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Sahlqvist Correspondence Theory for Modal Logic with Quantification over Relations*

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Abstract. Lehtinen (2008) introduced a new concept of validity of modal formulas, where quantification over binary relations is allowed for the so called "helper modalities", and the "boss modalities" are similar to ordinary modalities in modal logic in the sense that they are interpreted as a fixed binary relation in a Kripke frame. In the present paper, we study the correspondence theory for this validity notion. We define the class of Sahlqvist formulas for this validity notion, each formula of which has a first-order frame correspondent, and define the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ to compute the first-order correspondents of this class.

1 Introduction

Lehtinen ([6]) introduced a new concept of validity of modal formulas, which allows, from the perspective of second-order logic, quantification over binary relations. In this definition of validity, if the modal similarity type is $\tau = \{\Diamond_1, \ldots, \Diamond_n\}$, then we say that the modal formula φ is τ-valid in a set W (notation $W \Vdash_\tau \varphi$) iff it is valid in each frame $\mathbb{F} = (W, R_1, \ldots, R_n)$. With the help of the standard translation, assume that only $p_1, \ldots, p_k$ occur in φ, then the τ-validity in a set W can be equivalently written as:

$$W \Vdash_\tau \varphi \ \Leftrightarrow\ W \vDash \forall R_1 \ldots \forall R_n \forall P_1 \ldots \forall P_k \forall x ST_x(\varphi).$$

As is shown in [6, Example 5.1.2, 5.1.3], this notion of validity can be used to define the size of the domain. Indeed, take $\tau = \{\Diamond\}$,

$$W \Vdash_\tau \Diamond p \to \Box p \ \Leftrightarrow \mid W \mid \leq 1.$$

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$$W \Vdash_\tau (\Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3) \to \Diamond((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee (p_2 \wedge p_3)) \Leftrightarrow \mid W \mid \leq 2.$$

In this definition, set validity allows us to talk about the size of a domain, but we lose the possibility to talk about relations. Therefore, Lehtinen proposes a more general perspective by allowing some relations to be *helpers* and others to be *bosses*, such that we only quantify over the helpers and keep the bosses similar to the standard Kripke frame validity.

In the new definition, the similarity type τ is defined to be the disjoint union of τ_H and τ_B, where modalities in $\tau_H = \{\Diamond_1^H, \ldots, \Diamond_m^H\}$ are called *helpers*, and modalities in $\tau_B = \{\Diamond_1^B, \ldots, \Diamond_n^B\}$ are called *bosses*.

We say that a formula is τ_H-$valid$ in a frame $(W, R_1, \ldots, R_n)$, if

$$(W, R_1, \ldots, R_n, H_1, \ldots, H_m) \Vdash \varphi$$

for all helper relations $H_1, \ldots, H_m$. With the help of the standard translation, the τ_H-$validity$ in $\mathbb{F} = (W, R_1, \ldots, R_n)$ can be reformulated as

$$\mathbb{F} \vDash \forall H_1 \ldots \forall H_m \forall P_1 \ldots \forall P_k \forall x ST_x(\varphi).$$

With the notion of τ_H-validity, we can use modal formulas to define first-order properties of Kripke frames that cannot be defined using standard validity notion.

Example 1 (Example 5.1.7 in [6]). Let $\tau_B = \{\Diamond\}$, $\tau_H = \{\Diamond^H\}$, and $\mathbb{F} = (W, R)$. Then we have

$$\mathbb{F} \Vdash \Box p \to \Box^H p \text{ iff } R = W \times W.$$

In the present paper, we study the Sahlqvist correspondence theory of this validity notion, namely, we define a class of Sahlqvist formulas in the modal language of helpers and bosses, and define an **A**ckermann **L**emma **B**ased **A**lgorithm $\mathrm{ALBA}^{\mathrm{RQ}}$[1] to compute the first-order correspondents of Sahlqvist formulas.

The structure of the paper is organized as follows: Section 2 presents preliminaries on modal logic of helpers and bosses. Section 3 defines Sahlqvist formulas and inequalities. Section 4 defines the expanded modal language, the first-order correspondence language and the standard translation, which will be used in the algorithm. Section 5 defines the Ackermann Lemma Based Algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$. Section 6 proves the soundness of the algorithm. Section 7 shows that $\mathrm{ALBA}^{\mathrm{RQ}}$ succeeds on Sahlqvist formulas. Section 8 gives some examples. Section 9 gives conclusions.

2 Preliminaries

In the present section, we collect the preliminaries on modal logic with helpers and bosses. For more details, see [6, Section 5].

[1] Here RQ stands for "relation quantifier".

2.1 Language and Syntax

Definition 1. Given a set Prop of propositional variables, a finite set $\tau_H = \{\Diamond^H_1, \ldots, \Diamond^H_m\}$, a finite set $\tau_B = \{\Diamond^B_1, \ldots, \Diamond^B_n\}$ such that $\tau_H \cap \tau_B = \varnothing$, the modal language with helpers and bosses is defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Diamond\varphi,$$

where $p \in \mathrm{Prop}$, $\Diamond \in \tau_H \cup \tau_B$. $\Box$ and $\leftrightarrow$ are defined in the standard way. We call a formula *pure* if it contains no propositional variables. We use $\tau := (\tau_H, \tau_B)$ to denote the *similarity type* of the language. Throughout the article, we will also make substantial use of the following expressions:

(1) An *inequality* is of the form $\varphi \leq \psi$, where φ and ψ are formulas.
(2) A *quasi-inequality* is of the form $\varphi_1 \leq \psi_1$ & $\ldots$ & $\varphi_n \leq \psi_n \ \Rightarrow\ \varphi \leq \psi$.

We will find it easy to work with inequalities $\varphi \leq \psi$ in place of implicative formulas $\varphi \to \psi$ in Section 3.

