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Finite Axiomatizability of Transitive Logics of Finite Depth and of Finite Weak Width*

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Abstract. This paper presents a study of finite axiomatizability of transitive logics of finite depth and finite weak width. We prove the finite axiomatizability of each transitive logic of finite depth and of weak width 1 that are characterized by rooted transitive frames in which all antichains contain at most n irreflexive points. As to negative results, we show that there are non-finitely-axiomatizble transitive logics of depth n and of weak width k for each $n \geqslant 3$ and $k \geqslant 2$.

It is well known from [9] that all transitive logics of finite depth have the finite model property(f.m.p.). In [11], the authors prove the finite axiomatizability of transitive logics of finite depth that are characterized by frames with an upper-bounded number of equivalence classes modulo the relation of having the same proper successors. This result implies the finite axiomatizability of transitive logics of finite depth and of finite width (in the sense of [5]), that of transitive logics of depth at most 2, and that of weakly convergent transitive logics of depth at most 3, etc.

This paper presents a study of finite axiomatizability of transitive logics of finite depth and of finite weak width, i.e., containing a weak width formula Wid_n^+ for some $n \geqslant 1$. These formulas are weaker forms of width formulas in [5], and each Wid_n^+ $(n \geqslant 1)$ corresponds to the condition within rooted transitive frames that all subframes generated by some proper successor of a root are at most width n. As a negative results, we show that there are non-finitely-axiomatizble transitive logics of depth n and of weak width k for each $n \geqslant 3$ and $k \geqslant 2$, by a way of constructing infinite irreducible sequences of frames. As a positive result, we prove the finite axiomatizability of each transitive logic of finite depth and of weak width 1 that contains $\mathrm{Wid}_n^\bullet$ for an $n \geqslant 1$, in which $\mathrm{Wid}_n^\bullet$ corresponds to the condition within rooted transitive frames that each antichain in them contains at most n irreflexive points.

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Section 1 provides preliminary notions and facts, and section 2 gives criteria of finite axiomatizability of transitive logics whose extensions all have the f.m.p. In section 3, we introduce transitive logics of finite depth and of finite weak width, and prove the non-finite-axiomatizability result. We then present in section 4 our main result of finite axiomatizability of transitive logic of finite depth and of weak width 1, and concludes the paper in section 5.

1 Preliminaries

This section lists some standard preliminary notions and theorems, and a full account can be found in standard modal logic textbooks (e.g., [1, 2]). *Modal formulas* are built up from propositional variables, using truth-functional operators and the necessity operator $\Box$. We will simply call them formulas. A *normal modal logic* (or simply modal logic) is a set of modal formulas that contains all truth-functional tautologies and $\Box(p \to q) \to (\Box p \to \Box q)$, and is closed under modus ponens, substitution and necessitation. As usual, we use **K** (**K4**) for the smallest modal logic (containing $\Box p \to \Box\Box p$). For each modal logic **L**, an *extension* of **L** is a modal logic $\mathbf{L}'$ such that $\mathbf{L} \subseteq \mathbf{L}'$. Let **L** be any modal logic and let Δ be any set of formulas, $\mathbf{L} \oplus \Delta$ is the smallest modal logic including $\mathbf{L} \cup \Delta$; and for each formula ϕ, we use $\mathbf{L} \oplus \phi$ for $\mathbf{L} \oplus \{\phi\}$. As usual, we use **S4** for $\mathbf{K4} \oplus \Box p \to p$. A modal logic $\mathbf{L}'$ is *finitely axiomatizable over* **L** if $\mathbf{L}' = \mathbf{L} \oplus \Delta$ for a finite Δ, and is *finitely axiomatizable* if it is finitely axiomatizable over **K**.

Let $\mathfrak{F} = \langle W, R \rangle$ be any frame with $w \in W$, and let $\mathfrak{M}$ be any model on $\mathfrak{F}$. For each formula ϕ, we use $\mathfrak{M}, w \vDash \phi$ for that $\mathfrak{M}$ satisfies ϕ at w, use $\mathfrak{F}, w \vDash \phi$ for that $\mathfrak{M}', w \vDash \phi$ for each model $\mathfrak{M}'$ on $\mathfrak{F}$, and use $\mathfrak{F} \vDash \phi$ for that ϕ is valid in $\mathfrak{F}$ ($\mathfrak{F}$ is a frame for ϕ). For any set Δ of formulas, and any class $\mathscr{C}$ of frames, the validity-relation $\vDash$ between them are defined as usual, and we will use $\mathbf{Log}(\mathscr{C})$ for the modal logic $\{\phi : \mathscr{C} \vDash \phi\}$. For all $u, v \in W$, let $\vec{R}uv$ iff Ruv but not Rvu, and let $u \perp_R v$ iff neither Ruv nor Rvu. For all $u, v \in W$, when Ruv, we say that u *sees* v, and call v a *successor of* u; and when $\vec{R}uv$, we call v a *proper successor* of u, and u a *proper predecessor* of v. For each $X \subseteq W$, let $X{\uparrow_R} = \{v : Ruv \text{ for a } u \in X\}$, $X{\downarrow_R} = \{v : Rvu \text{ for a } u \in X\}$, $X{\uparrow_R^-} = X{\uparrow_R} - X$ and $X{\downarrow_R^-} = X{\downarrow_R} - X$. When R is clear in the context, we drop "$_R$" and use "$X{\uparrow}$" and "$X{\downarrow}$" instead. For each $w \in W$, let $w{\uparrow} = \{w\}{\uparrow}$, $w{\downarrow} = \{w\}{\downarrow}$, $w{\uparrow^-} = \{w\}{\uparrow^-}$ and $w{\downarrow^-} = \{w\}{\downarrow^-}$.

For each family $\{\mathfrak{F}_i\}_{i \in I}$ ($\{\mathfrak{M}_i\}_{i \in I}$) of pairwise disjoint frames (models), we use $\biguplus_{i \in I} \mathfrak{F}_i$ ($\biguplus_{i \in I} \mathfrak{M}_i$) for the disjoint union of $\{\mathfrak{F}_i\}_{i \in I}$ ($\{\mathfrak{M}_i\}_{i \in I}$). For each frame $\mathfrak{F} = \langle W, R \rangle$ and each model $\mathfrak{M}$ on $\mathfrak{F}$, and for each nonempty $X \subseteq W$, we use $\mathfrak{F} \upharpoonright X$ ($\mathfrak{M} \upharpoonright X$) for the restriction of $\mathfrak{F}$ ($\mathfrak{M}$) to X, and use $\mathfrak{F}|_X$ ($\mathfrak{M}|_X$) for the subframe of $\mathfrak{F}$ (submodel of $\mathfrak{M}$) generated by X; and when $X = \{w\}$, we use $\mathfrak{F}|_w$ and $\mathfrak{M}|_w$ for $\mathfrak{F}|_{\{w\}}$ and $\mathfrak{M}|_{\{w\}}$ respectively. For frames $\mathfrak{F}$ and $\mathfrak{G}$ (models $\mathfrak{M}$ and $\mathfrak{M}'$), we say that a function f *reduces* $\mathfrak{F}$ ($\mathfrak{M}$) to $\mathfrak{G}$ ($\mathfrak{M}'$) when f is a reduction of $\mathfrak{F}$ ($\mathfrak{M}$) to $\mathfrak{G}$ ($\mathfrak{M}'$); and that $\mathfrak{F}(\mathfrak{M})$ is *reducible to* $\mathfrak{G}(\mathfrak{M}')$ if a function reduces $\mathfrak{F}(\mathfrak{M})$ to $\mathfrak{G}(\mathfrak{M}')$. We assume the reader's familiarity with the related theorems on preservation of truth and validity under these frame/model constructions.

