Studies in Logic, Vol. 16, No. 6 (2023): 58–73
PII: 1674-3202(2023)-06-0058-16

On Types over p-adically Closed Fields

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Abstract. The aim of this paper is to study types over a p-adically closed field. We classify the 1-types over an arbitrary p-adically closed field, which extends the previous work of Penazzi, Pillay and Yao (2019) on classifying 1-types over the standard model $\mathbb{Q}_p$ of the field of p-adic numbers. We also study the orthogonality of pseudo-limit types and distance types and yield an analogue of the dichotomy of "cuts" and "noncuts" in the o-minimal context.

1 Introduction

This paper presents several new results on types in p-adically closed fields, the structures (in the language of rings) which are elementarily equivalent to the field $\mathbb{Q}_p$ of p-adic numbers. We denote the theory of $\mathbb{Q}_p$ in the language of rings by pCF.

Delon showed in [3] that every type over $\mathbb{Q}_p$ is definable. In [10], the authors classified the complete 1-types over the standard model $\mathbb{Q}_p$ as follows:

Theorem 1. [10] *The complete* 1*-types over* $\mathbb{Q}_p$ *are precisely the following:*

(i) The realized types $\mathrm{tp}(a/\mathbb{Q}_p)$ *for each* $a \in \mathbb{Q}_p$*;*

(ii) For each $a \in \mathbb{Q}_p$ *and* C*, a coset of* $\mathbb{G}_m^0$ *in* $\mathbb{G}_m$*, the type* $p_{a,C}$ *saying that* x *is infinitesimally closed to* a *(i.e.* $v(x-a) > n$ *for each* $n \in \mathbb{N}$*), and* $x - a \in C$*;*

(iii) For each coset C *as above the type* $p_{\infty,C}$ *saying that* $x \in C$ *and* $v(x) < n$ *for all* $n \in \mathbb{Z}$*,*

where $\mathbb{G}_m$ *the is multiplicative group of a very saturated elementary extension of* $\mathbb{Q}_p$*, and*

$$\mathbb{G}_m^0 = \{b \in \mathbb{G}_m | \text{ for each } n \in \mathbb{N}^+, \exists x(x^n = b)\}.$$

In this paper, we extend the above result to an arbitrary p-adically closed field:

Theorem 2. *Let* K *be a model of* pCF*,* Γ_K *the value group of* K*, and* $\mathbb{G}_m^0$ *as above. Then the complete* 1*-types over* K *are precisely the following:*

(i) The realized types $\mathrm{tp}(a/K)$ *for each* $a \in K$*;*

Received 2022-10-31

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(ii) *(distance type around a point) For each cut* $\Lambda \subseteq \Gamma_K$ *(see Definition 2),* $c \in K$*, and coset* C *of* $\mathbb{G}_m^0$*, the type* $p_{\Lambda,c,C}$ *saying that* $\Lambda < v(x-c) < \Gamma_K \backslash \Lambda$ *and* $x - c \in C$*;*

(iii) *(Pseudo-limit type) For each pesudo-Cauchy sequence* $\{c_i\}_{i \in I}$ *(see Definition 1), the type* $p_{\{c_i\}_{i\in I}}$ *saying that* x *is a pseudo-limit of* $\{c_i\}_{i\in I}$*, and for any formula* $\phi(x)$ *over* K*,* $\phi(x) \in p_{\{c_i\}_{i\in I}}$ *iff* $\phi(x)$ *is eventually true (see Definition 1) on* $\{c_i\}_{i\in I}$*.*

Note that when K is $\mathbb{Q}_p$, any pseudo-limit type is realized since $\mathbb{Q}_p$ is complete as a metric space. As the value group of $\mathbb{Q}_p$ is $\mathbb{Z}$ and there are exactly two cut over $\mathbb{Z}$, namely, $\emptyset$ and $\mathbb{Z}$, we see that case (ii) of Theorem 2 corresponds to case (ii) and (iii) of Theorem 1.

Recall that an o-minimal structure is an ordered structure $(M, <, ...)$ in which every definable subset $X \subseteq M$ is a finite union of intervals and points. If $A \subseteq M$ and $c \in M$, we call tp(c/A) a cut iff there are $a, b \in \mathrm{dcl}(A)$, the definable closure of A in M, such that $a < c < b$, and for $a_0 \in \mathrm{dcl}(A)$ with $a_0 < c$, there is $a_1 \in \mathrm{dcl}(A)$ with $a_0 < a_1 < c$, and likewise for $c < b_0 \in \mathrm{dcl}(A)$. Say that tp$(c/A)$ is a noncut iff it is not algebraic and not a cut. Abusing terminology, we will also refer to c itself as a cut/noncut over A. In the topological view, tp(c/A) is a noncut iff there is $a \in \mathrm{dcl}(A)$ such that the "distance" of c and a is minimal among the "distance" of c and other points in dcl(A), and tp(c/A) is a cut iff there is no such $a \in \mathrm{dcl}(A)$.

In the pCF environment, it is reasonable to consider the pseudo-limit types and distance types as the analogues of "cut" and "noncut" in the o-minimal context, respectively.

Assuming o-minimality, a result of [8] shows that if tp(c/A) is a type of cut over A, and tp(d/A) is a type of noncut over A, then c and d are algebraically independent over A, namely, $c \notin \mathrm{dcl}(A, d)$ and $d \notin \mathrm{dcl}(A, c)$. Recall from [14] that two types $p(x)$ and $q(y)$ over A are weakly orthogonal if they implies a complete type $r(x, y)$ over A. It is easy to see that if tp(c/A) and tp(d/A) are weakly orthogonal, then c and d are algebraically independent over A. We extend Marker's result to pCF environment, showing that

Theorem 3. *Let* K *be a model of* pCF*,* $p(x) \in S_1(K)$ *a pseudo-limit type, and* $q(y) \in S_1(K)$ *a distance type, then* p *and* q *are weakly orthogonal.*

The paper is organized as follows: For the rest of the introduction we give precise definition and preliminaries relevant to our results.

In Section 2, we study the the elementary extensions of p-adically closed fields and their value groups.

In Section 3, we will prove Theorem 2, classifying the 1-types over an arbitrary model of pCF.