2.2 Semantics

Definition 2. Given a similarity type $\tau = (\tau_H, \tau_B)$, a τ-*Kripke frame* is a tuple $\mathbb{F} = (W, R_1, \ldots, R_n, H_1, \ldots, H_m)$ where $W \neq \varnothing$ is the *domain* of $\mathbb{F}$, $R_1, \ldots, R_n$, $H_1, \ldots, H_m$ are *accessibility relations* which are binary relations on W, and each R_i corresponds to $\Diamond^B_i$, each H_i corresponds to $\Diamond^H_i$. The underlying τ_B-*Kripke frame* of a τ-Kripke frame is a tuple $\mathbb{F} = (W, R_1, \ldots, R_n)$ where each R_i corresponds to $\Diamond^B_i$ respectively and no relations for $\Diamond^H_i$ are there. τ_B-Kripke frames are used to define validity. A τ-*Kripke model* is a pair $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a τ-Kripke frame and $V : \mathrm{Prop} \to P(W)$ is a *valuation* on $\mathbb{F}$. Now the satisfaction relation is defined as follows[2]: given any τ-Kripke model $\mathbb{M} = (W, R_1, \ldots, R_n, H_1, \ldots, H_m, V)$, any $w \in W$,

$$\begin{array}{lll}
\mathbb{M}, w \Vdash \Box^B_i\varphi & \text{iff} & \forall v(R_i wv \ \Rightarrow\ \mathbb{M}, v \Vdash \varphi);\\
\mathbb{M}, w \Vdash \Diamond^B_i\varphi & \text{iff} & \exists v(R_i wv \ \text{and}\ \mathbb{M}, v \Vdash \varphi).\\
\mathbb{M}, w \Vdash \Box^H_i\varphi & \text{iff} & \forall v(H_i wv \ \Rightarrow\ \mathbb{M}, v \Vdash \varphi);\\
\mathbb{M}, w \Vdash \Diamond^H_i\varphi & \text{iff} & \exists v(H_i wv \ \text{and}\ \mathbb{M}, v \Vdash \varphi).
\end{array}$$

For any formula φ, we let $[\![\varphi]\!]^{\mathbb{M}} = \{w \in W \mid \mathbb{M}, w \Vdash \varphi\}$ denote the *truth set* of φ in $\mathbb{M}$. The formula φ is *globally true* on $\mathbb{M}$ (notation: $\mathbb{M} \Vdash \varphi$) if $[\![\varphi]\!]^{\mathbb{M}} = W$. The crucial difference between modal logic with helpers and bosses and ordinary modal logic is the definition of validity. Validity in the former is only defined on τ_B-Kripke frames:

[2]The basic case and the Boolean cases are defined as usual, and here we only give the clauses for the modalities.

A τ-formula φ is *valid* on a τ_B-Kripke frame $\mathbb{F} = (W, R_1, \ldots, R_n)$ (notation: $\mathbb{F} \Vdash \varphi$) if φ is globally true on $(\mathbb{F}, H_1, \ldots, H_m, V)$ for all helper relations $H_1, \ldots, H_m$ and all valuations V. The semantics of inequalities and quasi-inequalities are given as follows:

$$\mathbb{M} \Vdash \varphi \leq \psi \text{ iff (for all } w \in W, \text{ if } \mathbb{M}, w \Vdash \varphi, \text{ then } \mathbb{M}, w \Vdash \psi).$$

$$\mathbb{M} \Vdash \varphi_1 \leq \psi_1 \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n \ \Rightarrow \ \varphi \leq \psi \text{ iff}$$

$$\mathbb{M} \Vdash \varphi \leq \psi \text{ holds whenever } \mathbb{M} \Vdash \varphi_i \leq \psi_i \text{ for all } 1 \leq i \leq n.$$

The definitions of validity are similar to formulas. It is easy to see thatt $\mathbb{M} \Vdash \varphi \leq \psi$ iff $\mathbb{M} \Vdash \varphi \to \psi$.

3 Sahlqvist Formulas and Inequalities

In this section, we define Sahlqvist formulas and inequalities in the similarity type τ, in the style of unified correspondence [2]. We collect preliminaries here.

Definition 3 (Order-type)**.** (cf. [4, p. 346]) For an n-tuple $(p_1, \ldots, p_n)$ of propositional variables, an order-type ε is an element in $\{1, \partial\}^n$. We say that p_i has order-type 1 (resp. ∂) with respect to ε if $\varepsilon_i = 1$ (resp. $\varepsilon_i = \partial$), and denote $\varepsilon(p_i) = 1$ (resp. $\varepsilon(p_i) = \partial$). We use ε^∂ to denote the order-type where $\varepsilon^\partial(p_i) = 1$ (resp. $\varepsilon^\partial(p_i) = \partial$) iff $\varepsilon(p_i) = \partial$ (resp. $\varepsilon(p_i) = 1$).

Definition 4 (Signed generation tree)**.** (cf. [5, Definition 4]) The *positive* (resp. *negative*) *generation tree* of any τ-formula φ is defined by first labelling the root of the generation tree of φ with $+$ (resp. $-$) and then labelling the children nodes as follows:

- Assign the same sign to the children nodes of any node labelled with $\vee, \wedge, \Box_i^H, \Diamond_i^H, \Box_i^B, \Diamond_i^B$;
- Assign the opposite sign to the child node of any node labelled with $\neg$;
- Assign the opposite sign to the first child node and the same sign to the second child node of any node labelled with $\to$;

Nodes in signed generation trees are called *positive* (resp. *negative*) if they are signed $+$ (resp. $-$).

We give an example of signed generation tree in Figure 1.

For any τ-formula $\varphi(p_1, \ldots p_n)$, any order-type ε over n, and any $i = 1, \ldots, n$, an *ε-critical node* in a signed generation tree of φ is a leaf node $+p_i$ when $\varepsilon_i = 1$ or $-p_i$ when $\varepsilon_i = \partial$. An *ε-critical branch* in a signed generation tree is a branch from an ε-critical node. The ε-critical occurrences are intended to be those which the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ will solve for.

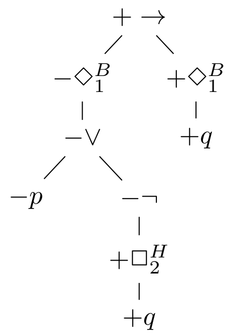

Figure 1: Positive generation tree for $\Diamond_1^B(p \vee \neg\Box_2^H q) \rightarrow \Diamond_1^B q$

Outer						Inner			
$+$	$\vee$	$\wedge$	$\Diamond$	$\neg$		$+$	$\wedge$	$\Box$	$\neg$
$-$	$\wedge$	$\vee$	$\Box$	$\neg$	$\rightarrow$	$-$	$\vee$	$\Diamond$	$\neg$

Table 1: Outer and Inner nodes.

We use $+p \prec +\varphi$ (resp. $-p \prec +\varphi$) to indicate that an occurrence of a propositional variable p inherits the positive (resp. negative) sign from the positive generation tree $+\varphi$, and say that p is *positive* (resp. *negative*) in φ if $+p \prec +\varphi$ (resp. $-p \prec +\varphi$) for all occurrences of p in φ.

Definition 5. (cf. [5, Definition 5]) Nodes in signed generation trees are called *outer nodes* and *inner nodes*, according to Table 1. Here $\Box$ stands for $\Box_i^H$ or $\Box_i^B$, $\Diamond$ stands for $\Diamond_i^H$ or $\Diamond_i^B$.

A branch in a signed generation tree is *excellent* if it is the concatenation of two paths P_1 and P_2, one of which might be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) of inner nodes only, and P_2 consists (apart from variable nodes) of outer nodes only.