Let $\mathfrak{F} = \langle W, R\rangle$ be a transitive frame. For all $w, u \in W$, $w \sim_R u$ iff either $w = u$, or Rwu and Ruw. A *cluster* in $\mathfrak{F}$ is an equivalence class modulo $\sim_R$. For each $w \in W$, we use $\boldsymbol{c}_{(w)}$ for the cluster containing w. For each cluster $\boldsymbol{c}$ in $\mathfrak{F}$, $\boldsymbol{c}$ is *degenerate* if it is a singleton of an irreflexive point in $\mathfrak{F}$, otherwise, it is *nondegenerate.* Let $k \geqslant 1$. A point u_1 in $\mathfrak{F}$ is *of rank greater than* k if there is an $\vec{R}$-chain $\{u_1, \ldots, u_n\}$ with $n > k$, and is of *rank* k if there is an $\vec{R}$-chain $\{u_1, \ldots, u_k\}$ and u_1 is not of rank greater than k. $\mathfrak{F}$ is of *rank* k if it contains a point of rank k but no point of rank greater than k, and is *of finite rank* if it is of rank k for some $k \geqslant 1$. The following formulas are from [9], where $i \geqslant 1$:

$$\mathrm{B}_1 = \Diamond\Box p_1 \rightarrow p_1,$$
$$\mathrm{B}_{i+1} = \Diamond(\Box p_{i+1} \wedge \neg \mathrm{B}_i) \rightarrow p_{i+1}.$$

We use $\mathbf{K4B}_n$ ($\mathbf{S4B}_n$) for $\mathbf{K4} \oplus \mathrm{B}_n$ ($\mathbf{S4} \oplus \mathrm{B}_n$), where $n \geqslant 1$. A transitive logic is *of depth* n ($n \geqslant 1$) if it contains B_n but not B_{n-1} (it is assumed that $\mathrm{B}_0 = \bot$), and is *of finite depth* if it contains B_k for a $k \geqslant 1$. The following are established in [9]:

Proposition 1 For each transitive frame $\mathfrak{F}$ and each $n \geqslant 1$, $\mathfrak{F} \vDash \mathrm{B}_n$ iff $\mathfrak{F}$ is of rank at most n.

Theorem 2. *All transitive logics of finite depth have the f.m.p.*

An *antichain* in a transitive frame $\mathfrak{F} = \langle W, R\rangle$ is a nonempty $A \subseteq W$ such that for all $u, v \in A$, $u \neq v$ only if $u \perp_R v$. Whenever we speak of an antichain $\{u_0, \ldots, u_n\}$ in a frame, we presuppose that $u_0, \ldots, u_n$ are distinct. A transitive frame is *of width at most* n ($n \geqslant 1$) if $|A| \leqslant n$ for each antichain A in the frame. The following formulas are from [5], where $n \geqslant 1$:

$$\mathrm{Wid}_n = \textstyle\bigwedge_{i\leqslant n}\Diamond p_i \rightarrow \bigvee_{0\leqslant i\neq j\leqslant n}\Diamond(p_i \wedge (p_j \vee \Diamond p_j)).$$

A transitive logic is *of width* n ($n \geqslant 1$) if it contains Wid_n but not Wid_{n-1} (it is assumed that $\mathrm{Wid}_0 = \bot$), and is *of finite width* if it contains Wid_k for a $k \geqslant 1$. The following proposition is from [5]:

Proposition 3 For each rooted transitive frame $\mathfrak{F}$ and each $n \geqslant 1$, $\mathfrak{F} \vDash \mathrm{Wid}_n$ iff $\mathfrak{F}$ is of width at most n.

2 Criteria of Finite Axiomatizability

In this section, we present necessary and sufficent conditions for all extensions of a modal logic $\mathbf{L}$ to be finitely axiomatizable over $\mathbf{L}$. For each family $\{\mathbf{L}_i\}_{i\in I}$ of modal logics, $\bigoplus_{i\in I}\mathbf{L}_i$ is the smallest modal logic including $\bigcup_{i\in I}\mathbf{L}_i$. The following is a well-known theorem from Tarski (see, e.g., [2]):

Theorem 4. *Let* $\mathbf{L}$ *and* $\mathbf{L}'$ *be any modal logics such that* $\mathbf{L}\subseteq\mathbf{L}'$. *Then* $\mathbf{L}'$ *is finitely axiomatizable over* $\mathbf{L}$ *iff there is no infinite ascending* $\subset$*-chain* $\mathbf{L}_0\subset\mathbf{L}_1\subset\mathbf{L}_2\subset\cdots$ *of extensions of* $\mathbf{L}$ *such that* $\mathbf{L}'=\bigoplus_{i\in\omega}\mathbf{L}_i$.

Let $\{\mathfrak{F}_i\}_{i\in\omega}$ be any infinite sequence of frames. $\{\mathfrak{F}_i\}_{i\in\omega}$ is *backward irreducible* (*forward-backward irreducible*, or simply *irreducible*) if for all $i,j\in I$ with $i<j$ ($i\neq j$), no point-generated subframe of $\mathfrak{F}_j$ is reducible to $\mathfrak{F}_i$. For each class $\mathscr{C}$ of frames, $\{\mathfrak{F}_i\}_{i\in\omega}$ is a *backward irreducible (or irreducible) sequence w.r.t.* $\mathscr{C}$ if $\{\mathfrak{F}_i\}_{i\in\omega}$ is backward irreducible (or irreducible) and $\mathfrak{F}_i\in\mathscr{C}$ for each $i\in\omega$. A modal logic $\mathbf{L}$ is *characterized* by a class $\mathscr{C}$ of frames if $\mathbf{L}=\mathbf{Log}(\mathscr{C})$.

The following theorem provides a sufficient condition of finite axiomatizability in terms of backward irreducible sequences, and is proved by applying Theorem 4.

Theorem 5. *Let* $\mathbf{L}$ *be a modal logic, and let* $\mathscr{C}$ *be a class of frames for* $\mathbf{L}$ *such that each extension of* $\mathbf{L}$ *is characterized by a subclass of* $\mathscr{C}$. *Then all extensions of* $\mathbf{L}$ *are finitely axiomatizable over* $\mathbf{L}$ *if there is no backward irreducible sequence w.r.t.* $\mathscr{C}$.

Proof Suppose that $\mathbf{L}'$ extends $\mathbf{L}$ but is not finitely axiomatizable over $\mathbf{L}$. By Theorem 4, there is an infinite ascending $\subset$-chain $\mathbf{L}_0\subset\mathbf{L}_1\subset\cdots$ of extensions of $\mathbf{L}$, and then for each $i\in\omega$, there is a $\phi_i\in\mathbf{L}_{i+1}-\mathbf{L}_i$, and hence by hypothesis, $\mathfrak{F}_i\nvDash\phi_i$ for a member $\mathfrak{F}_i$ of $\mathscr{C}$ such that $\mathfrak{F}_i\vDash\mathbf{L}_i$, which implies that for each $i,j\in\omega$ with $i<j$, $\mathfrak{F}_i\nvDash\phi_i$ and $\mathfrak{F}_j\nvDash\phi_i$. Therefore, $\{\mathfrak{F}_i\}_{i\in\omega}$ is a backward irreducible sequence w.r.t. $\mathscr{C}$. □

From now on, whenever we speak of an backward irreducible (or irreducible) sequence *of such and such frames* (*for* $\mathbf{L}$), we mean an backward irreducible (or irreducible) sequence w.r.t. the class of such and such frames (for $\mathbf{L}$). The following corollary is often applied in studies of finite axiomatizability of modal logics whose extensions have the f.m.p. (see, e.g., [3], [7] and [10])

Corollary 1 Let $\mathbf{L}$ be any modal logic whose extensions all have the f.m.p. Then all extensions of $\mathbf{L}$ are finitely axiomatizable over $\mathbf{L}$ if there is no backward irreducible sequence of finite rooted frames for $\mathbf{L}$.

Proof Let $\mathscr{C}$ be the class of finite rooted frames for $\mathbf{L}$. It follows from hypothesis that each extension of $\mathbf{L}$ is characterized by a subclass of $\mathscr{C}$. Hence the conclusion follows from Theorem 5. □

In the following, we prove the converse of Corollary 1, and combined it with Corollary 1 to get our final criterion of finite axiomatizability in terms of (backward) irreducible sequences. Let $\mathfrak{F}=\langle W,R\rangle$ be a finite rooted transitive frame, where

$W = \{w_0, \ldots, w_n\}$ with w_0 to be a root of $\mathfrak{F}$, and $w_0, \ldots, w_n$ to be all distinct. We call $\langle w_0, \ldots, w_n \rangle$ an *ordering of points in* $\mathfrak{F}$. Let $p_0, \ldots, p_n$ be distinct propositional letters, and let us call a conjunction of the following formulas a *frame formula for* $\mathfrak{F}$ *w.r.t.* $\langle w_0, \ldots, w_n \rangle$:

- p_0,
- $\Box(p_0 \vee \cdots \vee p_n)$,
- $\bigwedge\{(p_i \to \neg p_j) \wedge \Box(p_i \to \neg p_j) : i, j \leqslant n$ and $i \neq j\}$,
- $\bigwedge\{(p_i \to \Diamond p_j) \wedge \Box(p_i \to \Diamond p_j) : i, j \leqslant n$ and $Rw_iw_j\}$,
- $\bigwedge\{(p_i \to \neg\Diamond p_j) \wedge \Box(p_i \to \neg\Diamond p_j) : i, j \leqslant n$ and not $Rw_iw_j\}$.

