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A Logic for Probabilities of Successive Events*

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Abstract. In the set language of probability theory, besides complement, intersection, and union, there is another important operation: product. The product of two basic events expresses that these events occur in succession. However, there is limited research about successive events in the literature on probability logic. In this paper, we propose a modal logic (called DML) to capture the reasoning about successive events in probability theory, and then we construct a probability logic (called $\mathrm{PL}_{\mathrm{DML}}$) based on DML. We compare DML with standard modal logic on Kripke semantics and show that DML is equivalent to the normal modal logic on deterministic models. We also give a deductive system of $\mathrm{PL}_{\mathrm{DML}}$ and show its completeness.

1 Introduction

In probability theory, an event is expressed by a set. In the literature on logic for probabilities, the basic events and their Boolean combinations are well-studied. For example, the negation of an event A means that A does not happen. The conjunction of two events A and B means both A and B happen. The disjunction of two events A and B means that either A or B happens. However, successive events cannot be expressed by Boolean operators. In set language, successive events can be expressed by the product of basic events.

Successive events are several events occurring in succession. It is worth pointing out that events occurring in succession might occur at the same time. Succession here is to indicate order not time. Consider the example of tossing a die, and we assume that all dies are fair. There are 6 possible results. The die might fall with 1, 2, 3, 4, 5, or 6 up. Let E be the event that the die falls with an even number up, that is,

$$E_e = \{2, 4, 6\}.$$

The probability of E_e is $\frac{1}{2}$. Let $E_{>3}$ be the event that the die falls with a number bigger than 3 up, that is,

$$E_{>3} = \{4, 5, 6\}.$$

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The probability of $E_{>3}$ is $\frac{1}{2}$. Now if two persons respectively toss a die at the same time, the result that the first die falls with an even number up and the second die falls with a number bigger than 3 up is successive events, which is represented by

$$\begin{aligned} E_e \times E_{>3} = \{&(2,4),(2,5),(2,6)\\ &(4,4),(4,5),(4,6)\\ &(6,4),(6,5),(6,6)\}. \end{aligned}$$

The probability of $E_e \times E_{>3}$ is the product of the probability of E_e and the probability of $E_{>3}$, namely $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

The research on logic for probabilities can date back to Fagin et al. ([3, 4]). They propose a probability logic where linear inequalities involving probabilities are allowed. For example, a typical formula "$w(\phi) - 2w(\psi) \geq 0$" (or equivalently $w(\phi) \geq 2w(\psi)$) means that "the probability of ϕ is at least twice the probability of ψ". The deductive system of the probability logic given by [3, 4] is an extension of both propositional logic and linear inequality logic with some probability axioms and rules. The deductive system is weakly complete.

Another way of formalizing probability is to interpret modal operators of modal logic as probability (see [5, 7, 8]). A modal formula $\mathrm{P}_r^{>}\phi$ means that the probability of ϕ is strictly greater than r. In the model, the probability function assigns each measurable set a number in a base, where a base is a finite subset of the set $[0,1]$ that satisfies some conditions. Due to the fact that the base is finite, this probability has the property of compactness, and it also has strong completeness.

Zhou ([16, 17]) proposes a probability logic which is an extension of propositional logic with probability operators. A formula $L_r\phi$ means that the probability of ϕ is no less than r, where r is a rational number between 0 and 1. The deductive system of this probability logic is an extension of propositional logic with some probability axioms and rules. One of these probability rules is a ω-rule, which has an infinite number of premises and one conclusion. Zhou shows the weak completeness and confirms a conjecture of Larry Moss that the infinitary rule can be replaced by a finitary rule.

Different from [16, 17], Ognjanović et al. (see [10, 11, 12, 13]) proposes a probability logic with not only ω-rule but also infinitary derivations. A derivation (or a proof) in Ognjanović's system is a well-founded tree in which some nodes might have an infinite number of successors (see [9]). Based on infinitary derivations, this probability logic is shown to have strong completeness.

None of the papers mentioned above considers the probability of successive events in their logical language. In this paper, we propose a modal logic for reasoning about successive events and construct a probability logic based on it. The main contributions are listed as follows:

- We consider a fragment of the language of linear temporal logic, in which successive events can be expressed.
- We propose a semantics that combines the semantics of linear temporal logic (see [15]) and the update semantics of public announcement logic (see [6, 14]).
- We compare this semantics with standard Kripke semantics.
- We construct a probability logic based on this logic and give a complete deductive system.