In Section 4, we will prove Theorem 3, the orthogonality of pseudo-limit types and distance types.

1.1 Notations

Let T be a complete theory with infinite models in a countable language L and M a model of T. We usually write tuples as a, b, x, y rather than $\bar{a}, \bar{b}, \bar{x}, \bar{y}$. For A a subset of M, an L_A-formula is a formula with parameters from A. If $\phi(x)$ is an L_M-formula and $A \subseteq M$, then $\phi(A)$ is the collection of the realizations of $\phi(x)$ from A, namely, $\phi(A) = \{a \in A^{|x|} \mid M \models \phi(a)\}$. Similarly, if $X \subseteq M^{|x|}$ is a definable set defined by the formula $\phi(x)$, then we use $X(A)$ to denote the set $\phi(A) = X \cap A^{|x|}$. If $X \subseteq M^n$ is definable in M and $N \succ M$, we sometimes use $X(x)$ to denote the formula which defines X and $X(N)$ the subset of N^n defined by the formula $X(x)$.

Assume that $A \subseteq M$ and $a \in M$. We say that a is in the algebraic closure of A (in M), written $a \in \mathrm{acl}(A)$, if there is a formula $\varphi(x)$ over A (namely of L_A) such that $M \models \varphi(a)$, and moreover such that $\varphi(x)$ has only finitely many solutions in M. We say that a is in the definable closure of A, $a \in \mathrm{dcl}(A)$, if for some L_A-formula $\varphi(x)$, a is the unique solution of $\varphi(x)$ in M. Note that both $\mathrm{acl}(-)$ and $\mathrm{dcl}(-)$ are idempotent operators, namely, $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$ and $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$ for any $A \subseteq M$. For any n-tuple $a = (a_1, ..., a_n) \in M^n$, we denote $\mathrm{acl}(A \cup \{a_1, ..., a_n\})$ by $\mathrm{acl}(A, a)$. Similarly for $\mathrm{dcl}(A, a)$.

Our notation for model theory is standard, and we will assume familiarity with basic notions such as very saturated models (or monster models), skolem functions, partial types, type-definable etc. References are [11] as well as [9].

1.2 Background in p-adically Closed Fields

Let p be a prime and $\mathbb{Q}_p$ the field of p-adic numbers. We call the complete theory of $\mathbb{Q}_p$ (in the language of rings) the *theory of p-adically closed fields*, written pCF. A p-adic closed field is a model of pCF, or equivalently, a field which is elementarily equivalent to $\mathbb{Q}_p$. A key point is the Macintyre’ s theorem [7] that pCF has quantifier elimination in the language of rings together with new predicates $P_n(x)$ for the n-th powers for each $n \in \mathbb{N}^+$. Let K be a p-adically closed field, we denote its multiplicative group by K^*, its valuation ring by R_K, and its value group by Γ_K. The value group Γ_K is a model of Pr (Presburger arithmetic), namely, $(\Gamma_K, <, +)$ is elementary equivalent to $(\mathbb{Z}, +, <)$. The valuation on a K is the map v from K to $\Gamma_K \cup \{\infty\}$ satisfying:

- $\infty > y$ and $y + \infty = \infty + y = \infty$ for all $y \in \Gamma_K$
- $v(x) = \infty \iff x = 0$;
- $v(x + y) \geq \min\{v(x), v(y)\}$ and $v(x + y) = \min\{v(x), v(y)\}$ when $v(x) \neq v(y)$;

- $v(xy) = v(x) + v(y)$.

Note that the valuation ring $R_K = \{x \in K |\ v(x) \geq 0\}$ and the relation $v(x) \leq v(y)$ are definable in the language of rings (see [4]), so is quantifier-free definable in Macintyre' s language. We will freely use the variables and parameters from the value group sort.

Throughout this paper, $(\mathbb{K}, +, \times, 0, 1)$ will denote a very saturated model (or monster model) of pCF and K an (arbitrary) small elementary submodel of $\mathbb{K}$, where we say that a set X is *small* if $|X| < |\mathbb{K}|$. We use $\mathbb{G}_{\mathrm{m}}$ to denote the multiplicative group of $\mathbb{K}$, so $\mathbb{G}_{\mathrm{m}}(K) = K^*$ is the multiplicative group of K. Before getting into details we recall some basic facts which will be used freely in this section and the rest of the paper.

First, the topology on K is the valuation topology. Consider the formula

$$B(x, y, \gamma) := v(x - y) > \gamma$$

where x, y are of the home sort and γ is of the value group sort. For any $c \in K$ and $\delta \in \Gamma_K$, we call $B(c, K, \delta)$ an open ball of center c and radius δ.

Fact 1.

- The p-adic field $\mathbb{Q}_p$ is a complete, locally compact topological field, with basis given by the sets $B(c, \mathbb{Q}_p, n)$ for $c \in \mathbb{Q}_p$, $n \in \mathbb{Z}$.
- K is also a topological field, with basis given by the sets $B(c, K, \delta)$ for $c \in K$, $\delta \in \Gamma_K$, but not need to be complete or locally compact.
- For each $c \in K$ and $\gamma \in \Gamma_K$, $B(c, K, \delta) = B(c', K, \delta)$ whenever $c' \in B(c, K, \delta)$.
- For each $c \in K$ and $\gamma \in \Gamma_K$, $B(c, K, \delta)$ is clopen.
- For each $c \in K$ and $\gamma \in \Gamma_K$, $B(c, K, \delta)$ is a disjoint union of $B(c_0, K, \delta + 1), ..., B(c_{p-1}, K, \delta + 1)$ for some $c_0, ..., c_{p-1} \in B(c, K, \delta)$.

It is well-known that pCF satisfies *Hensel's Lemma*:

Fact 2 (Hensel)**.** Let $f(t)$ be a polynomial over R_K in one variable t, and let $\alpha \in R_K$, $e \in \mathbb{N}$. Suppose that $v(f(\alpha)) \geq 2e + 1$ and $v(f'(\alpha)) \leq e$, where f' denotes the derivative of f. Then there exists a unique $\epsilon \in R_K$ such that $v(\epsilon) \geq e + 1$ and $f(\alpha + \epsilon) = 0$.