A *frame formula*[1] *for* $\mathfrak{F}$ is a frame formula for $\mathfrak{F}$ w.r.t. an ordering $\langle u_0, \ldots, u_n \rangle$ of points in $\mathfrak{F}$, where u_0 is a root.

Lemma 1 Let $\mathfrak{F} = \langle W, R \rangle$ be a finite rooted transitive frame, for which ϕ is a frame formula w.r.t. an ordering $\langle w_0, \ldots, w_n \rangle$ of points in $\mathfrak{F}$. Then ϕ is satisfiable in $\mathfrak{F}$ at its root w_0.

Proof Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $V(p_i) = \{w_i\}$ for each $i \leqslant n$. It is routine to check that $\mathfrak{M}, w_0 \vDash \phi$. □

The following is Lemma 3.20 from [1], and the proof is left to the reader.

Lemma 2 Let $\mathfrak{F}$ be a finite rooted transitive frame, for which ϕ is a frame formula, and let $\mathfrak{G} = \langle U, S \rangle$ be any transitive frame with $u \in U$. Then ϕ is satisfiable in $\mathfrak{G}$ at u iff $\mathfrak{G}|_u$ is reducible to $\mathfrak{F}$.

Proposition 6 Let $\{\mathfrak{F}_i\}_{i\in\omega}$ be an irreducible sequence of finite rooted transitive frames. Then there is a continuum of extensions of $\mathbf{L} = \mathbf{Log}(\{\mathfrak{F}_i\}_{i\in\omega})$.

Proof For each $i \in \omega$, let ϕ_i be a frame formula for $\mathfrak{F}_i$; and for each $I \subseteq \omega$, let $\mathbf{L}_I = \mathbf{Log}(\{\mathfrak{F}_i\}_{i\in I})$. Consider any $I, J \subseteq \omega$ such that there is an $i \in I - J$. For each $k \in J$, because $i \neq k$, ϕ_i is by hypothesis and Lemma 2 not satisfiable in $\mathfrak{F}_k$, and hence $\mathfrak{F}_k \vDash \neg\phi_i$. It then follows that $\neg\phi_i \in \mathbf{L}_J$. By Lemma 1, $\mathfrak{F}_i \nvDash \neg\phi_i$, and then $\neg\phi_i \notin \mathbf{L}_I$, and hence $\mathbf{L}_I \neq \mathbf{L}_J$. A similar argument shows that $\mathbf{L}_I \neq \mathbf{L}_J$ if there is a $j \in J - I$. Hence $\mathbf{L}_I \neq \mathbf{L}_J$ for all $I, J \subseteq \omega$ such that $I \neq J$. It then follows that there is a continuum of extensions of $\mathbf{L}$. □

For each frame $\mathfrak{F} = \langle W, R \rangle$, we use $\|\mathfrak{F}\|$ for $|W|$. The following is easily verifiable:

[1] A frame formula for $\mathfrak{F}$ is also known as a *Jankov-Fine formula* for $\mathfrak{F}$ (see [1]). The term "frame formula" goes back to [4].

Fact 7 Let $\{\mathfrak{F}_i\}_{i\in\omega}$ be an infinite sequence of frames such that for an $m \geqslant 1$, $\|\mathfrak{F}_i\| \leqslant m$ for all $i \in \omega$. Then there is an infinite $I \subseteq \omega$ such that all frames in $\{\mathfrak{F}_i\}_{i\in I}$ are isomorphic.

Proposition 8 Each infinite backward irreducible sequence of finite frames has an infinite irreducible subsequence.

Proof Let $\{\mathfrak{F}_i\}_{i\in\omega}$ be a backward irreducible sequence of finite frames. By Fact 7, there is no $m \in \omega$ such that $\|\mathfrak{F}_i\| \leqslant m$ for all $i \in \omega$. Then there is an infinite $I \subseteq \omega$ such that for all $i, j \in I$ with $i < j$, $\|\mathfrak{F}_i\| < \|\mathfrak{F}_j\|$, and hence no point-generated subframe of $\mathfrak{F}_i$ is reducible to $\mathfrak{F}_j$. It then follows that $\{\mathfrak{F}_i\}_{i\in I}$ is irreducible. □

Theorem 9. *Let* $\mathbf{L}$ *be a transitive logic whose extensions all have the f.m.p. Then the following are equivalent:*[2]

(i) all extensions of $\mathbf{L}$ *are finitely axiomatizable over* $\mathbf{L}$*;*
(ii) there is no infinite backward irreducible sequence of finite rooted frames for $\mathbf{L}$*;*
(iii) there is no infinite irreducible sequence of finite rooted frames for $\mathbf{L}$*.*

Proof By definition of irreducible sequences and Proposition 8, (ii) is equivalent to (iii). According to Corollary 1 and Proposition 6, we have that (ii) implies (i) and (i) implies (iii), and hence (i) is equivalent to (ii). □

3 Transitive Logics of Finite Depth and of Finite Weak Width

In this section, we present weak width formulas Wid_n^+ $(n \geqslant 1)$, discuss their frame conditions, and then show that there are non-finitely-axiomatizble extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$ whenever $n \geqslant 3$ and $k \geqslant 2$.

For each $n \geqslant 1$, let Wid_n^+ be the following formula:

$$\mathrm{Wid}_n^+ = q \wedge \Diamond(\Box\neg q \wedge (\textstyle\bigwedge_{i\leqslant n}\Diamond p_i)) \rightarrow \textstyle\bigvee_{0\leqslant i\neq j\leqslant n}\Diamond(p_i \wedge (p_j \vee \Diamond p_j)).$$

A transitive logic is *of weak width* n $(n \geqslant 1)$ if it contains Wid_n^+ but not Wid_{n-1}^+, and is *of finite weak width* if it contains Wid_k^+ for a $k \geqslant 1$.

Proposition 10 Let $\mathfrak{F} = \langle W, R\rangle$ be a transitive frame, and let $w \in W$ and $n \geqslant 1$. Then $\mathfrak{F}, w \vDash \mathrm{Wid}_n^+$ iff for each u with $\vec{R}wu$, $\mathfrak{F}|_u$ is of width at most n.

Proof Suppose that $\mathfrak{M}, w \nvDash \mathrm{Wid}_n^+$ for a model $\mathfrak{M}$ on $\mathfrak{F}$. Because $\mathfrak{M}, w \vDash q \wedge \Diamond(\Box\neg q \wedge (\bigwedge_{i\leqslant n}\Diamond p_i))$, there is a $u \in w\!\uparrow$ such that $\mathfrak{M}, u \vDash \Box\neg q \wedge (\bigwedge_{i\leqslant n}\Diamond p_i)$, and

[2] Since a continuum of extensions of $\mathbf{L}$ can be constructed from an infinite irreducible sequence of finite rooted frames for $\mathbf{L}$, we also have the following equivalences: there is a continuum of non-finitely-axiomatizble extensions of $\mathbf{L}$ iff there is an infinite backward irreducible sequence of finite rooted frames for $\mathbf{L}$ iff there is an infinite irreducible sequence of finite rooted frames for $\mathbf{L}$.

then $\vec{R}wu$, and for each $i \leqslant n$, $\mathfrak{M}, v_i \vDash p_i$ for a $v_i \in u{\uparrow}$. Consider any $i, j \leqslant n$ such that $i \neq j$. Because $\mathfrak{M}, w \nvDash \bigvee_{0 \leqslant i \neq j \leqslant n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$, and because Rwv_i by the transitivity of R, it then follows that $\mathfrak{M}, v_i \vDash p_i$ and $\mathfrak{M}, v_i \nvDash p_j \vee \Diamond p_j$, and $\mathfrak{M}, v_j \vDash p_j$ and $\mathfrak{M}, v_j \nvDash p_i \vee \Diamond p_i$, and then neither $v_i = v_j$ nor Rv_iv_j nor Rv_jv_i. Hence $\{v_0, \ldots, v_n\}$ is an antichain, and then $\mathfrak{F}|_u$ is of width greater than n because $\{v_0, \ldots, v_n\} \subseteq u{\uparrow}$.

