$

where $\lambda \in \mathbb{Q}_p$ and $\alpha^{\tau^*}\lambda^{-1} \in P_k(\bar{\mathbb{K}}^*)$. Now for any $\beta \in (H^n)^{00}$, it is easy to see that $\beta^{\tau^*} \in P_k(\mathbb{K}^*)$. So

$$\alpha^{\tau^*}\lambda^{-1} \in P_k(\bar{\mathbb{K}}^*) \iff (\beta\alpha)^{\tau^*}\lambda^{-1} \in P_k(\bar{\mathbb{K}}^*),$$

where $\beta\alpha = (\beta_0\alpha_0, ..., \beta_{n-1}\alpha_{n-1})$. So we have shown that

$$\bar{K} \models P_k(f(\alpha)) \iff \bar{\mathbb{K}} \models P_k(f(\beta\alpha)).$$

So p is invariant under the $(H^n)^{00}$-translate, and hence an f-generic type of H^n

By (*), we see that $P_k(f(\bar{x})) \in p(\bar{x})$ if and only if $\mathbb{K} \models \lambda^{-1} P_k(d_{\tau^*})$. So $\lambda^{-1} P_k(y)$ is the formula defines "$P_k(f(\bar{x})) \in p$". Since P_k defines a finite index subgroup of H, $\{\lambda\}$ is $\emptyset$-definable. So p is $\emptyset$-definable as required. □

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Presburger 算数中的 f-generic 型

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摘 要

设有序加法群 $(G, +, <, 0)$ 是一个 Presburger 算数理论的模型。本文研究了 G^n 上的 f-generic 型，并且对其给出了一个分类定理。利用这个分类定理，我们证明了所有的 f-generic 型都是 $\emptyset$-可定义的。此外，文章的最后一部分研究了 p-adic 域 $\mathbb{Q}_p$ 的乘法群 H。我们证明了 H^n 的 f-generic 型也都是 $\emptyset$-可定义的。

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