Recall that $P_n(x)$ denotes the formula saying that x is an n-th power, and that pCF has quantifier elimination after adding predicates for all $P_n(x)$. It is easy to see from the Hensel's Lemma that each $P_n(K^*)$ is an open subgroup of the multiplicative group K^* with finite index, and each coset of $P_n(K^*)$ contains representatives from $\mathbb{Z}$ (see [1, 7] for details).

The following lemma can also be conclude directly from the Hensel's Lemma. Nevertheless we give a proof here for convenience.

Lemma 1. *Let $a, b \in K^*$. If $v(a) \geq v(b) + 2v(n) + 1$, then a and $a - b$ are in the same coset of $P_n(K^*)$ in K^*, namely, $K \models \forall\lambda(P_n(\lambda b) \leftrightarrow P_n(\lambda(b-a)))$.*

Proof. Let $\epsilon \in K$ such that $v(\epsilon) \geq 2v(n)+1$. Consider the polynomial $f(t) = t^n - (1+\epsilon)$. Since $f(1) = \epsilon$ and $f'(1) = n$. It follows from the Hensel's Lemma that $f(t)$ has a root in K, which means that $1+\epsilon$ is an n-th power whenever $v(\epsilon) \geq 2v(n)+1$.

Now suppose that $v(a) > v(b)+2v(n)+1$, then $v(b-a) = \min\{v(a), v(b)\} = v(b)$. Let $\epsilon = a/(b-a)$, we see that

$$v(\epsilon) = v(a/(b-a)) = v(a) - v(b-a) \geq 2v(n) + 1,$$

so $1 + \epsilon$ is an n-th power. Since

$$b = (b-a) + a = (b-a)(1 + a/(b-a)) = (b-a)(1+\epsilon),$$

we see that b and $b - a$ are in the same coset of $P_n(K^*)$. □

The partial type $\{P_n(x) |\ n \in \mathbb{N}^+\}$ defines a subgroup of $\mathbb{G}_\mathrm{m}$, we call it the *definable connected component* of $\mathbb{G}_\mathrm{m}$, and denote it by $\mathbb{G}_\mathrm{m}^0$. Note that every coset of $\mathbb{G}_\mathrm{m}^0$ is type-definable over $\emptyset$.

Recall that a *well-indexed sequence* in K is a sequence $\{a_i\}_{i\in I}$ in K whose terms a_i are indexed by the elements i of an infinite well-ordered set $(I, <)$ without a last element.

Definition 1. Let $\{a_i\}_{i\in I}$ be a well-indexed sequence in K.

- We say that $\{a_i\}_{i\in I}$ is a *pseudo-Cauchy sequence* if for some index i_0 we have that $v(a_k - a_j) > v(a_j - a_i)$ whenever $k > j > i > i_0$.
- We say that $a^* \in \mathbb{K}$ is a *pseudo-limit* of $\{a_i\}_{i\in I}$ if $v(a_i - a^*)$ is eventually strictly increasing, that is, for some index i_0, we have that $v(a_k - a^*) > v(a_j - a^*)$ whenever $k > j > i_0$.
- We say that a formula $\phi(x)$ over K is *eventually true* on $\{a_i\}_{i\in I}$ if there is some $i_0 \in I$ such that $K \models \phi(a_i)$ for all $i > i_0$.

Remark 1. Since $\mathbb{K}$ is a monster model, a compactness argument shows that any pseudo-Cauchy sequence $\{a_i\}_{i\in I}$ in K has a (not necessary unique) pseudo-limit in $\mathbb{K}$.

Using cell decomposition in the form of Denef [4] or [5], the following can be easily derived, cf. [2], Lemma 4:

Fact 3. Let $X \subseteq K^{m+1}$ be a definable subset, and $b_j : X \to K$ definable functions, for $j = 1, ..., r$. Then there exists a finite partition of X such that each part A has form

$$A = \{(x, y) \in K^m \times K | \, x \in D, v(a_1(x)) \square_1 v(y - c(x)) \square_2 v(a_2(x)), \, (y - c(x)) \in \lambda P_n\}$$

and for each $(x, y) \in A$, we have

$$v(b_j(x, y)) = \frac{1}{e_j} v((y - c(x))^{\mu_j} d_j(x)),$$

with $x \in K^m$, $D \subseteq K^m$ definable, $0 < e_j \in \mathbb{Z}$, $\mu_j \in \mathbb{Z}$, $\lambda \in \mathbb{Z}$, c, a_i, d_j definable functions from K^m to K and $\square_i$ either $<$, $\leq$ or no condition.

Remark 2. It is easy to see from Fact 3 that

- Every one-variable formula $\phi(x)$ over K is equivalent to a disjoint disjunction of the formulas of the form

$$(\gamma_1 \square_1 v(x - c) \square_2 \gamma_2) \wedge P_n(\lambda(x - c))$$

  with $c \in K$, $\gamma_i \in \Gamma_K$, $\lambda \in \mathbb{Z}$, and $\square_i$ either $<$, $\leq$ or no condition.
- If $s(x)$ is a K-definable function and $a^* \in \mathbb{K}$, then there are $0 < e \in \mathbb{Z}$, $n \in \mathbb{Z}$, $d \in K$, and $\gamma \in \Gamma_K$ such that

$$v(s(a^*)) = 1/e(v((a^* - d)^n) + \gamma).$$

Another goodness of pCF is that it has definable Skolem functions(See [13]), i.e. for any formula $\varphi(x, y)$ over K with $K \models \forall x \exists y \varphi(x, y)$, we can find a K-definable function f_φ such that $K \models \forall x \varphi(x, f_\varphi(x))$. So for any $A \subseteq K$, $\mathrm{dcl}(A)$ is an elementary substructure of K. The reader is referred to [1] for additional details of p-adically closed fields.

2 Extensions of Models

Lemma 2. *Suppose that S is a small subset of $\mathbb{K}$, then* $\mathrm{dcl}(S) = \mathrm{acl}(S)$ *in* $\mathbb{K}$.