Suppose that there is a $u \in \mathbf{c}_{(w)}{\uparrow}^-$ such that $\mathfrak{F}|_u$ is of width greater than n. Then there is an antichain $\{v_0, \ldots, v_n\} \subseteq u{\uparrow}$. Let $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where $V(q) = \{w\}$, and $V(p_i) = v_i$ for each $i \leqslant n$. Since $\vec{R}wu$ and $\{v_0, \ldots, v_n\} \subseteq u{\uparrow}$, it is easy to see that $\mathfrak{M}, u \vDash \Box\neg q \wedge (\bigwedge_{i \leqslant n} \Diamond p_i)$, and then $\mathfrak{M}, w \vDash q \wedge \Diamond(\Box\neg q \wedge (\bigwedge_{i \leqslant n} \Diamond p_i))$. For each $v \in w{\uparrow}$ and each $i \leqslant n$, if $\mathfrak{M}, v \vDash p_i$, we know by definition of V that $v = v_i$ and $\mathfrak{M}, v \nvDash p_j \vee \Diamond p_j$ for each $j \leqslant n$ with $j \neq i$. Hence for each $v \in w{\uparrow}$, $\mathfrak{M}, v \nvDash \bigvee_{0 \leqslant i \neq j \leqslant n}(p_i \wedge (p_j \vee \Diamond p_j))$, from which it follows that $\mathfrak{M}, w \nvDash \bigvee_{0 \leqslant i \neq j \leqslant n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$, and hence $\mathfrak{M}, w \nvDash \mathrm{Wid}_n^+$. $\Box$

In what follows, we show that there are non-finitely-axiomatizble extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$ whenever $n \geqslant 3$ and $k \geqslant 2$, by way of constructing irreducible sequences of finite rooted transitive frames of rank 3.

We now construct irreducible sequences of finite rooted transitive frames of rank 3, in each of which all points of rank 2 have exactly two proper successors. For each $n \in \omega$, let $B_n = \{X \in \mathscr{P}(C_n) : |X| = 2\}$, where $C_n = \{k : k \leqslant n+1\}$, and let $\mathfrak{H}_n = \langle W_n, E_n\rangle$, where

$$W_n = \{a\} \cup B_n \cup C_n,$$
$$E_n = \{\langle a, u\rangle : u \in B_n \cup C_n\} \cup \{\langle b, c\rangle \in B_n \times C_n : c \in b\}.$$

It is easy to see that for each $n \in \omega$ and in each of $\mathfrak{H}_n$, a is of rank 3, and members of B_n are of rank 2 while those of C_n are of rank 1. Note that for each $n \in \omega$, $\mathfrak{H}_n$ is a finite strict partial order. Since all points of rank 2 in these frames have exactly two proper successors, the following Fact holds:

Fact 11 For each $n \geqslant 2$, Wid_n^+ is valid in all members of $\{\mathfrak{H}_n\}_{n\in\omega}$.

In our proof of Lemma 3, we make use of the following simple fact about reduction:

Fact 12 Let f be a reduction of $\mathfrak{F}$ to $\mathfrak{G}$, where both $\mathfrak{F}$ and $\mathfrak{G}$ are transitive, and let w be a point in $\mathfrak{F}$. Then the following hold:

(i) w is a dead-end in $\mathfrak{F}$ iff $f(w)$ is a dead-end in $\mathfrak{G}$;

(ii) for each $n \geqslant 1$, if $f(w)$ is of rank n in $\mathfrak{G}$, then w is of rank at least n in $\mathfrak{F}$.

Lemma 3 $\{\mathfrak{H}_n\}_{n\in\omega}$ is irreducible.

Proof Let $k, n \in \omega$ with $k < n$. We only show that $\mathfrak{H}_n$ is not reducible to $\mathfrak{H}_k$, the other direction is trivial because $|W_k| < |W_n|$. Let us use R for E_n and S for E_k. By definition, $b{\uparrow}_{E_n} = b$ for each $b \in B_n$, and hence by hypothesis,

$$b{\uparrow}_R = b \text{ for each } b \in B_n. \tag{1}$$

Suppose for reductio that f reduces $\mathfrak{H}_n$ to $\mathfrak{H}_k$. It follows from Fact 12 that $f(a) = a$, $f[B_n] = B_k$ and $f[C_n] = C_k$. Since $k < n$, $C_k \subset C_n$, and then there are distinct $c, c' \in C_n$ such that $f(c) = f(c')$. Let $b = \{c, c'\} \in B_n$. Then $f(b) = \{v, v'\} \in B_k$ for some distinct $v, v' \in C_k$. By definition,

$$Sf(b)v,\ Sf(b)v' \text{ and } f(b) \neq v, v'. \tag{2}$$

Since $f(c) = f(c')$, either $v \neq f(c), f(c')$ or $v' \neq f(c), f(c')$. If $v \neq f(c), f(c')$, then by (1) and (2), $Sf(b)v$ but $f(u) \neq v$ for each $u \in b{\uparrow}_R = \{c, c'\}$, contrary to the supposition that f reduces $\mathfrak{H}_n$ to $\mathfrak{H}_k$. By the same token, if $v' \neq f(c), f(c')$, then $Sf(b)v'$ but $f(u) \neq v'$ for each $u \in b{\uparrow}_R$, contrary to the supposition again. □

Theorem 13. *Let $n \geqslant 3$ and $k \geqslant 2$. There are non-finitely-axiomatizble extensions of* $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$.[3]

Proof By Proposition 1 and Fact 11, we have that for each $n \in \omega$, $\mathfrak{H}_n$ is a frame for $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$. It then follows from Lemma 3 and Theorem 9, there are non-finitely-axiomatizble extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$. □

4 Finite Axiomatizability of Transitive Logics of Finite Depth and of Weak Width 1

Consider the following formulas, where $n \geqslant 1$:

$$\mathrm{Wid}_n^\bullet = \bigwedge_{i \leqslant n} \Diamond(p_i \wedge \Box\neg p_i) \to \bigvee_{0 \leqslant i \neq j \leqslant n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j)).$$

In this section, we discuss the frame conditions for $\mathrm{Wid}_n^\bullet$ with $n \geqslant 1$, provide a study of well-quasi-orders on trees, and then prove the finite axiomatizability of each transitive logic of finite depth and of weak width 1 that contains $\mathrm{Wid}_n^\bullet$ for an $n \geqslant 1$.

4.1 Transitive Frames for $\mathrm{Wid}_n^\bullet$

Let $\mathfrak{F} = \langle W, R \rangle$ be any frame, and let A be an antichain in $\mathfrak{F}$. We say A is *irreflexive* if for all $w \in A$, Rww fails.

[3] According to footnote 2, we can actually show that there is a continuum of extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$ whenever $n \geqslant 3$ and $k \geqslant 2$.

Proposition 14 Let $\mathfrak{F} = \langle W, R\rangle$ be any transitive frame, and let $w \in W$ and $n \geqslant 1$. Then $\mathfrak{F}, w \vDash \mathrm{Wid}_n^\bullet$ iff $|A| \leqslant n$ for each irreflexive antichain A in $\mathfrak{F}|_w$.

Proof Suppose that $\mathfrak{M}, w \nvDash \mathrm{Wid}_n^\bullet$ for a model $\mathfrak{M}$ on $\mathfrak{F}$. Because $\mathfrak{M}, w \vDash \bigwedge_{i\leqslant n}\Diamond(p_i \wedge\Box\neg p_i)$, we have that for each $i \leqslant n$, $\mathfrak{M}, u_i \vDash p_i$ for an irreflexive point $u_i \in w\!\uparrow$. Consider any $i, j \leqslant n$ such that $i \neq j$. Because $\mathfrak{M}, w \nvDash \bigvee_{0\leqslant i\neq j\leqslant n}\Diamond(p_i\wedge(p_j\vee\Diamond p_j))$, it then follows from Rwu_i and $\mathfrak{M}, u_i \vDash p_i$ that $\mathfrak{M}, u_i \nvDash p_j \vee \Diamond p_j$; it further follows from Rwu_j and $\mathfrak{M}, u_j \vDash p_j$ that $\mathfrak{M}, u_j \vDash p_j$ and $\mathfrak{M}, u_j \nvDash p_i \vee \Diamond p_i$. So we have that neither $u_i = u_j$ nor Ru_iu_j nor Ru_ju_i. Hence $\{u_0, \ldots, u_n\}$ is an irreflexive antichain in $\mathfrak{F}|_w$ whose cardinality is greater than n.