The paper is organized as follows. Section 2 introduces the modal logic DML to capture the reasoning about successive events. Section 3 gives the alternative Kripke semantics of DML. Section 4 proposes a probability logic based on DML and gives a deductive system $\mathbb{PL}_{\mathbb{DML}}$. Section 5 shows the weak completeness of $\mathbb{PL}_{\mathbb{DML}}$. Section 6 concludes with some remarks.

2 A Modal Logic for Sequential Events

In this section, we introduce the logic, called DML (Deterministic Modal Logic), to capture the reasoning about successive events in probability theory.

Let **P** be a set of propositional letters.

Definition 1 (Language of DML)**.** The language of DML, denoted by $\mathcal{L}_{\text{DML}}$, is defined by the following BNF (where $p \in \mathbf{P}$):

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \bigcirc\phi.$$

The auxiliary connectives $\bot, \rightarrow, \vee$ are defined as abbreviations as usual.

The formula $\bigcirc\phi$ means that the event ϕ will happen in the next step. What is more, the formula $\phi \wedge \bigcirc^n\psi$ where n is the modal depth of ϕ means that the events ϕ and ψ successively happen. We use $(\phi; \psi)$ to denote the formula $\phi \wedge \bigcirc\psi$.

The language $\mathcal{L}_{\text{DML}}$ is a fragment of linear temporal logic without the *until* modality U.

Definition 2 (Model of DML)**.** A DML-model (or simply a model) is a triple $\mathcal{M} = \langle S, \Omega, V\rangle$ where

- S is a non-empty set of states;
- $\Omega \subseteq S^*$ is a non-empty set of sequences over S that is prefix-free;
- $V : \mathbf{P} \rightarrow 2^S$ is a valuation that labels each propositional letter with a set of states.

For each $\rho \in \Omega$, $(\mathcal{M}, \rho)$ is called a pointed model.

Intuitively, each $s \in S$ stands for a basic event, and each $\rho \in \Omega$ stands for a sequence of successive events. Please note that Ω is a subset of S^*. This means that only some successive events are allowed, which is in line with practice. For example, in sampling without replacement, only some basic events (not all possible basic events) can occur in succession.

The length of the sequence ρ is denoted as $|\rho|$. The n-th character of the sequence ρ is denoted as $\rho[n]$. The suffix of ρ starting at the i-th character is denoted as ρ^i. For example, let $\rho = s_1s_2s_3s_4$. We then have that $|\rho| = 4$, $\rho[1] = s_1$, and $\rho^3 = s_3s_4$.

Given a model $\mathcal{M} = \langle S, \Omega, V\rangle$, we use $\mathcal{M}^{-n}$ to denote the model $\langle S, \Omega^{-n}, V\rangle$ where $\Omega^{-n} = \{\rho^{n+1} \mid \rho \in \Omega\}$. It is obvious that $(\mathcal{M}^{-n})^{-m} = \mathcal{M}^{-(n+m)}$. What is more, we use $[\rho]^{\mathcal{M}}$ (or simply $[\rho]$) to denote the set $\{\sigma \in \Omega \mid \rho \text{ is a prefix of } \sigma\}$.

The intuition of the updated model $\mathcal{M}^{-n}$ is that after moving forward n steps, we will only consider the sequence of successive events that could be generated from this moment. In spirit, it is similar to the update model in public announcement logic.

Definition 3 (Semantics of DML)**.**

$$
\begin{array}{rcl}
\mathcal{M}, \rho \vDash \top & & \text{always} \\
\mathcal{M}, \rho \vDash p & \iff & \rho[1] \in V(p) \\
\mathcal{M}, \rho \vDash \neg\phi & \iff & \mathcal{M}, \rho \nvDash \phi \\
\mathcal{M}, \rho \vDash \phi \wedge \psi & \iff & \mathcal{M}, \rho \vDash \phi \text{ and } \mathcal{M}, \rho \vDash \psi \\
\mathcal{M}, \rho \vDash \bigcirc\phi & \iff & |\rho| > 1 \text{ and } \mathcal{M}^{-1}, \rho^2 \vDash \phi
\end{array}
$$

We use $[\![\phi]\!]^{\mathcal{M}}$ (or simply $[\![\phi]\!]$) to denote the set $\{\rho \mid \mathcal{M}, \rho \vDash \phi\}$.

This semantics is different from the semantics of linear temporal logic. The key feature of this semantics is that a formula captures to a set of sequences of successive events. This makes DML to be a natural generalization of propositional logic for basic events, where a propositional formula corresponds to a set of basic events. This feature can be illustrated more clearly by the following examples.