Proof Assume that $a \in \mathrm{acl}(S)$. There is a finite set D defined over S with the smallest cardinality such that $a \in D$. Then let $f(x) = \prod_{d \in D}(x - d)$ and $\mathrm{Aut}(\mathbb{K}/S)$ the group of automorphisms of $\mathbb{K}$ fixing S point-wise. Since every $\sigma \in \mathrm{Aut}(\mathbb{K}/S)$ fixes D set-wisely, we see that each coefficient of f is $\mathrm{Aut}(\mathbb{K}/S)$-invariant. Thus, all coefficients of f are in $\mathrm{dcl}(S)$ by the saturation of $\mathbb{K}$. As definable Skolem functions exist, some root b of f is in $\mathrm{dcl}(\mathrm{dcl}(S)) = \mathrm{dcl}(S)$. If $b \neq a$, then $D \backslash \{b\}$ is defined over S with cardinality $< |D|$ and $a \in D \backslash \{b\}$, which is impossible. Thus, $a = b \in \mathrm{dcl}(S)$. $\square$

For any $S \subseteq K$, $\mathrm{acl}(S)$ is the same whether computed in K or $\mathbb{K}$. So we have that

Corollary 1. *If* $S \subseteq K$*, then* $\mathrm{dcl}(S) = \mathrm{acl}(S)$ *in* K.

By [6], the algebraic closure operation $\mathrm{acl}(-)$ in any model K of pCF defines a *pre-geometry*, namely the exchange axiom is satisfied: if $a, b \in K$, $A \subseteq K$ and $b \in \mathrm{acl}(A, a) \backslash \mathrm{acl}(A)$, then $a \in \mathrm{acl}(A, b)$. For $a \in \mathbb{K}$, we write $\mathrm{dcl}(K, a)$ as $K\langle a\rangle$, which is an elementary extension of K. It is easy to see from Lemma 1 and the exchange axiom that:

Lemma 3. *Let* $a \in \mathbb{K}\backslash K$*. Then there is no proper middle extension between* $K \prec K\langle a\rangle$*, i.e. no* L *such that* $K \precneqq L \precneqq K\langle a\rangle$.

Recall from [12] that $\mathrm{Pr} = \mathrm{Th}(\mathbb{Z}, +, <, 0, \{D_n\}_{n>0})$ has definable Skolem functions, quantifier elimination in the language $\{+, <, 0, \{D_n\}_{n>0}\}$, and is decidable, where each D_n is a unary predicate symbol for the set of elements divisible by n. For any $A \subseteq M \models \mathrm{Pr}$, we see that $\mathrm{dcl}(A)$ is an elementary substructure of M. Clearly, the value group $\Gamma_{\mathbb{K}}$ of $\mathbb{K}$ is a monster model of Pr.

Lemma 4. *Let* $K_0 \prec K$*, and* G *an elementary substructure of* Γ_K *extending* Γ_{K_0}*. Then there is* K_1 *such that* $K_0 \prec K_1 \prec K$ *and* $G = \Gamma_{K_1}$.

Proof. Let

$$\mathcal{K} = \{L \models p\mathrm{CF} \mid K_0 \prec L \prec K \text{ and } \Gamma_L \subseteq G\}.$$

Then $\mathcal{K}$ is not empty since $K_0 \in \mathcal{K}$. Applying Zorn's Lemma to $(\mathcal{K}, \subseteq)$, and let K_1 be a maximal element of $\mathcal{K}$. We claim that G is the value group of K_1. Otherwise, there is $\alpha \in G\backslash\Gamma_{K_1}$. Take any $a \in K$ such that $v(a) = \alpha$. Then $v(a - c) = \min\{v(a), v(c)\} \in G$ for each $c \in K_1$. By Remark 2, for each $b \in \mathrm{dcl}(K_1, a) = K_1\langle a\rangle$, there are $0 < e \in \mathbb{Z}$, $n \in \mathbb{Z}$, $c \in K$, and $\gamma \in \Gamma_{K_1}$ such that

$$v(b) = 1/e(v((a - c)^n) + \gamma).$$

So $v(K_1\langle a\rangle) \subseteq G$ and thus the proper extension $K_1\langle a\rangle$ of K_1 is also in $\mathcal{K}$. A contradiction. □

We see from Lemma 4 that any $M \models \mathrm{Pr}$ is isomorphic to a value group of some $K \prec \mathbb{K}$. In this paper, we consider any (small) model of Pr as a value group of of some $K \prec \mathbb{K}$. We also write $\mathrm{dcl}(M, \alpha)$ as $M\langle\alpha\rangle$ for $\alpha \in \Gamma_{\mathbb{K}}$.

Lemma 5. *Let* $K' \succ K$ *and* $a \in K'\backslash K$ *such that* $v(a) = \alpha \notin \Gamma_K$*. Then* $\Gamma_{K\langle a\rangle} = \Gamma_K\langle\alpha\rangle$.

Proof. It is easy to see that

$$\Gamma_K\langle\alpha\rangle = \mathrm{dcl}(\Gamma_K,\alpha) \subseteq v(\mathrm{dcl}(K,a)) = \Gamma_{K\langle a\rangle}$$

since $\Gamma_K \cup \{\alpha\} \subseteq v(\mathrm{dcl}(K,a))$. Suppose for a contradiction that $\Gamma_K\langle\alpha\rangle$ is a proper subset of $\Gamma_{K\langle a\rangle}$. Then by Lemma 4, there is K' such that $K \prec K' \prec K\langle a\rangle$ such that $\Gamma_{K'} = \Gamma_K\langle\alpha\rangle$. Since $\Gamma_{K'}$ is proper middle extension between Γ_K and $\Gamma_{K\langle a\rangle}$, it follows that K' is a proper middle extension between K and $K\langle a\rangle$. This contradicts to Lemma 3. □

Corollary 2. *Let $M \models \mathrm{Pr}$ and $\alpha \in \Gamma_{\mathbb{K}}\backslash M$, then there is no middle extension between M and $M\langle\alpha\rangle$, i.e. there is no N such that $M \precneqq N \precneqq M\langle\alpha\rangle$.*