Suppose that there is an irreflexive antichain $\{u_0, \ldots, u_n\}$ in $\mathfrak{F}|_w$. Let $\mathfrak{M} = \langle\mathfrak{F}, V\rangle$ where $V(p_i) = u_i$ for each $i \leqslant n$. It is easy to see that $\mathfrak{M}, u_i \vDash p_i \wedge \Box\neg p_i$ for each $i \leqslant n$, and hence $\mathfrak{M}, w \vDash \bigwedge_{i\leqslant n}\Diamond(p_i \wedge \Box\neg p_i)$. For each $v \in w\!\uparrow$ and each $i \leqslant n$, if $\mathfrak{M}, v \vDash p_i$, we know by definition of V that $v = u_i$ and $\mathfrak{M}, v \nvDash p_j \vee \Diamond p_j$ for each $j \leqslant n$ with $j \neq i$. Hence for each $v \in w\!\uparrow$, $\mathfrak{M}, v \nvDash \bigvee_{0\leqslant i\neq j\leqslant n}(p_i \wedge (p_j \vee \Diamond p_j))$, from which it follows that $\mathfrak{M}, w \nvDash \bigvee_{0\leqslant i\neq j\leqslant n} \Diamond(p_i\wedge(p_j\vee\Diamond p_j))$, and hence $\mathfrak{M}, w \nvDash \mathrm{Wid}_n^\bullet$. □

The following proposition is a direct consequence of Proposition 14.

Proposition 15 For each rooted transitive frame $\mathfrak{F}$ and each $n \geqslant 1$, $\mathfrak{F} \vDash \mathrm{Wid}_n^\bullet$ iff $|A| \leqslant n$ for each irreflexive antichain A in $\mathfrak{F}$.

4.2 Well-quasi-orders

Let A be any set. A binary relation R on A is a *quasi-order* iff it is reflexive and transitive. Let $\preceq$ be a quasi-order on A. We say $\preceq$ is a *well-quasi-order* (in short: *wqo*) iff every infinite sequence $(a_k)_{k\in\omega}$ of elements of A contains an infinite subsequence $(a_k)_{k\in I\subseteq\omega}$ of it such that $a_i \preceq a_j$ for all $i, j \in I$ with $i < j$.[4] Note that any quasi-order on A is *wqo* if A is finite, and that $\preceq$ is a *wqo* on any $A' \subseteq A$ if $\preceq$ is a *wqo* on A. Let $\leqslant$ be the usual less-than-order on ω. We fix a new order $\preccurlyeq$ on ω as follows: $m \preccurlyeq n$ iff either $m = n = 0$ or $0 < m \leqslant n$.

Fact 16 Both $\leqslant$ and $\preccurlyeq$ are *wqo* on ω.

The following lemma is from [8], and the reader can also refer to Lemma 2.6 in [6].

[4]Another well-known definition of *well-quasi-order* is as follows: $\preceq$ is a *well-quasi-order* iff every infinite sequence $(a_k)_{k\in\omega}$ of elements of A contains two element a_i, a_j such that $a_i \preceq a_j$ with $i < j$. These two definitions are equivalent, and a proof of their equivalence can be found in Lemma 2.5 in [6].

Lemma 4 Let $\preceq_1$ and $\preceq_2$ be *wqo* on set A_1 and A_2 respectively, and let $\preceq$ be the order on $A_1 \times A_2$ defined as follows: $\langle a_1, a_2 \rangle \preceq \langle a_1', a_2' \rangle$ iff $a_1 \preceq_1 a_1'$ and $a_2 \preceq_2 a_2'$. Then $\preceq$ is a *wqo* on $A_1 \times A_2$.

Let A be any set. We use A^* for set of all finite sequences (or strings) over A, use $\ell(s)$ for the length of the sequence s, and for each $i \leqslant k$, we will use $\#_i(s)$ for the i-th member of s, starting from 0. For each $n \geqslant 0$, we fix $\mathrm{Seq}_{\leqslant n}(A) = \{s \in A^* : \ell(s) \leqslant n\}$. Let $\preceq$ be a quasi-order on A. We define the orders $\trianglelefteq$ and $\ll$ on A^* as follows:

- for all $s, t \in A^*$, $s \trianglelefteq t$ iff $\ell(s) = \ell(t)$, and for each $i < \ell(s)$, $\#_i(s) \preceq \#_i(t)$.
- for all $s, t \in A^*$ where $s = (a_i)_{i<k}$ and $t = (b_i)_{i<n}$, $t \ll s$ iff either $n = k = 0$, or $n \geqslant k > 0$ and $a_k \preceq b_n$ and $s \trianglelefteq t'$ for a subsequence t' of t.

It is easy to see that both $\trianglelefteq$ and $\ll$ are quasi-orders on A^*. Furthermore, Lemma 4 can be applied to show the following Lemma by a trivial induction.

Lemma 5 If $\preceq$ is a *wqo* on A, then $\trianglelefteq$ is a *wqo* on $\mathrm{Seq}_{\leqslant n}(A)$ for all $n \geqslant 0$.

The following theorem is a slightly stronger formulation of Theorem 3.2 in [6], however the same proof can be applied here. A restricted version of the theorem, where A is the set of natural number, is proved in [3] along the same line as [6].

Theorem 17. *If $\preceq$ is a* wqo *on A, then $\ll$ is a* wqo *on A^*.*

A *tree* is a pair $\langle T, \leq \rangle$, in which T is a nonempty set and $\leq$ is a partial ordering on T satisfying *downward connectedness* ($\forall m \forall m' \exists w (w \leq m \wedge w \leq m')$) and *no downward branching* ($\forall m \forall w \forall w' (w \leq m \wedge w' \leq m \rightarrow w \leq w' \vee w' \leq w)$). $w < u$ is introduced as $w \leq u \wedge w \neq u$. Let $\mathfrak{T} = \langle T, \leq \rangle$ be any tree. Note that the set $anc_{\mathfrak{T}}(w) = \{u \in T : u \leq w\}$ is a chain under $\leq$, and a finite tree always has a unique root. We use $dom(\mathfrak{T})$ for the domain of $\mathfrak{T}$, and use $root(\mathfrak{T})$ for the root of $\mathfrak{T}$ when it exists. For any $w \in T$, *the level of* w in $\mathfrak{T}$ is $lev_{\mathfrak{T}}(w) = |anc_{\mathfrak{T}}(w)|$, the *set of immediate successors* of w is $suc_{\mathfrak{T}}(w) = \{u \in T : w < u \wedge \neg \exists v (w < v < u)\}$, and the *height* of $\mathfrak{T}$ is $heit(\mathfrak{T}) = max\{lev_{\mathfrak{T}}(w) : w \in T\}$. Given a set Σ of labels, a Σ-*tree* is a pair $\langle \mathfrak{T}, \tau \rangle$, where $\mathfrak{T}$ is a tree and τ is a labeling function on $\mathfrak{T}$ from $dom(\mathfrak{T})$ to Σ. Let $\mathfrak{t} = \langle \mathfrak{T}, \tau \rangle$ be any Σ-tree where $\mathfrak{T} = \langle T, \leq \rangle$. *A* Σ-tree $\mathfrak{t}$ is *finite* if its underlying tree $\mathfrak{T}$ is finite, and the *height (domain, root, etc.)* can be level up to Σ-trees from their underlying trees naturally. For each $\Delta \subseteq \Sigma$, $dom(\mathfrak{t})^{\Delta} = \{w \in dom(\mathfrak{t}) : \tau(w) \in \Delta\}$, and we use $dom(\mathfrak{t})^{l}$ for $dom(\mathfrak{t})^{\{l\}}$.