Definition 6 (Sahlqvist inequalities)**.** (cf. [5, Definition 6]) For any order-type ε, the signed generation tree $*\varphi$ (where $* \in \{+, -\}$) of a formula $\varphi(p_1, \ldots p_n)$ is ε-*Sahlqvist* if

- for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is excellent;
- for every branch (notice that here it might not be ε-critical) with occurrences of $+\Diamond^H$ or $-\Box^H$, every node from the root to this occurrence of $+\Diamond^H$ or $-\Box^H$ in the signed generation tree is an outer node.

An inequality $\varphi \leq \psi$ is ε-*Sahlqvist* if the signed generation trees $+\varphi$ and $-\psi$ are ε-Sahlqvist. An inequality $\varphi \leq \psi$ is *Sahlqvist* if it is ε-Sahlqvist for some ε. A formula

$\varphi \to \psi$ is Sahlqvist if the inequality $\varphi \leq \psi$ is a Sahlqvist inequality.

Example 2. An example of Sahlqvist formula in our language is $\Diamond^H \Box^B p \to \Box^B \Diamond^H p$, which is similar to the Geach formula in ordinary modal logic. Notice that here we have position restrictions on the first occurrence of $\Diamond^H$.

The classification of outer nodes and inner nodes is based on how different connectives behave in the algorithm. When the input inequality is a Sahlqvist inequality, the algorithm first decompose the outer part of the formula, and then decompose the inner part of the formula, which will be shown in the success proof of the algorithm in Section 7.

The difference between the present setting and ordinary modal logic is that we have additional requirement of the positions of helper modalities, which will be clear from the execution of the algorithm.

4 The Expanded Modal Language, First-Order Correspondence Language and Standard Translation

4.1 The Expanded Modal Language

In the present subsection, we define the expanded modal language , which will be used in the execution of the algorithm:

$$\varphi ::= p \mid \bot \mid \top \mid \mathbf{i} \mid \neg\varphi \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi \to \varphi \mid \Box^H_i\varphi \mid \Diamond^H_i\varphi \mid \Box^B_j\varphi \mid \Diamond^B_j\varphi \mid \Box^{\mathrm{S}}\varphi \mid$$

$$\Diamond^{\mathrm{S}}\varphi \mid \blacksquare^H_i\varphi \mid \blacklozenge^H_i\varphi \mid \blacksquare^B_j\varphi \mid \blacklozenge^B_j\varphi \mid \blacksquare^{\mathrm{S}}\varphi \mid \blacklozenge^{\mathrm{S}}\varphi$$

where $\mathbf{i} \in \mathrm{Nom}$ are *nominals* as in hybrid logic which are interpreted as singleton sets, $\Diamond^H_i \in \tau_H$, $\Diamond^B_j \in \tau_B$, $\mathrm{S} = \{(\mathbf{i}_1, \mathbf{j}_1), \ldots, (\mathbf{i}_k, \mathbf{j}_k)\}$ for some pairs $(\mathbf{i}_1, \mathbf{j}_1), \ldots, (\mathbf{i}_k, \mathbf{j}_k)$.

The reason for introducing the nominals and S-modalities is to compute the minimal valuations for propositional variables and for the H-modalities (which are essentially quantified by second-order quantifiers in the validity definition), therefore we can eliminate them to get a quasi-inequality which is essentially quantified by first-order quantifiers.

$\Box^{\mathrm{S}}$ and $\Diamond^{\mathrm{S}}$ are interpreted on the relation $S := \{(V(\mathbf{i}_1), V(\mathbf{j}_1)), \ldots, (V(\mathbf{i}_k), V(\mathbf{j}_k))\}$. For $\blacksquare$ and $\blacklozenge$, they are interpreted as the box and diamond modality on the inverse relation $H_i^{-1}, R_j^{-1}, S^{-1}$, according to the superscipt and subscript, respectively. The S-modalities are interpreted as the computation result of the minimal relations for the helper modalities, which is similar to the minimal valuations of propositional variables in the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$.

For the semantics of the expanded modal language, the valuation is defined as $V : \mathrm{Prop} \cup \mathrm{Nom} \to \mathcal{P}(W)$ where $V(\mathbf{i})$ is defined as a singleton as in hybrid logic, and the additional semantic clauses can be given as follows:

$\mathbb{M}, w \Vdash \mathbf{i}$	iff	$w \in V(\mathbf{i})$
$\mathbb{M}, w \Vdash \Box^{\mathrm{S}}\varphi$	iff	for all v, $(w, v) \in S \Rightarrow \mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \Diamond^{\mathrm{S}}\varphi$	iff	there exists a v s.t. $(w, v) \in S$ and $\mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacksquare^{H}_{i}\varphi$	iff	for all v, $(v, w) \in H_i \Rightarrow \mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacklozenge^{H}_{i}\varphi$	iff	there exists a v s.t. $(v, w) \in H_i$ and $\mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacksquare^{B}_{j}\varphi$	iff	for all v, $(v, w) \in R_j \Rightarrow \mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacklozenge^{B}_{j}\varphi$	iff	there exists a v s.t. $(v, w) \in R_j$ and $\mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacksquare^{\mathrm{S}}\varphi$	iff	for all v, $(v, w) \in S \Rightarrow \mathbb{M}, v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacklozenge^{\mathrm{S}}\varphi$	iff	there exists a v s.t. $(v, w) \in S$ and $\mathbb{M}, v \Vdash \varphi$

4.2 The first-order correspondence language and the standard translation

In the first-order correspondence language, we have a binary predicate symbol H_i corresponding to the binary relation H_i, a binary predicate symbol R_j corresponding to the binary relation R_j, a set of constant symbols i corresponding to each nominal **i**, a set of unary predicate symbols P corresponding to each propositional variable p. Notice that we do not have binary predicate symbols for the S relations.

Definition 7. For the standard translation of the expanded modal language, the basic propositional cases and the Boolean cases as well as the modal cases for boss modalities are defined as usual and hence omitted, the other cases are defined as follows:

- $ST_x(\mathbf{i}) := x = i$;
- $ST_x(\Box^{H}_{i}\varphi) := \forall y(H_i xy \to ST_y(\varphi))$;
- $ST_x(\Diamond^{H}_{i}\varphi) := \exists y(H_i xy \wedge ST_y(\varphi))$;
- $ST_x(\Diamond^{\mathrm{S}}\varphi) := \bigvee_{l=1}^{k}(x = i_l \wedge ST_{j_l}(\varphi))$;
- $ST_x(\Box^{\mathrm{S}}\varphi) := \neg ST_x(\Diamond^{\mathrm{S}}\neg\varphi)$;
- $ST_x(\blacksquare^{H}_{i}\varphi) := \forall y(H_i yx \to ST_y(\varphi))$;
- $ST_x(\blacklozenge^{H}_{i}\varphi) := \exists y(H_i yx \wedge ST_y(\varphi))$;
- $ST_x(\blacklozenge^{\mathrm{S}}\varphi) := \bigvee_{l=1}^{k}(x = j_l \wedge ST_{i_l}(\varphi))$;
- $ST_x(\blacksquare^{\mathrm{S}}\varphi) := \neg ST_x(\blacklozenge^{\mathrm{S}}\neg\varphi)$.