Example 1 (Sampling with replacement)**.** Imagine there is an opaque box, containing 4 red balls (R) and 1 black ball (B). You draw one ball from the box per time with a replacement. Now consider the case you draw from the box twice, which can be depicted by Figure 1.

Let the propositional letter p_R denote "draw a red ball", and let p_B denote "draw a black ball". So we can construct the model $\mathcal{M} = \langle S, \Omega, V\rangle$ as follows:

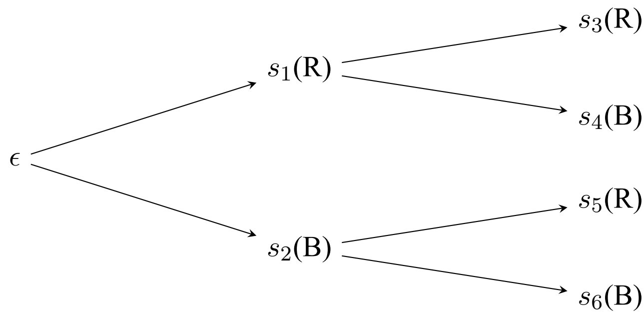
- $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$,
- $\Omega = \{s_1s_3, s_1s_4, s_2s_5, s_2s_6\}$,
- $V(p_R) = \{s_1, s_3, s_5\}$, and $V(p_B) = \{s_2, s_4, s_6\}$.

The formula p_R represents the event that draws a red ball at the first time. By the semantics of DML, we get that $[\![p_R]\!] = \{s_1s_3, s_1s_4\} = \{[s_1]\}$.

The formula $\bigcirc p_R$ represents the event that draws a red ball at the second time. By the semantics of DML, we get that $[\![\bigcirc p_R]\!] = \{s_1s_3, s_2s_5\}$.

The formula $p_R; p_B$ represents the successive events that firstly draw a red ball and secondly draw a black ball. By the semantics of DML, we get that $[\![p_R; p_B]\!] = \{s_1 s_3\}$.

Figure 1: Sampling with replacement



In the remainder of this section, we will consider two operations on models: generated submodel and disjoint union. These two operations will play important roles in the proof in Section 5.

Definition 4 (Generated submodel)**.** Given $\mathcal{M} = \langle S, \Omega, V \rangle$ and $\rho \in \Omega$, the model $\mathcal{M}|_\rho = \langle S', \Omega', V' \rangle$ is defined as follows:

- $S' = \{s \in S \mid s \text{ occurs in } \rho\}$;
- $\Omega' = \{\rho\}$;
- $s \in V'(p)$ if and only if $s \in V(p)$.

To show that the generated submodel preserves the truth of formulas, we will need the following proposition which can be easily checked.

Proposition 1. *Given two models $\mathcal{M}_1 = \langle S_1, \Omega_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle S_2, \Omega_2, V_2 \rangle$, we have that $\mathcal{M}_1, \rho \vDash \phi$ if and only if $\mathcal{M}_2, \rho \vDash \phi$ if the following conditions are satisfied:*

- $\rho \in \Omega_1 \cap \Omega_2$
- *$s \in V_1(p)$ if and only if $s \in V_2(p)$ for each s occurs in ρ and each p.*

Proposition 2. *For each formula $\phi \in \mathcal{L}_{\text{DML}}$, we have that $\mathcal{M}, \rho \vDash \phi$ if and only if $\mathcal{M}|_\rho, \rho \vDash \phi$.*

Proof. It can be proved by induction on ϕ. The base step and Boolean cases are straightforward. We will only consider the case that ϕ is of the form $\bigcirc\psi$.

Left-to-right: Assume that $\mathcal{M}, \rho \vDash \bigcirc\psi$. This means $|\rho| > 1$ and $\mathcal{M}^{-1}, \rho^2 \vDash \psi$. By inductive hypothesis, $(\mathcal{M}^{-1})|_{\rho^2}, \rho^2 \vDash \psi$. Then by proposition 1, we can get $(\mathcal{M}|_\rho)^{-1}, \rho^2 \vDash \psi$. By the semantics, we then have that $\mathcal{M}|_\rho, \rho \vDash \bigcirc\psi$.