In the following, we consider only finite ω-trees, and use $\mathbf{T}^{\omega}$ for the set of all finite ω-trees. For each $m, n \geqslant 1$, we fix

$$\mathbf{T}^{\omega}_{=m,<n} = \{\mathfrak{t} \in \mathbf{T}^{\omega} : heit(\mathfrak{t}) = m \wedge \left|dom(\mathfrak{t})^0\right| < n\},$$
$$\mathbf{T}^{\omega}_{\leqslant m,<n} = \bigcup_{1\leqslant i\leqslant n} \mathbf{T}^{\omega}_{=m,<n}.$$

Note that $\mathbf{T}^{\omega}_{=1,<n} = \mathbf{T}^{\omega}_{\leqslant 1,<n}$ and all ω-trees in them have only one node, i.e. the root. It is convenient for our discussion to represent a Σ-tree $\mathfrak{t} = \langle \mathfrak{T}, \tau\rangle$ as the following triple:

$$\mathfrak{t} = \langle (root(\mathfrak{t}), \tau(root(\mathfrak{t}))), (\mathfrak{t}_1, \ldots, \mathfrak{t}_m), (\mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n})\rangle, \tag{3}$$

where

- $\mathfrak{t}_1, \ldots, \mathfrak{t}_m$ are all subtrees of $\mathfrak{t}$ generated by an element of $\{w \in suc_{\mathfrak{t}}(root(\mathfrak{t})) : \tau(w) = 0\}$,
- $\mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n}$ are all subtrees of $\mathfrak{t}$ generated by an element of $\{w \in suc_{\mathfrak{t}}(root(\mathfrak{t})) : \tau(w) > 0\}$,
- $\tau_{m+n}(root(\mathfrak{t}_{m+n})) = min\{\tau_i(root(\mathfrak{t}_i)) : m \leqslant i \leqslant m+n\}$, in which τ_i is the labeling function in $\mathfrak{t}_i$.

We call the triple above a *standard representation triple of* $\mathfrak{t}$. Note that the last two elements of a standard representation triple could be the empty sequence, such as when the represented tree has only one-node. Recall that $m \preccurlyeq n$ iff either $m = n = 0$ or $0 < m \leqslant n$. We define $\sqsubseteq$ on $\mathbf{T}^{\omega}$ inductively as follows:

(i) for any ω-tree $\mathfrak{t} = \langle (r, s), (), ()\rangle$ and any ω-tree $\mathfrak{t}'$, $\mathfrak{t} \sqsubseteq \mathfrak{t}'$ iff $\mathfrak{t}'$ is a one-node tree and $s \preccurlyeq \tau'(root(\mathfrak{t}'))$, where τ' is the labeling function in $\mathfrak{t}'$;

(ii) for any ω-tree $\mathfrak{t} = \langle (r, s), (\mathfrak{t}_1, \ldots, \mathfrak{t}_m), (\mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n})\rangle$ and any ω-tree $\mathfrak{t}' = \left\langle (r', s'), (\mathfrak{t}'_1, \ldots, \mathfrak{t}'_k), \left(\mathfrak{t}'_{k+1}, \ldots, \mathfrak{t}'_{k+l}\right)\right\rangle$, $\mathfrak{t} \sqsubseteq \mathfrak{t}'$ iff $s \preccurlyeq s'$, and

  (a) $m = k$ and for each $1 \leqslant i \leqslant m$, $\mathfrak{t}_i \sqsubseteq \mathfrak{t}'_i$;

  (b) either $l = n = 0$, or $l \geqslant n > 0$ and $\mathfrak{t}_{m+n} \sqsubseteq \mathfrak{t}'_{k+l}$ and there are $j_{m+1}, \ldots, j_{m+n}$ such that $k+1 \leqslant j_{m+1} < \cdots < j_{m+n} \leqslant k+l$, and $\mathfrak{t}_h \sqsubseteq \mathfrak{t}'_{j_h}$ for each h with $m+1 \leqslant h \leqslant m+n$.

Note that if we replace $\sqsubseteq$ with $\preceq$ in (a) and (b), then they become the exactly same as definition of $\trianglelefteq$ and definition of $\ll$ respectively.

Theorem 18. *For all* $m, n \geqslant 1$, $\sqsubseteq$ *is a* wqo *on* $\mathbf{T}^{\omega}_{\leqslant m,<n}$.

Proof It suffices to show that for all $m, n \geqslant 1$, $\sqsubseteq$ is a *wqo* on $\mathbf{T}^{\omega}_{=m,<n}$. We prove it by induction on m. The base case ($m = 1$) holds because of Fact 16. Consider $m = k+1$. Suppose that for all $n \geqslant 1$, $\sqsubseteq$ is a *wqo* on $\mathbf{T}^{\omega}_{=k,<n}$. Let $n \geqslant 1$ and let $(\mathfrak{t}_i)_{i\in\omega}$ be any infinite sequence of elements from $\mathbf{T}^{\omega}_{=k+1,<n}$, where $\mathfrak{t}_i = \left\langle (r_i, s_i), \left(\mathfrak{t}^i_1, \ldots, \mathfrak{t}^i_{m_i}\right), \left(\mathfrak{t}^i_{m_i+1}, \ldots, \mathfrak{t}^i_{m_i+n_i}\right)\right\rangle$ for each $i \in \omega$. We have by Fact 16 that there is an infinite subsequence $(\mathfrak{t}_i)_{i\in I_1}$ of $(\mathfrak{t}_i)_{i\in\omega}$ such that $(s_i)_{i\in I_1}$

is an infinite $\preccurlyeq$-chain. Since $\left|dom(\mathfrak{t}_i)^0\right| < n$ for each $i \in \omega$, there is an infinite subsequence $(\mathfrak{t}_i)_{i\in I_2}$ of $(\mathfrak{t}_i)_{i\in I_1}$ such that $m_i = m_j$ for all $i, j \in I_2$. It then follows from Lemma 16 and supposition that there is an infinite subsequence $(\mathfrak{t}_i)_{i\in I_3}$ of $(\mathfrak{t}_i)_{i\in I_2}$ such that for each $i < j \in I_3$, $m_i = m_j$ and for each $1 \leqslant h \leqslant m_i$, $\mathfrak{t}^i_h \sqsubseteq \mathfrak{t}^j_h$. Apply Theorem 17 and supposition, we obtain that there is an infinite subsequence $(\mathfrak{t}_i)_{i\in I_4}$ of $(\mathfrak{t}_i)_{i\in I_3}$ such that for each $i < j \in I_4$, either $n_i = n_j = 0$, or $n_j \geqslant n_i > 0$ and $\mathfrak{t}^i_{m_i+n_i} \sqsubseteq \mathfrak{t}^j_{m_j+n_j}$ and there are $j_{m_i+1}, \ldots, j_{m_i+n_i}$ such that $m_j + 1 \leqslant j_{m_i+1} < \cdots < j_{m_i+n_i} \leqslant m_j + n_j$, and $\mathfrak{t}^i_h \sqsubseteq \mathfrak{t}^j{}_{j_h}$ for each h with $m_i + 1 \leqslant h \leqslant m_i + n_i$. By definition of $\sqsubseteq$, $(\mathfrak{t}_i)_{i\in I_4}$ is an infinite $\sqsubseteq$-chain, and hence we have that for all $n \geqslant 1$, $\sqsubseteq$ is a *wqo* on $\mathbf{T}^{\omega}_{=k+1,<n}$. □

4.3 Finite Axiomatizability

Recall that a transitive logic is of weak width 1 if it contains Wid^+_1. In the subsection, we show the finite axiomatizability of all transitive logics of finite depth and of finite weak width 1 that contains $\mathrm{Wid}^\bullet_n$ for an $n \geqslant 1$ (Theorem 20).

Let $\mathfrak{F} = \langle W, R\rangle$ be a transitive frame. Then $\mathfrak{sk}(\mathfrak{F}) = \langle \mathfrak{sk}(W), \mathfrak{sk}(R)\rangle$ is the *skeleton of* $\mathfrak{F}$, where $\mathfrak{sk}(W)$ is the set of clusters in $\mathfrak{F}$, and for all $\mathbf{c}, \mathbf{d} \in \mathfrak{sk}(W)$, $\langle \mathbf{c}, \mathbf{d}\rangle \in \mathfrak{sk}(R)$ iff Rwu for some $w \in \mathbf{c}$ and $u \in \mathbf{d}$ (in fact, iff Rwu for all $w \in \mathbf{c}$ and $u \in \mathbf{d}$). For any binary relation R on a set W, we use R^* for the reflexive closure of R, i.e., $R \cup \{\langle w, w\rangle : w \in W\}$, and use R^{-1} for the inverse of R, i.e., $\{\langle w, u\rangle : \langle u, w\rangle \in R\}$. We fix $\mathfrak{sk}(\mathfrak{F})^* = \langle \mathfrak{sk}(W), \mathfrak{sk}(R)^*\rangle$ and $\mathfrak{sk}(\mathfrak{F})^{-1} = \left\langle \mathfrak{sk}(W), (\mathfrak{sk}(R)^*)^{-1}\right\rangle$.