It is easy to see that this translation is correct:

Proposition 1 (Folklore.). *For any Kripke model $\mathbb{M}$, any $w \in W$ and any expanded modal formula φ,*

$$\mathbb{M}, w \Vdash \varphi \textit{ iff } \mathbb{M} \vDash ST_x(\varphi)[x := w].$$

For inequalities, quasi-inequalities, the standard translation is given in a global way:

Definition 8.

- $ST(\varphi \leq \psi) := \forall x(ST_x(\varphi) \to ST_x(\psi))$;
- $ST(\varphi_1 \leq \psi_1 \& \ldots \& \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi) := ST(\varphi_1 \leq \psi_1) \wedge \ldots \wedge ST(\varphi_n \leq \psi_n) \to ST(\varphi \leq \psi)$.

Proposition 2 (Folklore.). *For any Kripke model* $\mathbb{M}$*, any inequality* Ineq*, any quasi-inequality* Quasi*,*

$$\mathbb{M} \Vdash \mathrm{Ineq} \textit{ iff } \mathbb{M} \vDash ST(\mathrm{Ineq});$$

$$\mathbb{M} \Vdash \mathrm{Quasi} \textit{ iff } \mathbb{M} \vDash ST(\mathrm{Quasi}).$$

5 The Algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$

In this section, we define the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ which computes the first-order correspondents of input Sahlqvist formulas, in the style of [3, 4]. The algorithm receives an input formula $\varphi \to \psi$ and transforms it into an inequality $\varphi \leq \psi$. Then the algorithm goes in three steps.

1. **Preprocessing and first approximation**:
   In the generation tree of $+\varphi$ and $-\psi$[3],
   (a) Apply the distribution rules:
      i. Push down $+\Diamond_i^H, +\Diamond_j^B, -\neg, +\wedge, - \to$ by distributing them over nodes labelled with $+\vee$ which are outer nodes, and
      ii. Push down $-\Box_i^H, -\Box_j^B, +\neg, -\vee, - \to$ by distributing them over nodes labelled with $-\wedge$ which are outer nodes.
   (b) Apply the splitting rules: rewrite $\alpha \leq \beta \wedge \gamma$ as $\alpha \leq \beta$ and $\alpha \leq \gamma$; rewrite $\alpha \vee \beta \leq \gamma$ as $\alpha \leq \gamma$ and $\beta \leq \gamma$;
   (c) Apply the monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

   for $\beta(p)$ positive in p and $\alpha(p)$ negative in p.

   We denote by $\mathrm{Preprocess}(\varphi \to \psi)$ the finite set $\{\varphi_i \leq \psi_i\}_{i \in I}$ of inequalities obtained after the exhaustive application of the previous rules. Then we apply the following first approximation rule to every inequality in $\mathrm{Preprocess}(\varphi \to \psi)$:

$$\frac{\varphi_i \leq \psi_i}{\mathbf{i}_0 \leq \varphi_i \quad \psi_i \leq \neg \mathbf{i}_1}$$

   Here, $\mathbf{i}_0$ and $\mathbf{i}_1$ are special fresh nominals. Now we get a set of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}_{i \in I}$.
2. **The reduction stage**:
   In this stage, for each $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}$, we apply the following rules to prepare for eliminating all the propositional variables and helper modalities:

[3] The discussion below relies on the definition of signed generation tree in Section 3. In what follows, we identify a formula with its signed generation tree.

(a) Splitting rules (similar to the splitting rules in Stage 1);

(b) Approximation rules:

$$\frac{\mathbf{i} \leq \Diamond\alpha}{\mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond\mathbf{j}} \qquad \frac{\Box\alpha \leq \neg\mathbf{i}}{\alpha \leq \neg\mathbf{j} \quad \Box\neg\mathbf{j} \leq \neg\mathbf{i}}$$

$$\frac{\alpha \to \beta \leq \neg\mathbf{i}}{\mathbf{j} \leq \alpha \qquad \beta \leq \neg\mathbf{k} \qquad \mathbf{j} \to \neg\mathbf{k} \leq \neg\mathbf{i}}$$

The nominals introduced by the approximation rules must not occur in the system before applying the rule, and $\Diamond$ stands for $\Diamond^H_i$, $\Diamond^B_j$ or $\Diamond^{\mathrm{S}}$, $\Box$ stands for $\Box^H_i$, $\Box^B_j$ or $\Box^{\mathrm{S}}$.

(c) Residuation rules:

$$\frac{\alpha \leq \neg\beta}{\beta \leq \neg\alpha} \qquad \frac{\neg\alpha \leq \beta}{\neg\beta \leq \alpha} \qquad \frac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta} \qquad \frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta}$$

Here $\Diamond$ stands for $\Diamond^H_i$, $\Diamond^B_j$ or $\Diamond^{\mathrm{S}}$, $\Box$ stands for $\Box^H_i$, $\Box^B_j$ or $\Box^{\mathrm{S}}$, $\blacklozenge$ stands for $\blacklozenge^H_i$, $\blacklozenge^B_j$ or $\blacklozenge^{\mathrm{S}}$ according to the superscript and subscript of the corresponding $\Box$, and $\blacksquare$ stands for $\blacksquare^H_i$, $\blacksquare^B_j$ or $\blacksquare^{\mathrm{S}}$ according to the superscript and subscript of the corresponding $\Diamond$.

(d) Ackermann rules:

By the Ackermann rules, we compute the minimal/maximal valuation for propositional variables and minimal valuation for helper modalities and use the Ackermann rules to eliminate all the propositional variables and helper modalities. These three rules are the core of $\mathrm{ALBA}^{\mathrm{RQ}}$, since their application eliminates propositional variables and helper modalities. In fact, all the preceding steps are aimed at reaching a shape in which the Ackermann rules can be applied. Notice that an important feature of these rules is that they are executed on the whole set of inequalities, and not on a single inequality.

The right-handed Ackermann rule for propositional variables:

The system $\left\{\begin{array}{l} \alpha_1 \leq p, \ldots, \alpha_n \leq p \\ \beta_1 \leq \gamma_1, \ldots, \beta_m \leq \gamma_m \end{array}\right.$ is replaced by

$\left\{\begin{array}{l} \beta_1((\alpha_1 \vee \ldots \vee \alpha_n)/p) \leq \gamma_1((\alpha_1 \vee \ldots \vee \alpha_n)/p), \ldots, \\ \beta_m((\alpha_1 \vee \ldots \vee \alpha_n)/p) \leq \gamma_m((\alpha_1 \vee \ldots \vee \alpha_n)/p) \end{array}\right.$

where:

i. Each β_i is positive in p, and each γ_i negative in p, for $1 \leq i \leq m$;

ii. Each α_i is pure.