Right-to-left: Assume $\mathcal{M}|_\rho, \rho \vDash \bigcirc\psi$. This means $|\rho| > 1$ and $(\mathcal{M}|_\rho)^{-1}$, $\rho^2 \vDash \psi$. Then by proposition 1, we can get $(\mathcal{M}^{-1})|_{\rho^2}, \rho^2 \vDash \psi$. By inductive hypothesis, $\mathcal{M}^{-1}, \rho^2 \vDash \psi$. By the semantics, we then have that $\mathcal{M}, \rho \vDash \bigcirc\psi$. □

Definition 5 (Disjoint union)**.** Let $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ be a finite set of models such that there is no common state between each two of them. The model $\biguplus_{1\le j\le n}\mathcal{M}_j = \langle S', \Omega', V'\rangle$ is defined as follows:

- $S' = \bigcup_{1\le j\le n} S_j$;
- $\Omega' = \bigcup_{1\le j\le n} \Omega_j$;
- $s \in V'(p)$ if and only if $s \in V_j(p)$ where s is a state of $\mathcal{M}_j$.

Proposition 3. *Given a finite set of models, $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$, for each formula $\phi \in \mathcal{L}_{\mathrm{DML}}$, we have that $\mathcal{M}_i, \rho \vDash \phi$ if and only if $\biguplus_{1\le j\le n}\mathcal{M}_j, \rho \vDash \phi$ where $1 \le i \le n$.*

Proof. It can be proved by induction on ϕ. We will only consider the case that ϕ is of the form $\bigcirc\psi$.

Left-to-right: Assume that $\mathcal{M}_i, \rho \vDash \bigcirc\psi$. This means $|\rho| > 1$ and $\mathcal{M}_i^{-1}, \rho^2 \vDash \psi$. By the inductive hypothesis, $(\biguplus_{1\le j\le n}\mathcal{M}_j)^{-1}, \rho^2 \vDash \psi$. Thus it follows by the semantics that $\biguplus_{1\le j\le n}\mathcal{M}_j, \rho \vDash \bigcirc\psi$.

Right-to-left: Assume that $\biguplus_{1\le j\le n}\mathcal{M}_j, \rho \vDash \bigcirc\psi$. By the semantics, it follows that $|\rho| > 1$ and $(\biguplus_{1\le j\le n}\mathcal{M}_j)^{-1}, \rho^2 \vDash \psi$. Please note that $(\biguplus_{1\le j\le n}\mathcal{M}_j)^{-1} = (\biguplus_{1\le j\le n}\mathcal{M}_j^{-1})$. By the inductive hypothesis, we have that $\mathcal{M}_i^{-1}, \rho^2 \vDash \psi$. Thus It follows by the semantics that $\mathcal{M}_i, \rho \vDash \bigcirc\psi$. □

3 Kripke Semantics of $\mathcal{L}_{\mathrm{DML}}$

In this section, we consider the standard Kripke semantics of the language $\mathcal{L}_{\mathrm{DML}}$, and show that DML is equivalent to the normal modal logic on the class of deterministic models.

Definition 6 (Kripke model)**.** A Kripke model for $\mathcal{L}_{\mathrm{DML}}$ is a triple $\mathcal{K} = \langle S, R, V\rangle$ where S and V are the same as Definition 2, and R is a deterministic binary relation on S, that is, if sRt and sRv then $t = v$. For each $s \in S$, $\langle \mathcal{K}, s\rangle$ is called a pointed Kripke model.

From the definition above, it can be seen that in this paper we assume that all Kripke models are deterministic. This is because we only consider deterministic Kripke models in this paper.

Definition 7 (Kripke semantics)**.** The Kripke semantics of $\mathcal{L}_{\mathrm{DML}}$, which in this paper is denoted as $\Vdash$, is standard (cf. [1]), where the modal formula $\bigcirc\phi$ is interpreted as an existential modal formula :

$$\mathcal{K}, s \Vdash \bigcirc\phi \Leftrightarrow \text{ there exists } t \text{ such that } sRt \text{ and } \mathcal{K}, t \Vdash \phi.$$

From the definition above, it can be seen that the interpretation of $\bigcirc\phi$ is the same as the existential modal formula (normally denoted as $\Diamond\phi$) in standard modal logic. So, in this paper, we will abbreviate the formula $\neg \bigcirc \neg\phi$ as $\Box\phi$.

Given a Kripke model $\mathcal{K} = \langle S, R, V\rangle$, a (possibly infinite) sequence of states $s_1 s_2 \cdots$ is called a $\mathcal{K}$-path if and only if $s_n R s_{n+1}$ for all $n \geq 1$. Especially, each $s \in S$ is a $\mathcal{K}$-path. A $\mathcal{K}$-path ρ is called a full $\mathcal{K}$-path if and only if either it is of infinite length or $\rho = s_1 \cdots s_n$ and there is no such $t \in S$ that $s_n Rt$.