Let $\mathfrak{F} = \langle W, R\rangle$ be any finite transitive frame for Wid^+_1 such that $\mathfrak{sk}(\mathfrak{F})^{-1}$ is a finite tree. The *representation tree* of $\mathfrak{F}$ is the following ω-tree:

$$\mathfrak{rt}(\mathfrak{F}) = \left\langle \mathfrak{sk}(\mathfrak{F})^{-1}, \tau\right\rangle, \tag{4}$$

where for each $\mathbf{c} \in \mathfrak{sk}(W)$, $\tau(\mathbf{c}) = |\mathbf{c}|$ if $\mathbf{c}$ is a nondegenerate cluster in $\mathfrak{F}$, otherwise $\tau(\mathbf{c}) = 0$.

Lemma 6 For any finite transitive frames $\mathfrak{F}$ and $\mathfrak{G}$ for Wid^+_1 such that $\mathfrak{sk}(\mathfrak{F})^{-1}$ and $\mathfrak{sk}(\mathfrak{G})^{-1}$ are finite trees, if $\mathfrak{rt}(\mathfrak{F}) \sqsubseteq \mathfrak{rt}(\mathfrak{G})$, then $\mathfrak{G}$ is reducible to $\mathfrak{F}$.

Proof We prove it by induction on the height of $\mathfrak{rt}(\mathfrak{F})$. Let $\mathfrak{rt}(\mathfrak{F}) = \left\langle \mathfrak{sk}(\mathfrak{F})^{-1}, \tau\right\rangle$ and $\mathfrak{rt}(\mathfrak{G}) = \left\langle \mathfrak{sk}(\mathfrak{G})^{-1}, \sigma\right\rangle$, and suppose that $\mathfrak{rt}(\mathfrak{F}) \sqsubseteq \mathfrak{rt}(\mathfrak{G})$. Consider $heit(\mathfrak{rt}(\mathfrak{F})) = 1$. By definition of $\sqsubseteq$, we have that

$$heit(\mathfrak{rt}(\mathfrak{G})) = 1 \text{ and} \tag{5}$$

$$\tau(root(\mathfrak{sk}(\mathfrak{F})^{-1})) \preccurlyeq \sigma(root(\mathfrak{sk}(\mathfrak{G})^{-1})). \tag{6}$$

By (5), both $\mathfrak{F}$ and $\mathfrak{G}$ are universal frames, i.e., containing only one cluster. Assume that $\mathbf{c}$ and $\mathbf{d}$ is the unique cluster in $\mathfrak{F}$ and $\mathfrak{G}$, respectively. It follows from (6) that $\tau(\mathbf{c}) \preccurlyeq \tau(\mathbf{d})$. By definition of $\preccurlyeq$, either $\tau(\mathbf{c}) = \tau(\mathbf{d}) = 0$ or $0 < \tau(\mathbf{c}) \leqslant \tau(\mathbf{d})$. If the former holds, then we have by (4) that both **c** and **d** are degenerate clusters; if the latter holds, then we have by (4) that both **c** and **d** are nondegenerate clusters and $|\mathbf{c}| < |\mathbf{d}|$. In either case, there is a function f from **d** onto **c** that reduces $\mathfrak{G}$ to $\mathfrak{F}$.

Consider $heit(\mathfrak{rt}(\mathfrak{F})) = k$. Let $\mathfrak{rt}(\mathfrak{F}) = \langle (\mathbf{r}, s), (\mathfrak{t}_1, \ldots, \mathfrak{t}_m), (\mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n}) \rangle$ and $\mathfrak{rt}(\mathfrak{G}) = \langle (\mathbf{r}', s'), (\mathfrak{t}'_1, \ldots, \mathfrak{t}'_k), (\mathfrak{t}'_{k+1}, \ldots, \mathfrak{t}'_{k+l}) \rangle$. Since $\mathfrak{rt}(\mathfrak{F}) \sqsubseteq \mathfrak{rt}(\mathfrak{G})$, we have that

(i) $s \preccurlyeq s'$,

(ii) $m = k$ and for each $1 \leqslant i \leqslant m$, $\mathfrak{t}_i \sqsubseteq \mathfrak{t}'_i$,

(iii) either $l = n = 0$, or $l \geqslant n > 0$ and $\mathfrak{t}_{m+n} \sqsubseteq \mathfrak{t}'_{k+l}$ and there are $j_{m+1}, \ldots, j_{m+n}$ such that $k + 1 \leqslant j_{m+1} < \cdots < j_{m+n} \leqslant k + l$, and $\mathfrak{t}_h \sqsubseteq \mathfrak{t}'_{j_h}$ for each h with $m + 1 \leqslant h \leqslant m + n$.

Apply the same reason as the base case, we have by (i) that there is a function f from $\mathbf{r}'$ onto $\mathbf{r}$ such that f reduces $\mathfrak{G} \upharpoonright \mathbf{r}'$ to $\mathfrak{F} \upharpoonright \mathbf{r}$. Since $heit(\mathfrak{rt}(\mathfrak{F})) = k$, the heights of $\mathfrak{t}_1, \ldots, \mathfrak{t}_m, \mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n}$ are all less than k, and hence by (ii), (iii) and induction hypothesis, we have that for each $1 \leqslant i \leqslant m$, there is a function f_i that reduces $\mathfrak{G} \upharpoonright (\bigcup dom(\mathfrak{t}'_i))$ to $\mathfrak{F} \upharpoonright (\bigcup dom(\mathfrak{t}_i))$, and for each h with $m + 1 \leqslant h \leqslant m + n$, there is a function f_h that reduces $\mathfrak{G} \upharpoonright (\bigcup dom(\mathfrak{t}'_{j_h}))$ to $\mathfrak{F} \upharpoonright (\bigcup dom(\mathfrak{t}_h))$. Let $\neg J = \{k + 1, \ldots, k + l\} - \{j_{m+1}, \ldots, j_{m+n}\}$. It follows from (iii) that $\mathfrak{t}_{m+n} \sqsubseteq \mathfrak{t}'_{k+l}$, and hence $|root(\mathfrak{t}_{m+n})| \preccurlyeq \left|root(\mathfrak{t}'_{k+l})\right|$. We then have by (3) that $0 < |root(\mathfrak{t}_{m+n})| \leqslant min\{\left|root(\mathfrak{t}'_j)\right| : j \in \neg J\}$, and thus $root(\mathfrak{t}_{m+n})$ and elements of $\{root(\mathfrak{t}'_j) : j \in \neg J\}$ are nondegenerate clusters. Let g be any function from $\bigcup \bigcup_{j \in \neg J} dom(\mathfrak{t}'_j)$ onto $\bigcup dom(\mathfrak{t}_{m+n})$ such that $g[root(\mathfrak{t}'_j)] = root(\mathfrak{t}_{m+n})$ for each $j \in \neg J$. This is possible because of $|root(\mathfrak{t}_{m+n})| \leqslant min\{\left|root(\mathfrak{t}'_j)\right| : j \in \neg J\}$. It is easy to see that g reduces $\mathfrak{G} \upharpoonright (\bigcup \bigcup_{j \in \neg J} dom(\mathfrak{t}'_j))$ to $\mathfrak{F} \upharpoonright (\bigcup dom(\mathfrak{t}_{m+n}))$. Finally, let $h = f \cup g \cup \{f_i\}_{1 \leqslant i \leqslant m+n}$. It is routine to check that h reduces $\mathfrak{G}$ to $\mathfrak{F}$. □

Recall that for each nonempty $X \subseteq W$, we use $\mathfrak{F} \upharpoonright X$ for the restriction of $\mathfrak{F}$ to X. Apply Proposition 10, the following fact is easily verifiable.

Fact 19 Let $\mathfrak{F} = \langle W, R \rangle$ be any rooted finite transitive frame for Wid_1^+, and let **c** be the initial cluster in $\mathfrak{F}$. Then there are disjoint subframes $\mathfrak{F}_1, \ldots \mathfrak{F}_n$ of $\mathfrak{F}$ such that $\mathfrak{F} \upharpoonright (\mathbf{c}{\uparrow}^-) = \biguplus_{1 \leqslant i \leqslant n} \mathfrak{F}_i$ and $\mathfrak{sk}(\mathfrak{F}_i)^{-1}$ is a finite tree for each $1 \leqslant i \leqslant n$.