The left-handed Ackermann rule for propositional variables:

The system $\left\{\begin{array}{l} p \leq \alpha_1, \ldots, p \leq \alpha_n \\ \beta_1 \leq \gamma_1, \ldots, \beta_m \leq \gamma_m \end{array}\right.$ is replaced by

$$\begin{cases} \beta_1((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \leq \gamma_1((\alpha_1 \wedge \ldots \wedge \alpha_n)/p), \ldots, \\ \beta_m((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \leq \gamma_m((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \end{cases}$$

where:

i. Each β_i is negative in p, and each γ_i positive in p, for $1 \leq i \leq m$;
ii. Each α_i is pure.

The right-handed Ackermann rule for helper modalities:

The system $\begin{cases} \mathbf{i}_1 \leq \Diamond_i^H \mathbf{j}_1, \ldots, \mathbf{i}_{k_1} \leq \Diamond_i^H \mathbf{j}_{k_1} \\ \Box_i^H \neg \mathbf{j}'_1 \leq \neg \mathbf{i}'_1, \ldots, \Box_i^H \neg \mathbf{j}'_{k_2} \leq \neg \mathbf{i}'_{k_2} \\ \beta_1 \leq \gamma_1, \ldots, \beta_m \leq \gamma_m \end{cases}$

is replaced by $\{\ \beta_1(\mathrm{S}/H_i) \leq \gamma_1(\mathrm{S}/H_i), \ldots, \beta_m(\mathrm{S}/H_i) \leq \gamma_m(\mathrm{S}/H_i)$

where:

i. $\mathrm{S} = \{(\mathbf{i}_1, \mathbf{j}_1), \ldots, (\mathbf{i}_{k_1}, \mathbf{j}_{k_1}), (\mathbf{i}'_1, \mathbf{j}'_1), \ldots, (\mathbf{i}'_{k_2}, \mathbf{j}'_{k_2})\}$;
ii. Each β_j is positive in $\Diamond_i^H$, $\blacklozenge_i^H$ and negative in $\Box_i^H$, $\blacksquare_i^H$, and each γ_j is negative in $\Diamond_i^H$, $\blacklozenge_i^H$ and positive in $\Box_i^H$, $\blacksquare_i^H$;
iii. (S/H_i) stands for uniformly replacing $\Box_i^H$, $\Diamond_i^H$, $\blacksquare_i^H$, $\blacklozenge_i^H$ by $\Box^{\mathrm{S}}$, $\Diamond^{\mathrm{S}}$, $\blacksquare^{\mathrm{S}}$, $\blacklozenge^{\mathrm{S}}$ respectively.

3. **Output**: If in the previous stage, for some $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}$, the algorithm gets stuck, i.e. some propositional variables or helper modalities cannot be eliminated by the application of the reduction rules, then the algorithm halts and output "failure". Otherwise, each initial tuple $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}$ of inequalities after the first approximation has been reduced to a set of pure inequalities $\mathrm{Reduce}(\varphi_i \leq \psi_i)$ without helper modalities, and then the output is a set of quasi-inequalities $\{\&\mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_1 : \varphi_i \leq \psi_i \in \mathrm{Preprocess}(\varphi \to \psi)\}$ without helper modalities, where & is the big meta-conjunction in quasi-inequalities. Then the algorithm use the standard translation to transform the quasi-inequalities into first-order formulas.

6 Soundness of $\mathrm{ALBA}^{\mathrm{RQ}}$

In the present section, we will prove the soundness of the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ with respect to Kripke frames. The basic proof structure is similar to [7].

Theorem 3 (Soundness). *If* $\mathrm{ALBA}^{\mathrm{RQ}}$ *runs successfully on* $\varphi \to \psi$ *and outputs* $\mathrm{FO}(\varphi \to \psi)$*, then for any* τ_B*-Kripke frame* $\mathbb{F} = (W, R_1, \ldots, R_n)$*,*

$$\mathbb{F} \Vdash \varphi \to \psi \textit{ iff } \mathbb{F} \models \mathrm{FO}(\varphi \to \psi).$$

Proof. The proof goes similarly to [4, Theorem 8.1]. Let $\varphi_i \leq \psi_i$, $1 \leq i \leq n$ denote the inequalities produced by preprocessing $\varphi \to \psi$ after Stage 1, and $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}$ denote the inequalities after the first-approximation rule, $\mathrm{Reduce}(\varphi_i \leq$

ψ_i) denote the set of pure inequalities after Stage 2, and $\mathrm{FO}(\varphi \to \psi)$ denote the standard translation of the quasi-inequalities into first-order formulas, then we have the following chain of equivalences:

$$\mathbb{F} \Vdash \varphi \to \psi \tag{1}$$
$$\mathbb{F} \Vdash \varphi_i \leq \psi_i, \text{ for all } 1 \leq i \leq n \tag{2}$$
$$\mathbb{F} \Vdash (\mathbf{i}_0 \leq \varphi_i \ \&\ \psi_i \leq \neg\mathbf{i}_1) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1 \text{ for all } 1 \leq i \leq n \tag{3}$$
$$\mathbb{F} \Vdash \mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1 \text{ for all } 1 \leq i \leq n \tag{4}$$
$$\mathbb{F} \Vdash \mathrm{FO}(\varphi \to \psi) \tag{5}$$

- The equivalence between (1) and (2) follows from Proposition 4;
- the equivalence between (2) and (3) follows from Proposition 5;
- the equivalence between (3) and (4) follows from Propositions 6, 7 and 8;
- the equivalence between (4) and (5) follows from Proposition 2.

□

In the remainder of this section, we prove the soundness of the rules in Stage 1, 2 and 3.