The following proposition states that for each Kripke model, there are equivalent DML-models.

Proposition 4. *Given a Kripke model $\mathcal{K} = \langle S, R, V\rangle$, for each set Ω of full $\mathcal{K}$-paths, we have that $\langle S, \Omega, V\rangle, \rho \vDash \phi$ if and only if $\mathcal{K}, s \Vdash \phi$ where $\rho[1] = s$.*

Proof. We prove it by induction on ϕ. The basic step and Boolean cases are straightforward. If $\phi := \bigcirc\psi$, there are two cases: $|\rho| = 1$ or $|\rho| > 1$.

For the case of $|\rho| = 1$, by the DML-semantics, we always have that $\langle S, \Omega, V\rangle, \rho \nvDash \bigcirc\psi$. Meanwhile, since ρ is a full $\mathcal{K}$-path and $|\rho| = 1$, this follows that there is no such state t that sRt in $\mathcal{K}$. Thus, we also always have that $\mathcal{K}, s \nVdash \bigcirc\psi$.

For the case of $|\rho| > 1$, the proof is as follows:

If $\langle S, \Omega, V\rangle, \rho \vDash \bigcirc\psi$, it follows that $\langle S, \Omega^{-1}, V\rangle, \rho^2 \vDash \psi$. By the inductive hypothesis, $\mathcal{K}, s_2 \Vdash \psi$ where $s_2 = \rho[2]$. Due to $\rho[1] = s \in S$, we can get sRs_2. Then by Kripke semantics, we have that $\mathcal{K}, s \Vdash \bigcirc\psi$.

If $\mathcal{K}, s \Vdash \bigcirc\psi$, it follows that there exists a state s_2 such that sRs_2 and $\mathcal{K}, s_2 \Vdash \psi$. By the inductive hypothesis, $\langle S, \Omega', V\rangle, \rho' \vDash \psi$ where $\rho'[1] = s_2$. By Proposition 1, we then have that $\langle S, \{\rho'\}, V\rangle, \rho' \vDash \psi$. Since ρ' is a full $\mathcal{K}$-path starting from s_2 and sRs_2, it follows that $s\rho'$ is also a full $\mathcal{K}$-path. Then by the DML-semantics, we have that $\langle S, \{s\rho'\}, V\rangle, s\rho' \vDash \bigcirc\psi$. $\Box$

Next, we will show that for each DML-model, there is an equivalent Kripke model.

Definition 8 ($\mathcal{M}^\bullet$). Given a DML-model $\mathcal{M} = \langle S, \Omega, V\rangle$, a Kripke model $\mathcal{M}^\bullet$ is defined as $\langle S^\bullet, R^\Omega, V^\bullet\rangle$ where

- $S^\bullet = \{\sigma \mid \text{there exists } \rho \in \Omega \text{ such that } \sigma \text{ is a suffix of } \rho\}$. In other words, $S^\bullet$ is the suffix-closure of Ω.
- $\rho R^\bullet \sigma$ if and only if $\sigma = \rho^2$. It is obvious that $R^\bullet$ is deterministic.
- $\sigma \in V^\bullet(p)$ if and only if $\sigma[1] \in V(p)$.

Proposition 5. *Given a* DML*-model* $\mathcal{M}$*, we have that the Kripke model* $(\mathcal{M}^{-1})^\bullet$ *is a generated submodel of the Kripke model* $\mathcal{M}^\bullet$.

Proof. The proof is omitted due to space limitations. $\square$

Proposition 6. $\mathcal{M}, \rho \vDash \phi$ *if and only if* $\mathcal{M}^\bullet, \rho \Vdash \phi$.

Proof. It can be proved by induction on ϕ. We will only consider the case that ϕ is of the form $\bigcirc\psi$.

If $\mathcal{M}, \rho \vDash \bigcirc\psi$, it follows that $\mathcal{M}^{-1}, \rho^2 \vDash \psi$. By inductive hypothesis, we get $(\mathcal{M}^{-1})^\bullet, \rho^2 \vDash \psi$. By proposition 5, $(\mathcal{M}^{-1})^\bullet$ is generated submodel of $\mathcal{M}^\bullet$. So $\mathcal{M}^\bullet, \rho^2 \Vdash \psi$ (cf. [1]). Due to $\rho R^\bullet \rho^2$ in $\mathcal{M}^\bullet$, thus we can get $\mathcal{M}^\bullet, \rho \Vdash \bigcirc\psi$.