Proof. Suppose Not, then there is $\beta \in M\langle\alpha\rangle$ such that

$$M \precneqq M\langle\beta\rangle \precneqq M\langle\alpha\rangle.$$

By Lemma 4, there is $K' \models p\mathrm{CF}$ such that $\Gamma_{K'} = M$. Take any $a \in \mathbb{K}$ such that $v(a) = \alpha$, then $M\langle\alpha\rangle$ is the value group of $K'\langle a\rangle$ by Lemma 5. Take any $b \in K'\langle a\rangle$ such that $v(b) = \beta$, then applying Lemma 5 again, $M\langle\beta\rangle$ is the value group of $K'\langle b\rangle$. We conclude that $K'\langle b\rangle$ is a proper middle extension between K' and $K'\langle a\rangle$. A contradiction. □

3 Classification of 1-types

Recall that $\mathbb{K}$ is the monster model of pCF and $\mathbb{G}_{\mathrm{m}}$ is the multiplicative group of $\mathbb{K}$. From now on, we fix K as a small elementary submodel of $\mathbb{K}$ and $K^* = \mathbb{G}_{\mathrm{m}}(K)$ the multiplicative group of K.

Definition 2. Suppose that $(\Gamma, +, <, 0)$ is a model of Presburger arithmetic. We say that $\Lambda \subseteq \Gamma$ is a *cut* of Γ if

- For each $\gamma, \beta \in \Lambda$, if $\gamma \in \Lambda$ and $\beta < \gamma$, then $\beta \in \Lambda$;
- For each $n \in \mathbb{Z}$ and $\gamma \in \Lambda$, $\gamma + n \in \Lambda$.

i.e. Λ is downward closed and satisfying $\Lambda + \mathbb{Z} = \Lambda$.

Lemma 6. *Let $a^* \in \mathbb{K}\backslash K$, and*

$$\Lambda := \{\alpha \in \Gamma_K : \textit{there is } c \in K \textit{ such that } v(a^* - c) > \alpha\}.$$

Then Λ is a cut of Γ_K

Proof. It is easy to see that Λ is downward closed. It suffices to show that $\delta \in \Lambda$ implies $\delta + 1 \in \Lambda$. Suppose that $v(c - a^*) > \delta$ for some $c \in K$ and $\delta \in \Gamma_K$, we see from Fact 1 that there are $c_0, ..., c_{p-1} \in K$ such that $B(c, K, \delta)$ is a disjoint union of $B(c_0, K, \delta+1), ..., B(c_{p-1}, K, \delta+1)$. Since $\mathbb{K}$ is an elementary extension of K, $B(c, \mathbb{K}, \delta)$ is also a disjoint union of $B(c_0, \mathbb{K}, \delta+1), ..., B(c_{p-1}, \mathbb{K}, \delta+1)$. As $a^* \in B(c, \mathbb{K}, \delta)$, there is $i < p$ such that $a^* \in B(c_i, \mathbb{K}, \delta+1)$. So $v(a^* - c_i) > \delta + 1$ as required. $\square$

Theorem 4. *Let K be a model of pCF and $\mathbb{G}_m^0 = \bigcap_{n\in\mathbb{N}^+} P_n(\mathbb{G}_m)$ be the definable connected component of $\mathbb{G}_m$. Then the complete 1-types over K are precisely the following:*

(a) The realized types $\mathrm{tp}(a/K)$ for each $a \in K$;
(b) (distance type around a point) For each cut $\Lambda \subseteq \Gamma_K$, $c \in K$, and coset C of $\mathbb{G}_m^0$, the type $p_{\Lambda,c,C}$ saying that $\Lambda < v(x-c) < \Gamma_K \backslash \Lambda$ and $x - c \in C$. We call $p_{\Lambda,c,C}$ a Λ-distance (or distance) type around point c.
(c) (Pseudo-limit type) For each pesudo-Cauchy sequence $\{c_i\}_{i\in I}$, the type $p_{\{c_i\}_{i\in I}}(x)$ saying that x is a pseudo-limit of $\{c_i\}_{i\in I}$. In this case, $p_{\{c_i\}_{i\in I}}(x)$ is determined by the sequence $\{c_i\}_{i\in I}$: For each formula $\phi(x)$ over K, $\phi(x) \in p_{\{c_i\}_{i\in I}}(x)$ iff $\phi(x)$ is eventually true on $\{c_i\}_{i\in I}$.

Proof. Let $p(x) \in S_1(K)$ be a non-realized type and $a^* \models p$. Let

$$\Lambda := \{\alpha \in \Gamma_K : \text{there is } c \in K \text{ such that } v(a^* - c) > \alpha\}.$$

Then Λ is a cut of Γ_K by Lemma 6. Let $a^* \models p$. Now we have two cases:

• **Case 1**: There is $c \in K$ such that $v(a^* - c)$ is maximal among the set $\{v(a^* - d) : d \in K\}$. Then $v(a^* - c) \in \Gamma_{\mathbb{K}} \backslash \Gamma_K$ realizes the cut Λ, i.e. $\Lambda < v(a^* - c) < \Gamma_K \backslash \Lambda$. Let C be the coset of $\mathbb{G}_{\mathrm{m}}^0$ such that $a^* - c \in C$. We claim that

Claim 1. *Let $\Sigma_{\Lambda,C}(x, c)$ be the partial type saying that $\Lambda < v(x - c) < \Gamma_K$ and $x - c \in C$, then for any formula $\phi(x)$ over K, $\phi(x) \in p$ iff $\phi(x)$ is consistent with $\Sigma_{\Lambda,C}(x, c)$.*

Proof. Clearly, every formula $\phi(x) \in p(x)$ is consistent with $\Sigma_{\Lambda,C}(x, c)$ since $\Sigma_{\Lambda,C}(x, c) \subseteq p$.