Let $\mathfrak{F} = \langle W, R \rangle$ be any rooted finite transitive frame for Wid_1^+ and let **c** be the initial cluster in $\mathfrak{F}$. According to Fact 19, there there are disjoint subframes $\mathfrak{F}_1, \ldots \mathfrak{F}_h$ of $\mathfrak{F}$ such that $\mathfrak{F} \upharpoonright \mathbf{c}{\uparrow}^- = \biguplus_{1 \leqslant i \leqslant h} \mathfrak{F}_i$ and $\mathfrak{sk}(\mathfrak{F}_i)^{-1}$ is a finite tree for each $1 \leqslant i \leqslant h$. Let $\mathbf{T} = \{\mathfrak{rt}(\mathfrak{F}_1), \ldots, \mathfrak{rt}(\mathfrak{F}_h)\}$, and assume that $\{\mathfrak{t}_1, \ldots, \mathfrak{t}_m\} = \{\mathfrak{t} \in \mathbf{T} : \mathfrak{t} = \langle \mathfrak{T}, \tau \rangle \wedge \tau(root(\mathfrak{t})) = 0\}$ and $\{\mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n}\} = \{\mathfrak{t} \in \mathbf{T} : \mathfrak{t} = \langle \mathfrak{T}, \tau \rangle \wedge \tau(root(\mathfrak{t})) > 0\}$ with $\tau_{m+n}(root(\mathfrak{t}_{m+n})) = min\{\tau_i(root(\mathfrak{t}_i)) : m \leqslant i \leqslant m + n\}$, in which τ_i is the labeling function in $\mathfrak{t}_i$. The *standard representation tree* of $\mathfrak{F}$ is the following ω-tree:

$$\mathfrak{srt}(\mathfrak{F}) = \langle (\mathbf{c}, \tau(\mathbf{c})), (\mathfrak{t}_1, \ldots, \mathfrak{t}_m), (\mathfrak{t}_{m+1}, \ldots, \mathfrak{t}_{m+n}) \rangle,$$

where for each $\mathbf{c} \in \mathfrak{sk}(W)$, $\tau(\mathbf{c}) = |\mathbf{c}|$ if $\mathbf{c}$ is a nondegenerate cluster in $\mathfrak{F}$, otherwise $\tau(\mathbf{c}) = 0$. Note that for any finite transitive frame $\mathfrak{F}$ that both $\mathfrak{srt}(\mathfrak{F})$ and $\mathfrak{rt}(\mathfrak{F})$ are well-defined, they are always different from each other, since the root of $\mathfrak{srt}(\mathfrak{F})$ is the initial cluster in $\mathfrak{F}$ and the root of $\mathfrak{rt}(\mathfrak{F})$ is the final cluster in $\mathfrak{F}$. Apply Lemma 6, the following Lemma can be proved in a similar way as the inductive case in Lemma 6.

Lemma 7 Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite transitive frames for Wid_1^+, and let $\mathfrak{srt}(\mathfrak{F}) \sqsubseteq \mathfrak{srt}(\mathfrak{G})$. Then $\mathfrak{G}$ is reducible to $\mathfrak{F}$.

Lemma 8 Let $n, k \geqslant 1$ and let $(\mathfrak{F}_k)_{k\in\omega}$ be an infinite sequence of finite rooted transitive frames for $\mathrm{Wid}_k^\bullet$ of rank at most m and of weak width 1. Then there is an infinite $I \subseteq \omega$ such that for all $i, j \in I$ with $i < j$, $\mathfrak{F}_j$ is reducible to $\mathfrak{F}_i$.

Proof Since each $\mathfrak{F}_i$ is a frame for $\mathrm{Wid}_k^\bullet$ of rank at most m, we have by Proposition 14 that there are at most $m \times k$ degenerate clusters in $\mathfrak{F}_i$, and hence $\mathfrak{srt}(\mathfrak{F}_i) \in \mathbf{T}^{\omega}_{\leqslant m, < m\times k+1}$ for each $i \in \omega$. We then obtain by Theorem 18 that there is an infinite $I \subseteq \omega$ such that $(\mathfrak{srt}(\mathfrak{F}_i))_{i\in I}$ is an infinite $\sqsubseteq$-chain, and hence by Lemma 7, $\mathfrak{F}_j$ is reducible to $\mathfrak{F}_i$ for all $i, j \in I$ with $i < j$. □

Theorem 20. *For all $n, k \geqslant 1$, all extensions of* $\mathbf{K4B}_n \oplus \{\mathrm{Wid}_1^+, \mathrm{Wid}_k^\bullet\}$ *are finitely axiomatizable, and are hence decidable.*

Proof Let $\mathbf{L} = \mathbf{K4B}_n \oplus \{\mathrm{Wid}_1^+, \mathrm{Wid}_k^\bullet\}$ with $n, k \geqslant 1$. By Theorem 2, all extensions of $\mathbf{L}$ have the f.m.p. To show that all extensions of $\mathbf{L}$ are finitely axiomatizable, it then suffices by Theorem 9 to let $\{\mathfrak{F}_i\}_{i\in\omega}$ be any infinite sequence of finite rooted frames for $\mathbf{L}$ and show that it is not irreducible. For each $i \in \omega$, because $\mathfrak{F}_i$ is a frame for B_n and Wid_1^+, it is clear by Propositions 1 and 10 that $\mathfrak{F}_i$ is of rank at most n and of weak width 1. Then by Lemma 8, $\mathfrak{F}_j$ is reducible to $\mathfrak{F}_i$ for some $i, j \in \omega$ with $i < j$, and hence $\{\mathfrak{F}_i\}_{i\in\omega}$ is not irreducible. □

Since $\mathbf{S4B}_n$ is an extension of $\mathbf{K4B}_n \oplus \{\mathrm{Wid}_k^\bullet\}$ for all $n, k \geqslant 1$, the following Corollary follows immediately from Theorem 20:

Corollary 2 For all $n \geqslant 1$, all extensions of $\mathbf{S4B}_n \oplus \{\mathrm{Wid}_1^+\}$ are finitely axiomatizable, and are hence decidable.

5 Conclusion

In this paper, we proved as our negative result that there are non-finitely-axiomatizable extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^+$ for all $n \geqslant 3$ and $k \geqslant 2$, by a way of constructing infinite irreducible sequences of finite rooted transitive frames of depth

3 and of weak width 2. As our positive result, we showed that all extensions of $\mathbf{K4B}_n \oplus \{\mathrm{Wid}_1^+, \mathrm{Wid}_k^\bullet\}$ are finitely axiomatizable for all $n, k \geqslant 1$, by a way of applying *wqo* on finite height ω-trees. It is known from [11] that there are non-finitely-axiomatizble extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_k^\bullet$ for all $n \geqslant 3$ and $k \geqslant 1$. Therefore formulas Wid_1^+ play an essential role in our finite axiomatizability result. However, the following problem still remains open: for each $n \geqslant 1$, are all extensions of $\mathbf{K4B}_n \oplus \mathrm{Wid}_1^+$ finitely axiomatizable? Finally, since the infinite irreducible sequences of frames constructed in section 3 don't validate any formula $\mathrm{Wid}_k^\bullet$. So the following problem is unsettled: for each $n, k \geqslant 1$ and $m \geqslant 2$, are all extensions of $\mathbf{K4B}_n \oplus \{\mathrm{Wid}_m^+, \mathrm{Wid}_k^\bullet\}$ finitely axiomatizable?

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深度和弱宽度有穷的传递逻辑的有穷可公理化

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摘　要

这篇文章研究深度和弱宽度都有穷的传递逻辑类的可有穷公理化问题，并给出了正反两方面的结论。在正面方面，本文证明了对每个深度有穷且弱宽度为 1 的传递逻辑 **L**，如果 **L** 的框架中反链的禁自返点基数都不大于某个自然数 n，那么 **L** 是有穷可公理化的。对于反面结论，本文证明了对任意 $n \geqslant 3$ 和 $k \geqslant 2$，存在深度为 n 且弱宽度为 k 的传递逻辑是不可有穷公理化的。

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