If $\mathcal{M}^\bullet, \rho \Vdash \bigcirc\psi$, it follows that there exists ρ' such that $\rho R^\bullet \rho'$ and $\mathcal{M}^\bullet, \rho' \Vdash \psi$. By the definition of $R^\bullet$, we know that $\rho' = \rho[2]$. Thus, $\mathcal{M}^\bullet, \rho[2] \Vdash \psi$. What is more, by the definition of $S^\bullet$, we know that either $\rho \in \Omega$ or ρ is a suffix of some $\sigma \in \Omega$. Either way, we have that $\rho[2]$ is an element of the domain of $(\mathcal{M}^{-1})^\bullet$. Since $(\mathcal{M}^{-1})^\bullet$ is a generated submodel of $(\mathcal{M}^\bullet)$, we then have that $(\mathcal{M}^{-1})^\bullet, \rho[2] \Vdash \psi$. By inductive hypothesis, we then have that $\mathcal{M}^{-1}, \rho[2] \vDash \psi$. It follows that $\mathcal{M}, \rho \vDash \bigcirc\psi$. $\square$

The following lemma states that the logical consequence of DML is equivalent to that of standard modal logic on deterministic modal class.

Lemma 1. *For each* $\phi \in \mathcal{L}_{\mathrm{DML}}$*, we have that* $\Gamma \vDash \phi$ *if and only if* $\Gamma \Vdash \phi$.

Proof. Suppose $\Gamma \vDash \phi$, but $\Gamma \nVdash \phi$. This follows that there is a pointed Kripke model $\mathcal{K}, s$ such that $\mathcal{K}, s \Vdash \Gamma$ but $\mathcal{K}, s \nVdash \phi$. By proposition 4, we can get a DML-model $\langle S, R, V\rangle, \rho$ such that $\langle S, R, V\rangle, \rho \vDash \Gamma$ and $\langle S, R, V\rangle, \rho \nvDash \phi$, which is contradictory with the premise that $\Gamma \vDash \phi$. Thus, we have shown that if $\Gamma \vDash \phi$ then $\Gamma \Vdash \phi$.

Suppose $\Gamma \Vdash \phi$, but $\Gamma \nVdash \phi$. This follows that there is a pointed DML-model $\langle S, R, V\rangle, \rho$ such that $\langle S, R, V\rangle \vDash \Gamma$ but $\langle S, R, V\rangle \nvDash \phi$. We then can construct a Kripke model $\mathcal{M}^\bullet$ by definition 8. It follows from Proposition 6 that $\mathcal{M}^\bullet \Vdash \Gamma$ and $\mathcal{M}^\bullet \nVdash \phi$, which contradicts the premise that $\Gamma \Vdash \phi$. Thus, we have shown that if $\Gamma \Vdash \phi$ then $\Gamma \vDash \phi$. $\square$

Definition 9 (Deductive system of DML)**.** The deductive system $\mathbb{DML}$ is presented in Table 1.

It can be seen that $\mathbb{DML}$ is an extension of the **K** system of normal modal logic with the axiom Det which characterizes the class of deterministic Kripke models.

Theorem 7. *The system* $\mathbb{DML}$ *is sound and strongly complete with respect to the semantics of* DML.

Proof. Since the system $\mathbb{DML}$ is sound and strongly complete with respect to the class of deterministic Kripke models (see [1]), by Lemma 1, it follows that $\mathbb{DML}$ is sound and strongly complete with respect to the semantics of DML. $\square$

Table 1: The system $\mathbb{DML}$

Axioms	
Taut	All instances of propositional tautologies
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
Det	$\bigcirc\phi \to \Box\phi$
Rules	
MP	From ϕ and $\phi \to \psi$, infer ψ.
N	From ϕ, infer $\Box\phi$.

4 A Prbabilistic Logic Based on DML

In this section, we construct a probability logic based on the logic DML and give a deductive system of this probability logic.

Definition 10 (Language of $\mathrm{PL}_{\mathrm{DML}}$)**.** The language of $\mathrm{PL}_{\mathrm{DML}}$, denoted as $\mathcal{L}_{\mathrm{PL}_{\mathrm{DML}}}$, is defined as follows (where $\psi_1, \cdots, \psi_n \in \mathcal{L}_{\mathrm{DML}}$ and $a_1, \cdots, a_n, a \in \mathbb{Q}$)

$$\phi ::= a_1\mathrm{P}\psi_1 + \cdots + a_n\mathrm{P}\psi_n \geq a \mid \neg\phi \mid (\phi \wedge \phi)$$

Formulas of the forms $a_1\mathrm{P}\psi_1 + \cdots + a_n\mathrm{P}\psi_n \bowtie a$ where $\bowtie \in \{>, <, \leq, =\}$ can be defined in $\mathcal{L}_{\mathrm{PL}_{\mathrm{DML}}}$ (see [4]).