Now suppose that $\phi(x)$ is consistent with $\Sigma_{\Lambda,C}(x, c)$. We aim to show that $a^* \models \phi(x)$. By Remark 2, we can assume that $\phi(x)$ is of the form

$$\alpha_1 \square_1 v(x - d) \square_2 \alpha_2 \wedge P_n(s(x - d))$$

with $\alpha_1, \alpha_2 \in \Gamma_K$, $d \in K$, and $s \in \mathbb{Z}$. Let $b^* \in \mathbb{K}$ realize the partial type

$$\{\alpha_1 \square_1 v(x - d) \square_2 \alpha_2, P_n(s(x - d))\} \cup \Sigma_{\Lambda,C}(x, c).$$

Then both $v(a^* - c)$ and $v(b^* - c)$ realize the cut Λ. As both $v(a^* - c)$ and $v(b^* - c)$ are not in Γ_K, we have that

$$v(b^* - d) = \min(v(b^* - c), v(c - d)) = \begin{cases} v(c-d), & \text{if } v(c-d) \in \Lambda \\ v(b^* - c), & \text{if } v(c-d) \notin \Lambda \end{cases}$$

and

$$v(a^* - d) = \min(v(a^* - c), v(c - d)) = \begin{cases} v(c-d), & \text{if } v(c-d) \in \Lambda \\ v(a^* - c), & \text{if } v(c-d) \notin \Lambda \end{cases}$$

If $v(c - d) \in \Lambda$, then

$$v(a^* - d) = \min(v(a^* - c), v(c - d)) = v(c - d) = v(b^* - d),$$

which means that $a^* \models \alpha_1 \Box_1 v(x - d) \Box_2 \alpha_2$.

If $v(c - d) \notin \Lambda$, then both

$$v(a^* - d) = \min(v(a^* - c), v(c - d)) = v(a^* - c)$$

and

$$v(b^* - d) = \min(v(b^* - c), v(c - d)) = v(b^* - c)$$

realize the cut Λ, so

$$\alpha_1 \Box_1 v(b^* - d) \Box_2 \alpha_2 \iff \alpha_1 \Box_1 v(a^* - d) \Box_2 \alpha_2,$$

which also means that $a^* \models \alpha_1 \Box_1 v(x - d) \Box_2 \alpha_2$.

We now show that a^* also realizes $P_n(s(x - d))$. If $v(c - d) \in \Lambda$, we have

$$v(a^* - c) > v(c - d) + \mathbb{Z} \text{ and } v(b^* - c) > v(c - d) + \mathbb{Z}$$

By Lemma 1, we see that $(a^* - d)$, $(c - d)$ and $(b^* - d)$ are in the same coset of $P_n(\mathbb{G}_{\mathrm{m}})$. So $a^* \models P_n(s(x - d))$ as required. Similarly, if $v(c - d) \notin \Lambda$, then we have both

$$v(a^* - c) < v(c - d) + \mathbb{Z} \text{ and } v(b^* - c) < v(c - d) + \mathbb{Z}.$$

Which implies that $a^* - d$ and $a^* - c$ are in the same coset of $P_n(\mathbb{G}_{\mathrm{m}})$, also, $b^* - d$ and $b^* - c$ are in the same coset of $P_n(\mathbb{G}_{\mathrm{m}})$. Since $(b^* - c)$ and $(a^* - c)$ are in the same coset of $\mathbb{G}_{\mathrm{m}}^0$, we see that $\mathbb{K} \models P_n(s(b^* - d)) \leftrightarrow P_n(s(a^* - d))$. So $a^* \models P_n(s(x - d))$. This complete the proof of the Claim. $\Box$

Clearly, we see for the above Claim that p is determined by the partial type $\Sigma_{\Lambda,C}(x,c)$ when Case 1 happens.

• **Case 2**. There is no such c as in the previous case. First we show that $v(a^*-c)\in\Gamma_K$ for each $c\in K$. To see this, suppose that there is $c\in K$ such that $v(a^*-c)\notin\Gamma_K$, then for any $c\neq d\in K$, we have

$$v(a^*-d)=\min\{v(a^*-c),v(c-d)\}\leq v(a^*-c).$$

This contradicts our assumption. So we conclude that $\Lambda=\{v(a^*-c)|\ c\in K\}$. We claim that Λ has a well-ordered cofinal subet I. Let

$$W=\{J\subseteq\Lambda|\ J\text{ is a well-ordered subset}\}.$$

Applying Zorn's Lemma to $(W,\subseteq)$, and let I be a maximal element of W, it is easy to see that I is cofinal in Λ. Since Λ is a cut, it has no largest element, we see that I is infinite. Take a sequence $\{c_i\in K|\ i\in I\}$ such that $v(a^*-c_i)=i$, Then a^* is a pseudo-limit of $\{c_i\}_{i\in I}$.

Claim 2. *A formula $\phi(x)$ over K is in $p(x)$ iff $\phi(x)$ is eventually true on $\{c_i\in K|\ i\in I\}$.*

Proof. Assume again that the formula $\phi(x)$ is of the form

$$\alpha_1\Box_1 v(x-d)\Box_2\alpha_2\wedge P_n(s(x-d))$$

with $\alpha_1,\alpha_2\in\Gamma_K$, $d\in K$ and $s\in\mathbb{Z}$. Let $i_0\in I$ such that $v(a^*-c_i)>v(a^*-d)+2v(n)+1$ for all $i>i_0$. As

$$v(a^*-d)=\min\{v(a^*-c_i),v(c_i-d)\}=v(c_i-d)$$

for all $i>i_0$, we see that

$$a^*\models\alpha_1\Box_1 v(x-d)\Box_2\alpha_2 \iff c_i\models\alpha_1\Box_1 v(x-d)\Box_2\alpha_2 \text{ (for all } i>i_0).$$

Applying Lemma 1, we have that (a^*-d) and (c_i-d) are in the same coset of $P_n(\mathbb{G}_{\mathrm{m}})$. So

$$\mathbb{K}\models P_n(s(a^*-d))\leftrightarrow P_n(s(c_i-d))$$

for all $i>i_0$. We conclude that $\phi(x)\in p$ iff $\phi(x)$ is eventually true on $\{c_i\in K|\ i\in I\}$. This completes the proof. □

We see from Claim 1 and Claim 2 that each $p(x)\in S_1(K)$ is either a realized type, or a distance type determined by a cut Λ, a point $c\in K$ and, a coset C of $\mathbb{G}_{\mathrm{m}}^0$, or a pseudo-limit type determined by a pseudo-Cauchy sequence.