Proposition 4 (Soundness of the rules in Stage 1)**.** *For the distribution rules, the splitting rules and the monotone and antitone variable-elimination rules, they are sound in both directions in $\mathbb{F}$, i.e. the inequality before the rule is valid in $\mathbb{F}$ iff the inequality(-ies) after the rule is(are) valid in $\mathbb{F}$.*

Proof. The proof is the same as [7, Proposition 6.2]. □

Proposition 5. *(2) and (3) are equivalent, i.e. the first-approximation rule is sound in $\mathbb{F}$.*

Proof. The proof is the same as [7, Proposition 6.3]. □

The next step is to show the soundness of each rule of Stage 2. For each rule, before the application of this rule we have a set of inequalities S (which we call the *system*), after applying the rule we get a set of inequalities S', the soundness of Stage 2 is then the equivalence of the following two conditions:

- $\mathbb{F} \Vdash \mathop{\&} S \ \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1$;
- $\mathbb{F} \Vdash \mathop{\&} S' \ \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1$;

where $\mathop{\&} S$ denote the meta-conjunction of inequalities of S. It suffices to show the following property:

- For any τ_B-Kripke frame $\mathbb{F} = (W, R_1, \dots, R_n)$, any binary relations $H_1, \dots, H_m$, any valuation V on it, if $(\mathbb{F}, H_1, \dots, H_m, V) \Vdash S$, then there is a valuation V' and binary relations $H'_1, \dots, H'_m$ such that $V'(\mathbf{i}_0) = V(\mathbf{i}_0)$, $V'(\mathbf{i}_1) = V(\mathbf{i}_1)$ and $(\mathbb{F}, H'_1, \dots, H'_m, V') \Vdash S'$;
- For any τ_B-Kripke frame $\mathbb{F} = (W, R_1, \dots, R_n)$, any binary relations $H'_1, \dots, H'_m$, any valuation V' on it, if $(\mathbb{F}, H'_1, \dots, H'_m, V') \Vdash S'$, then there is a valuation V and binary relations $H_1, \dots, H_m$ such that $V(\mathbf{i}_0) = V'(\mathbf{i}_0)$, $V(\mathbf{i}_1) = V'(\mathbf{i}_1)$ and $(\mathbb{F}, H_1, \dots, H_m, V) \Vdash S$.

Proposition 6. *The splitting rules, the approximation rules for $\Diamond, \Box, \to$, the residuation rules for $\neg, \Diamond, \Box$ are sound in $\mathbb{F}$.*

Proof. The proof is similar to [7, Proposition 6.4 and 6.11]. $\Box$

Proposition 7. *The Ackermann rules for propositional variables are sound in $\mathbb{F}$.*

Proof. The proof is similar to [7, Proposition 6.17]. $\Box$

Proposition 8. *The right-handed Ackermann rule for helper modalities is sound in $\mathbb{F}$.*

This rule is the key rule of the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ since it eliminates helper modalities. The proof method is similar to the soundness proof of the right-handed Ackermann rule for propositional variables. Without loss of generality, we assume that $k_1 = k_2 = m = 1$. To prove Proposition 8, it suffices to prove the following right-handed Ackermann lemma for helpers:

Lemma 1. *Assume that β_1 is positive in $\Diamond^H_i, \blacklozenge^H_i$ and negative in $\Box^H_i, \blacksquare^H_i$, and γ_1 is negative in $\Diamond^H_i, \blacklozenge^H_i$ and positive in $\Box^H_i, \blacksquare^H_i$, then for any τ_B-Kripke frame $\mathbb{F} = (W, R_1, \dots, R_n)$, any binary relations $H_1, \dots, H_m$, any valuation V on it, the following are equivalent (where $\mathrm{S} = \{(\mathbf{i}_1, \mathbf{j}_1), (\mathbf{i}'_1, \mathbf{j}'_1)\}$):*

(1) $\mathbb{M} := (\mathbb{F}, H_1, \dots, H_m, V) \Vdash \beta_1(\mathrm{S}/H_i) \leq \gamma_1(\mathrm{S}/H_i)$;

(2) there is a binary relation H'_i such that $\mathbb{M}' := (\mathbb{F}, H_1, \dots, H_{i-1}, H'_i, H_{i+1}, \dots, H_m, V) \Vdash \mathbf{i}_1 \leq \Diamond^H_i \mathbf{j}_1, \Box^H_i \neg \mathbf{j}'_1 \leq \neg \mathbf{i}'_1, \beta_1 \leq \gamma_1$.

Proof. From (1) to (2), we can take $H'_i := \{(V(\mathbf{i}_1), V(\mathbf{j}_1)), (V(\mathbf{i}'_1), V(\mathbf{j}'_1))\}$, then since $S = \{(V(\mathbf{i}_1), V(\mathbf{j}_1)), (V(\mathbf{i}'_1), V(\mathbf{j}'_1))\}$, we have that $\mathbb{M}' \Vdash \beta_1 \leq \gamma_1$. It is easy to see that $\mathbb{M}' \Vdash \mathbf{i}_1 \leq \Diamond^H_i \mathbf{j}_1$ and $\mathbb{M}' \Vdash \Box^H_i \neg \mathbf{j}'_1 \leq \neg \mathbf{i}'_1$.

From (2) to (1), from $\mathbb{M}' \Vdash \mathbf{i}_1 \leq \Diamond^H_i \mathbf{j}_1$ and $\mathbb{M}' \Vdash \Box^H_i \neg \mathbf{j}'_1 \leq \neg \mathbf{i}'_1$, we have that $(V(\mathbf{i}_1), V(\mathbf{j}_1)), (V(\mathbf{i}'_1), V(\mathbf{j}'_1)) \in H'_i$, therefore $S \subseteq H'_i$, so by the monotonicity and antitonicity conditions of the helper modalities in β_1 and γ_1,

$$[\![\beta_1(\mathrm{S}/H_i)]\!]^{\mathbb{M}'} \subseteq [\![\beta_1]\!]^{\mathbb{M}'} \subseteq [\![\gamma_1]\!]^{\mathbb{M}'} \subseteq [\![\gamma_1(\mathrm{S}/H_i)]\!]^{\mathbb{M}'}.$$

Since helper modalities with subscript i do not occur in $\beta_1(\mathrm{S}/H_i)$ and $\gamma_1(\mathrm{S}/H_i)$, we have $\mathbb{M} \Vdash \beta_1(\mathrm{S}/H_i) \leq \gamma_1(\mathrm{S}/H_i)$. $\Box$

7 Success

In this section, we prove that $\mathrm{ALBA}^{\mathrm{RQ}}$ succeeds on all Sahlqvist formulas. The proof structure is similar to [7].

Theorem 9. $\mathrm{ALBA}^{\mathrm{RQ}}$ *succeeds on all Sahlqvist formulas.*

Definition 9 (Definite ε-Sahlqvist inequality, similar to Definition 7.2 in [7])**.** Given any order-type ε, $* \in \{-,+\}$, the signed generation tree $*\varphi$ of the term $\varphi(p_1,\ldots,p_n)$ is *definite ε-Sahlqvist* if there is no $+\vee$, $-\wedge$ occurring in the outer part on an ε-critical branch. An inequality $\varphi \leq \psi$ is definite ε-Sahlqvist if the trees $+\varphi$ and $-\psi$ are both definite ε-Sahlqvist.