Definition 11 (Probability distribution)**.** Let Ω be a finite set. A function $\mu : \Omega \to [0, 1]$ is called a *probability distribution* over Ω if and only if

$$\sum_{\rho \in \Omega} \mu(\rho) = 1.$$

Given a subset Θ of Ω, let $\mu(\Theta) = \sum_{\rho \in \Theta} \mu(\rho)$ if $\Theta \neq \emptyset$. If $\Theta = \emptyset$, let $\mu(\Theta) = 0$.

Definition 12 (Model of $\mathrm{PL}_{\mathrm{DML}}$)**.** A $\mathrm{PL}_{\mathrm{DML}}$-model is a pair $(\mathcal{M}, \mu)$ where

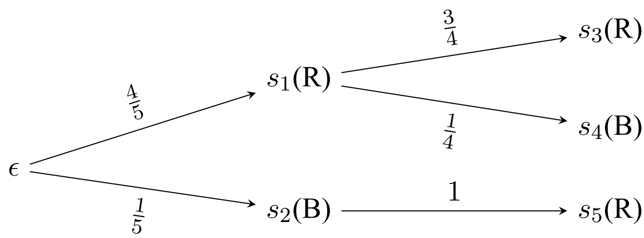
- $\mathcal{M} = \langle S, \Omega, V \rangle$ is a DML-model where Ω is finite;
- μ is a probability distribution over Ω.

Definition 13 (Semantics of $\mathrm{PL}_{\mathrm{DML}}$)**.** The satisfaction relation between a $\mathrm{PL}_{\mathrm{DML}}$-model $(\mathcal{M}, \mu)$ and a formula $\phi \in \mathcal{L}_{\mathrm{PL}_{\mathrm{DML}}}$, denoted as $\vDash$, is defined as follows:

$$\begin{aligned} \mathcal{M}, \mu \vDash \textstyle\sum_{i=1}^{n} a_i \psi_i \geq a \quad &\iff \quad \textstyle\sum_{i=1}^{n} a_i \mu([\![\psi_i]\!]^{\mathcal{M}}) \geq a \\ \mathcal{M}, \mu \vDash \neg\phi \quad &\iff \quad \mathcal{M}, \mu \nvDash \phi \\ \mathcal{M}, \mu \vDash \phi_1 \wedge \phi_2 \quad &\iff \quad \mathcal{M}, \mu \vDash \phi_1 \text{ and } \mathcal{M}, \mu \vDash \phi_2 \end{aligned}$$

Example 2. Imagine drawing a ball from the box in Example 1, but this time without replacement. Assume that these balls are exactly the same except for the color. So for your first draw, the probability of getting red is 0.8 (p_R), and 0.2 for black (p_B). For your second draw, the case will be: if you get a red ball on your first draw, the probability of getting a black ball increases to 0.75, since there are 3 red balls and 1 black ball left. If you get a black ball on your first draw, you will certainly draw a red ball in your next turn, because there is no black ball anymore. This sampling can be depicted by Figure 2.

Figure 2: Sampling without replacement



Let the DML-model $\mathcal{M} = \langle S, \Omega, V \rangle$ be defined as follows:

- $S = \{s_1, s_2, s_3, s_4, s_5\}$,
- $\Omega = \{s_1 s_3, s_1 s_4, s_2 s_5\}$,
- $V(p_R) = \{s_1, s_3, s_5\}$ and $V(p_B) = \{s_2, s_4\}$.

The probability distribution μ on Ω is defined as follows:

$$\begin{aligned} \mu(s_1 s_3) =& \frac{4}{5} \times \frac{3}{4} = \frac{3}{5} \\ \mu(s_1 s_4) =& \frac{4}{5} \times \frac{1}{4} = \frac{1}{5} \\ \mu(s_2 s_5) =& \frac{1}{5} \times 1 = \frac{1}{5} \end{aligned}$$

We then have the following:

- $\mathcal{M}, \mu \vDash \mathrm{P} p_R = \frac{4}{5}$, because of $\mu([\![p_R]\!]) = \mu(\{[s_1]\}) = \frac{4}{5}$. This means that the probability of the event that you draw a red ball at the first time is $\frac{4}{5}$.