Conversely, the proof of Claim 1 indicates that for each cut $\Lambda\subseteq K$, $c\in K$, and coset C of $\mathbb{G}_{\mathrm{m}}^0$, the partial type $\Sigma_{\Lambda,C}(x,c)$ determines a complete 1-type over K. Similarly, the proof of Claim 2 indicates that each pseudo-Cauchy sequence also determines a complete 1-type over K. □

4 Orthogonality of 1-types

As we mentioned in the introduction, distance types and pseudo-limit types are the analogues of "noncut" and "cut" in the o-minimal context respectively. We aim to show the orthogonality of distance types and pseudo-limit types in this section.

Lemma 7. *Let $a \in \mathbb{K}\backslash K$. Then*

- tp(a/K) *is a pseudo-limit type iff* $\Gamma_{K\langle a\rangle} = \Gamma_K$.
- tp(a/K) *is a distance type iff* $\Gamma_{K\langle a\rangle} \neq \Gamma_K$. *Moreover, if* tp$(a/K)$ *is a distance type around $c \in K$ then $\Gamma_{K\langle a\rangle} = \Gamma_K\langle v(a-c)\rangle$.*

Proof. It is easy to see from Theorem 4 that tp(a/K) is a pseudo-limit type iff $v(a-b) \in \Gamma_K$ for all $b \in K$. So $\Gamma_{K\langle a\rangle} = \Gamma_K$ implies that tp(a/K) is a pseudo-limit type.

Now suppose that $\Gamma_{K\langle a\rangle} \neq \Gamma_K$. To see that tp$(a/K)$ is a distance type, it suffices to show that $v(a-c) \notin \Gamma_K$ for some $c \in K$. Let s be a K-definable function such that $v(s(a)) \notin \Gamma_K$, then by Remark 2,

$$v(s(a)) = 1/e(v((a-d)^n) + \gamma)$$

for some $0 < e \in \mathbb{Z}$, $n \in \mathbb{Z}$, $d \in K$, and $\gamma \in \Gamma_K$. It is easy to see that $v(s(a)) \notin \Gamma_K$ implies $v(a-d) \notin \Gamma_K$. So tp(a/K) is a distance type as required.

For the "moreover" part, suppose that tp(a/K) is a distance type around $c \in K$, we have seen that every $\delta \in \Gamma_{K\langle a\rangle}$ is of the form $1/e(v((a-d)^n)+\gamma)$ with $0 < e \in \mathbb{Z}$, $n \in \mathbb{Z}$, $d \in K$, and $\gamma \in \Gamma_K$, whereas $v(a-d) = \min\{v(a-c), v(c-d)\}$. We conclude that $\delta \in \text{dcl}(\Gamma_K, v(a-c))$. So $\Gamma_{K\langle a\rangle} \subseteq \Gamma_K\langle v(a-c)\rangle$. As $\Gamma_K \subsetneqq \Gamma_{K\langle a\rangle} \prec \Gamma_K\langle v(a-c)\rangle$, we see that $\Gamma_{K\langle a\rangle} = \Gamma_K\langle v(a-c)\rangle$ by Lemma 5. □

Remark 3. It is easy to see from Lemma 7 that a non-realized type can not be both of distance and pseudo-limit simultaneously.

Lemma 8. *Let $a \in \mathbb{K}\backslash K$ and $b \in K\langle a\rangle\backslash K$, then* tp$(b/K)$ *is in the same case of* tp(a/K)*, i.e.* tp(b/K) *is distance type(resp. pseudo-limit type), if* tp(a/K) *is.*

Proof. We see from Lemma 3 that that $K\langle b\rangle = K\langle a\rangle$ and hence, $\Gamma_{K\langle a\rangle} = \Gamma_{K\langle b\rangle}$. By Lemma 7, tp$(a/K)$ is a distance type iff $\Gamma_{K\langle a\rangle} = \Gamma_{K\langle b\rangle} \neq \Gamma_K$ iff tp(b/K) is a distance type. □

Lemma 9. *If* tp(c/K) *is a distance type and* tp(d/K) *is a pseudo-limit type, then c and d are algebraic independent over K, i.e. $c \notin$ acl$(K,d) =$ dcl(K,d) and $d \notin$ acl$(K,c) =$ dcl(K,c).*

Proof. Suppose for a contradiction that $c \in \text{dcl}(K,d)$. As $c \notin K$, it follows from Lemma 8 that tp(c/K) is a also a pseudo-limit type, which is impossible by Lemma 7. Similarly, we have $d \notin \text{dcl}(K,c)$. □

For both a and b realize distance types over K, we have a rough relation between $\mathrm{tp}(a/K\langle b\rangle)$ and $\mathrm{tp}(b/K\langle a\rangle)$:

Proposition 1. *If both a and b realize distance types over K, then $\mathrm{tp}(a/K\langle b\rangle)$ is in the same case of $\mathrm{tp}(b/K\langle a\rangle)$, i.e. $\mathrm{tp}(a/K\langle b\rangle)$ is a realized (resp. distance , pesudo-limit) type if $\mathrm{tp}(b/K\langle a\rangle)$ is.*

Proof. Since $a, b \notin K$, we see that $b \in K\langle a\rangle$ iff $a \in K\langle b\rangle$ by Lemma 3. So $\mathrm{tp}(a/K)$ is realized iff $\mathrm{tp}(b/K)$ is realized.

Now we assume that $b \notin K\langle a\rangle$. Suppose for a contradiction that $\mathrm{tp}(b/K\langle a\rangle)$ is a distance type but $\mathrm{tp}(a/K\langle b\rangle)$ is a pseudo-limit type. Then by Lemma 7 we have that $\Gamma_{K\langle a\rangle} \precneqq \Gamma_{K\langle a,b\rangle}$ and $\Gamma_{K\langle b\rangle} = \Gamma_{K\langle a,b\rangle}$. Since a realizes a distance types over K, we see that $\Gamma_K \precneqq \Gamma_{K\langle a\rangle}$, and hence $\Gamma_{K\langle a\rangle}$ is a proper middle extension between Γ and $\Gamma_{K\langle b\rangle} = \Gamma_{K\langle a,b\rangle}$. Applying Lemma 7 again, there is $\alpha \in \Gamma_{K\langle b\rangle}$ such that $\Gamma_{K\langle b\rangle} = \Gamma_K\langle\alpha\rangle$. We conclude that $\Gamma_{K\langle a\rangle}$ is a proper middle extension between Γ and $\Gamma_K\langle\alpha\rangle$, this contradicts to Corollary 2.