Lemma 2. *Let* $\{\varphi_i \leq \psi_i\}_{i\in I} = \mathrm{Preprocess}(\varphi \to \psi)$ *obtained by exhaustive application of the rules in Stage 1 on an input ε-Sahlqvist formula $\varphi \to \psi$. Then each $\varphi_i \leq \psi_i$ is a definite ε-Sahlqvist inequality.*

Proof. Same as [7, Lemma 7.3]. □

Definition 10 (Inner ε-Sahlqvist signed generation tree, similar to Definition 7.4 in [7])**.** Given an order type ε, $* \in \{-,+\}$, the signed generation tree $*\varphi$ of the term $\varphi(p_1,\ldots,p_n)$ is *inner ε-Sahlqvist* if its outer part P_2 on an ε-critical branch is always empty, i.e. its ε-critical branches have inner nodes only.

Lemma 3. *Given inequalities $\mathbf{i}_0 \leq \varphi_i$ and $\psi_i \leq \neg\mathbf{i}_1$ obtained from Stage 1 where $+\varphi_i$ and $-\psi_i$ are definite ε-Sahlqvist, by applying the rules in Substage 1 of Stage 2 exhaustively, the inequalities that we get are in one of the following forms:*

1. *pure inequalities which does not have occurrences of propositional variables;*
2. *inequalities of the form $\mathbf{i} \leq \alpha$ where $+\alpha$ is inner ε-Sahlqvist;*
3. *inequalities of the form $\beta \leq \neg\mathbf{i}$ where $-\beta$ is inner ε-Sahlqvist.*

Proof. Similar to [7, Lemma 7.5]. For the sake of the proof of the next lemma we repeat the proof here. Indeed, the rules in the Substage 1 of Stage 2 deal with outer nodes in the signed generation trees $+\varphi_i$ and $-\psi_i$ except $+\vee$,$-\wedge$. For each rule, without loss of generality assume we start with an inequality of the form $\mathbf{i} \leq \alpha$, then by applying the approximation rules, splitting rules and the residuation rules for negation in Stage 2, the inequalities we get are either a pure inequality without propositional variables, or an inequality where the left-hand side (resp. right-hand side) is $\mathbf{i}$ (resp. $\neg\mathbf{i}$), and the other side is a formula α' which is a subformula of α, such that α' has one root connective less than α. Indeed, if α' is on the left-hand side (resp. right-hand side) then $-\alpha'$ ($+\alpha'$) is definite ε-Sahlqvist.

By applying the rules in the Substage 1 of Stage 2 exhaustively, we can eliminate all the outer connectives in the critical branches, so for non-pure inequalities, they become of form 2 or form 3. □

The next two lemmas are crucial to the success of the whole algorithm, which also justify the definition of Sahlqvist formulas and inequalities:

Lemma 4. *In Lemma 3, all the occurrences of* $+\Diamond^H$*'s and* $-\Box^H$*'s are in the form of* $\mathbf{i} \leq \Diamond^H\mathbf{j}$ *and* $\Box^H\neg\mathbf{j} \leq \neg\mathbf{i}$*, and in form 2 and 3,* $+\alpha$ *and* $-\beta$ *only contain positive occurrences of* $\Box^H$*'s and negative occurrences of* $\Diamond^H$*'s.*

Proof. As we can see from the proof of Lemma 3 and the second item of Definition 6 for Sahlqvist inequalities, during the decomposition of the outer part of the Sahlqvist signed generation trees, all occurrences of $+\Diamond^H$'s and $-\Box^H$'s are in the outer part of the signed generation tree, hence are treated by the approximation rules. Before the application of the approximation rules, the inequalities are of the form $\mathbf{i} \leq \Diamond^H\alpha$ or of the form $\Box^H\alpha \leq \neg\mathbf{i}$. By applying the approximation rules, they are in the form of $\mathbf{i} \leq \Diamond^H\mathbf{j}$ and $\Box^H\neg\mathbf{j} \leq \neg\mathbf{i}$. For the rest of occurrences of $\Diamond^H$'s and $\Box^H$'s, they could only be in form 2 and 3, and $\Diamond^H$'s occur only negatively and $\Box^H$'s occur only positively. ☐

Lemma 5. *Assume we have inequalities of the form as described in Lemma 3 and 4, the right-handed Ackermann rule for helper modalities is applicable and therefore all helper modalities can be eliminated.*

Proof. It is easy to check that the shape of the system exactly satisfies the requirement of the application of the right-handed Ackermann rule for helper modalities. In addition, since in the result of the rule, some inequalities are deleted and the other inequalities have helper modalities replaced by the same kind of modalities (e.g. diamond by diamond, box by box, white connectives by white connectives, black connectives by black connectives), we still have pure inequalities and inequalities of the form 2 and 3 as described in Lemma 3, but now without helper modalities. ☐

Lemma 6. *Assume we have an inequality* $\mathbf{i} \leq \alpha$ *or* $\beta \leq \neg\mathbf{i}$ *where* $+\alpha$ *and* $-\beta$ *are inner* ε*-Sahlqvist, by applying the splitting rules and the residuation rules in Stage 2, we have inequalities of the following form:*

1. $\alpha \leq p$*, where* $\varepsilon(p) = 1$*,* α *is pure;*
2. $p \leq \beta$*, where* $\varepsilon(p) = \partial$*,* β *is pure;*
3. $\alpha \leq \gamma$*, where* α *is pure and* $+\gamma$ *is* ε^∂*-uniform;*
4. $\gamma \leq \beta$*, where* β *is pure and* $-\gamma$ *is* ε^∂*-uniform.*

Proof. The proof is similar to [7, Lemma 7.6]. Notice that for each input inequality, it is of the form $\mathbf{i} \leq \alpha$ or $\beta \leq \neg\mathbf{i}$, where $+\alpha$ and $-\beta$ are inner ε-Sahlqvist. By applying the splitting rules and the residuation rules, it is easy to check that the inequality will have one side pure, and the other side still inner ε-Sahlqvist. By applying these rules exhaustively, one will either have p as the non-pure side (with this p on a critical branch), or have an inner ε-Sahlqvist signed generation tree with no critical branch, i.e., ε^∂-uniform. □

Lemma 7. *Assume we have inequalities of the form as described in Lemma 6, the Ackermann rules for propositional variables are applicable and therefore all propositional variables can be eliminated.*

Proof. Immediate observation from the requirements of the Ackermann rules. □

Proof of Theorem 9 Assume we have an Sahlqvist formula $\varphi \to \psi$ as input. By Lemma 2, we get a set of definite ε-Sahlqvist inequalities. Then by Lemma 3, we get inequalities as described in Lemma 3 and 4. By Lemma 5, all helper modalities are eliminated. By Lemma 6, we get the inequalities as described. Finally by Lemma 7, the inequalities are in the right shape to apply the Ackermann rules for propositional variables, and thus we can eliminate all the propositional variables and the algorithm succeeds on the input. □

8 Examples

In this section we show how to run the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ on some examples that we give in the introduction. By the Goldblatt-Thomason theorem [1, Theorem 3.19], a first-order definable class of Kripke frames is modally definable iff it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions. Since $|W| \leq 1$ and $R = W \times W$ are not closed under taking disjoint unions, they are not definable by ordinary modal formulas, so our results go beyond Sahlqvist theorem in ordinary modal logic.