Similarly, it is impossible that $\mathrm{tp}(b/K\langle a\rangle)$ is pseudo-limit but $\mathrm{tp}(a/K\langle b\rangle)$ is a distance type. $\square$

We now show that pseudo-limit types and distance types are weakly orthogonal.

Proposition 2. *Suppose that $\mathrm{tp}(a/K)$ is a distance type and $\mathrm{tp}(c/K)$ is a pseudo-limit of a sequence $\{c_i\}_{i\in I} \subseteq K$. Then $\mathrm{tp}(c/K\langle a\rangle)$ is also a pseudo-limit of the sequence $\{c_i\}_{i\in I}$.*

Proof. Firstly, $c \notin K\langle a\rangle$. Suppose not, $K \prec K\langle c\rangle \prec K\langle a\rangle$ implies that $K\langle c\rangle = K\langle a\rangle$, whereas $\Gamma_{K\langle c\rangle} = \Gamma_K$ and $\Gamma_K \neq \Gamma_{K\langle a\rangle}$.

Secondly, $\mathrm{tp}(c/K\langle a\rangle)$ can not be a distance type. Suppose not, we can assume that $\mathrm{tp}(c/K\langle a\rangle)$ is a distance type around some point $f \in K\langle a\rangle$. If $f \in K$, then $v(c-f)$ is maximal among $\{v(c-e)|\ e \in K\}$, and thus $\mathrm{tp}(c/K)$ is a distance type, which is a contradiction. So $f \in K\langle a\rangle\backslash K$, and by Lemma 8, we see that $\mathrm{tp}(f/K)$ is a distance type around a point $d \in K$. Since

$$v(c-f) > v(c-d) \in \Gamma_K,$$

we have that

$$v(f-d) = v((f-c)+(c-d)) = \min\{v(f-c), v(c-d)\} = v(c-d) \in \Gamma_K,$$

which is impossible because $v(f-d) \notin \Gamma_K$ by Lemma 7. Thus, we conclude that $\mathrm{tp}(c/K\langle a\rangle)$ is a pseudo-limit of a well-indexed sequence $\{f_j\}_{j\in J} \subseteq K\langle a\rangle$.

To see that $\mathrm{tp}(c/K\langle a\rangle)$ is a pseudo-limit of the sequence $\{c_i\}_{i\in I}$, it suffices to show that for each $j \in J$ there is $i \in I$ such that $v(c-c_i) \geq v(c-f_j)$. Suppose for a contradiction that there is $j_0 \in J$ such that $v(c - f_{j_0}) > v(c - c_i)$ for all $i \in I$. We see from Lemma 8 that tp(f_{j_0}/K) is a distance type. Suppose that tp(f_{j_0}/K) is around $e \in K$. Then, for all $i \in I$,

$$v(f_{j_0} - e) \geq v(f_{j_0} - c_i) \tag{1}$$

As tp(c/K) is a pseudo-limit of $\{c_i\}_{i \in I}$ and $e \in K$, there is $i_0 \in I$ such that $v(c - c_i) > v(c - e)$ for all $i \in I$ with $i > i_0$. Then we have that, for every $i \in I$ with $i > i_0$,

$$v(c - f_{j_0}) > v(c - c_i) > v(c - e) \tag{2}$$

We conclude from (1) and (2) that,

$$v(f_{j_0} - e) \geq v(f_{j_0} - c_i) = v(f_{j_0} - c + c - c_i) = \min\{v(f_{j_0} - c), v(c - c_i)\} = v(c - c_i),$$

and then

$$v(c - e) = v((c - f_{j_0}) + (f_{j_0} - e)) \geq \min(v(c - f_{j_0}), v(f_{j_0} - e)) \geq v(c - c_i),$$

for all $i \in I$ with $i > i_0$. Since c is a pseudo-limit of $\{c_i\}_{i \in I}$, we see that $v(c - e)$ is maximal among $\{v(c - d) |\ d \in K\}$, and hence tp(c/K) is a distance type. It is a contradiction. $\square$

We conclude the orthogonality of pseudo-limit types and distance types directly from Proposition 2:

Theorem 5. *Suppose that $p(x) \in S_1(K)$ is a pseudo-limit type and $q(y) \in S_1(K)$ is a distance type, then there is $r(x, y) \in S_2(K)$ such that $p(x) \cup q(y) \vdash r(x, y)$.*

Proof. Take any $r(x, y) \in S_2(K)$ such that $p(x) \cup q(x) \subseteq r(x, y)$. We now show that $p(x) \cup q(y) \vdash r(x, y)$. Let $a \models p(x)$ and $c \models q(y)$, then it suffices to show that $(a, c) \models r(x, y)$. Suppose that $(a', c') \models r(x, y)$. Since tp$(a/K) =$ tp(a'/K), by the saturation of $\mathbb{K}$, there is $c'' \in \mathbb{K}$ such that tp$(a, c''/K) =$ tp$(a', c'/K)$. So $r =$ tp$(a, c''/K)$ and $q =$ tp$(c/K) =$ tp(c''/K). Assume that q is a pseudo-limit of a sequence $(c_i)_{i \in I} \subseteq K$. We see from Proposition 2 that both tp$(c/K\langle a\rangle)$ and tp$(c''/K\langle a\rangle)$ are pseudo-limit of the sequence $(c_i)_{i \in I}$. By Lemma 4, tp$(c/K\langle a\rangle) =$ tp$(c''/K\langle a\rangle)$. So tp$(a, c/K) =$ tp$(a, c''/K) = r(x, y)$ as required. $\square$

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p-进闭域上型的研究

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摘 要

本文的目标是研究 p-进闭域上型。我们首先分类了 p-进闭域上的 1-型，该结果推广了 D. Penazzi，A. Pillay 和 N. Yao（2019）对于 p-进闭域的标准模型 $\mathbb{Q}_p$ 上的型的分类。我们还进一步研究了“伪极限型”与“距离型”的正交性，这种正交性类似于序-极小结构上 1-型的“切割”与“非切割”正交性和二歧性。

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