Example 3. We have input formula $\Diamond^H p \to \Box^H p$. To make the validity quantification pattern clear, we add quantifiers for the propositional variables, nominals and helper modalities:

$$\forall \Diamond^H \forall p(\Diamond^H p \to \Box^H p)$$

First we transform the input formula into inequality:

$$\forall \Diamond^H \forall p(\Diamond^H p \leq \Box^H p)$$

Stage 1: By first approximation, we have:

$$\forall \Diamond^H \forall p \forall \mathbf{i} \forall \mathbf{j}(\mathbf{i} \leq \Diamond^H p \;\&\; \Box^H p \leq \neg \mathbf{j} \;\Rightarrow\; \mathbf{i} \leq \neg \mathbf{j})$$

Stage 2: By the approximation rule for $\Diamond^H$, we have:

$$\forall \Diamond^H \forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k}(\mathbf{i} \leq \Diamond^H \mathbf{k} \;\&\; \mathbf{k} \leq p \;\&\; \Box^H p \leq \neg \mathbf{j} \;\Rightarrow\; \mathbf{i} \leq \neg \mathbf{j})$$

By the approximation rule for $\Box^H$, we have:

$$\forall\Diamond^H\forall p\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}\forall\mathbf{k}'(\mathbf{i}\leq\Diamond^H\mathbf{k}\ \&\ \mathbf{k}\leq p\ \&\ \Box^H\neg\mathbf{k}'\leq\neg\mathbf{j}\ \&\ p\leq\neg\mathbf{k}'\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

By the right-handed Ackermann rule for $\Diamond^H$ and $\Box^H$, we have (notice that there is no receiving inequalities, so we just eliminate the inequalities $\mathbf{i}\leq\Diamond^H\mathbf{k}$ and $\Box^H\neg\mathbf{k}'\leq\neg\mathbf{j}$):

$$\forall p\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}\forall\mathbf{k}'(\mathbf{k}\leq p\ \&\ p\leq\neg\mathbf{k}'\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

By the right-handed Ackermann rule for p, we have:

$$\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}\forall\mathbf{k}'(\mathbf{k}\leq\neg\mathbf{k}'\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

Stage 3:
By standard translation, we have:

$$\forall i\forall j\forall k\forall k'(k\neq k'\rightarrow i\neq j)$$

By first-order logic, we have:

$$\exists k\exists k'(k\neq k')\rightarrow\forall i\forall j(i\neq j)$$

By first-order logic, we have:

$$\forall k\forall k'(k=k')\vee\forall i\forall j(i\neq j)$$

which is:

$$|W|=1\vee|W|=0$$

which is:

$$|W|\leq 1$$

Example 4. We have input formula $\Box^B p\rightarrow\Box^H p$. To make the validity quantification pattern clear, we add quantifiers for the propositional variables, nominals and helper modalities:

$$\forall\Diamond^H\forall p(\Box^B p\rightarrow\Box^H p)$$

First we transform the input formula into inequality:

$$\forall\Diamond^H\forall p(\Box^B p\leq\Box^H p)$$

Stage 1:
By first approximation, we have:

$$\forall\Diamond^H\forall p\forall\mathbf{i}\forall\mathbf{j}(\mathbf{i}\leq\Box^B p\ \&\ \Box^H p\leq\neg\mathbf{j}\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

Stage 2:
By the approximation rule for $\Box^H$, we have:

$$\forall\Diamond^H\forall p\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}(\mathbf{i}\leq\Box^B p\ \&\ \Box^H\neg\mathbf{k}\leq\neg\mathbf{j}\ \&\ p\leq\neg\mathbf{k}\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

By the right-handed Ackermann rule for $\Box^H$, we have (notice that there is no receiving inequalities, so we just eliminate the inequality $\Box^H\neg\mathbf{k}\leq\neg\mathbf{j}$):

$$\forall p\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}(\mathbf{i}\leq\Box^B p\ \&\ p\leq\neg\mathbf{k}\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

By the left-handed Ackermann rule for p, we have:

$$\forall\mathbf{i}\forall\mathbf{j}\forall\mathbf{k}(\mathbf{i}\leq\Box^B\neg\mathbf{k}\ \Rightarrow\ \mathbf{i}\leq\neg\mathbf{j})$$

The following are not really obtained by rules in $\mathrm{ALBA}^{\mathrm{RQ}}$, but they are soundly obtained:

$$\forall\mathbf{j}\forall\mathbf{k}(\Box^B\neg\mathbf{k}\leq\neg\mathbf{j})$$

$$\forall\mathbf{j}\forall\mathbf{k}(\mathbf{j}\leq\Diamond^B\mathbf{k})$$

Stage 3:
By standard translation we have:

$$\forall j\forall kRjk$$

which is:

$$R=W\times W$$

9 Conclusion

In the present paper, we develop the correspondence theory for modal logic with helpers and bosses, define the Sahlqvist formulas in this setting, give an algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$ which transforms input Sahlqvist formulas into their first-order correspondents.

There is one issue remains to be dealt with. In the algorithm $\mathrm{ALBA}^{\mathrm{RQ}}$, we have the right-handed Ackermann rule for the helper modalities. It seems plausible to also have the left-handed Ackermann rule for the helper modalities, which is more difficult since $+\Box^H$'s and $-\Diamond^H$'s do not occur in the outer part of the signed generation tree, they cannot be in the form of $\mathbf{i}\leq\Diamond^H\mathbf{j}$ or $\Box^H\neg\mathbf{j}\leq\neg\mathbf{i}$, which makes it more difficult to compute the corresponding minimal/maximal relation. Therefore we leave it to future work.

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带关系量化的模态逻辑的萨奎斯特对应理论

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摘　要

Lehtinen（2008）引入了新的关于模态公式的有效性概念，其中允许对所谓的“helper modalities”所对应的二元关系进行量化，并且其中的“boss modalities”类似于模态逻辑中的普通模态词，即被解释为克里普克（S. Kripke）框架中确定的二元关系。本文研究了这一有效性概念的对应理论。针对这一有效性定义了一类萨奎斯特（Sahlqvist）公式，其中每个公式都存在其对应的一阶框架，并给出了相应的 $\mathrm{ALBA}^{\mathrm{RQ}}$ 算法来计算该类公式的一阶对应。

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