- $\mathcal{M}, \mu \vDash \mathrm{P} \bigcirc p_R = \frac{4}{5}$, because of $\mu([\![\bigcirc p_R]\!]) = \mu(\{s_1s_3, s_2s_5\}) = \frac{4}{5}$. This means that the probability of the event that you draw a red ball at the second time is $\frac{4}{5}$.
- $\mathcal{M}, \mu \vDash \mathrm{P}(p_R; p_B) = \frac{1}{5}$, because of $\mu([\![p_R; p_B]\!]) = \mu(\{s_1s_4\}) = \frac{1}{5}$. This means that the probability of the successive events that you firstly draw a red ball and secondly draw a black ball is $\frac{1}{5}$.

Definition 14 (Deductive system $\mathbb{PL}_{\mathbb{DML}}$). The deductive system $\mathbb{PL}_{\mathbb{DML}}$ is presented in Table 2.

The deductive system $\mathbb{DML}$ is similar to the deductive system proposed in [4] except for the rule Dst. The premise of the rule Dst in this paper is a formula provable in the system $\mathbb{DML}$.

Table 2: The system $\mathbb{PL}_{\mathbb{DML}}$

Axioms	
Taut	All instances of propositional tautologies
LinIne	All instances of linear inequality axioms
NonNeg	$\mathrm{P}\psi \geq 0$
Cert	$\mathrm{P}\top = 1$
Add	$\mathrm{P}(\psi_1 \wedge \psi_2) + \mathrm{P}(\psi_1 \wedge \neg\psi_2) = \mathrm{P}(\psi_1)$
Rules	
MP	From ϕ and $\phi \to \psi$, infer ψ.
Dst	From $\vdash_{\mathbb{DML}} \psi_1 \leftrightarrow \psi_2$, infer $\mathrm{P}(\psi_1) = \mathrm{P}(\psi_2)$.

Proposition 8. *If $\phi_1 \in \mathcal{L}_{\mathrm{DML}}$ and $\phi_2 \in \mathcal{L}_{\mathrm{DML}}$ are $\mathbb{DML}$-inconsistent, we then have that* $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1) + \mathrm{P}(\phi_2) = \mathrm{P}(\phi_1 \vee \phi_2)$.

Proof.

1. $\vdash_{\mathbb{DML}} \phi_1 \wedge \phi_2 \to \bot$ (inconsistent)
2. $\vdash_{\mathbb{DML}} \bot \to \phi_1 \wedge \phi_2$ (Axiom Taut)
3. $\vdash_{\mathbb{DML}} \phi_1 \wedge \phi_2 \leftrightarrow \bot$ (1.2.Axiom Taut)
4. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1 \wedge \phi_2) = \mathrm{P}(\bot)$ (3.Rule Dst)
5. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\top \wedge \bot) + \mathrm{P}(\top \wedge \neg\bot) = \mathrm{P}(\top)$ (Axiom Add)
6. $\vdash_{\mathbb{DML}} \top \wedge \bot \leftrightarrow \bot$ (Axiom Taut)
7. $\vdash_{\mathbb{DML}} \top \wedge \neg\bot \leftrightarrow \top$ (Axiom Taut)
8. $\vdash_{\mathbb{DML}} \mathrm{P}(\top \wedge \bot) + \mathrm{P}(\top \wedge \neg\bot) = \mathrm{P}(\bot) + \mathrm{P}(\top)$ (6.7.Axiom LinIne)
9. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\top) = 1$ (Axiom Cert)
10. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\bot) = 0$ (5.8.9.Axiom LinIne)
11. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1 \wedge \phi_2) = 0$ (4.10.Axiom LinIne)
12. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1) = \mathrm{P}(\phi_1 \wedge \phi_2) + \mathrm{P}(\phi_1 \wedge \neg\phi_2)$ (Axiom Add)

13. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1) = \mathrm{P}(\phi_1 \wedge \neg\phi_2)$ (11.12.Axiom LinIne)
14. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1 \vee \phi_2) = \mathrm{P}((\phi_1 \vee \phi_2) \wedge \phi_2) + \mathrm{P}(\phi_1 \vee \phi_2) \wedge \neg\phi_2)$ (Axiom Add)
15. $\vdash_{\mathbb{DML}} (\phi_1 \vee \phi_2) \wedge \phi_2 \leftrightarrow \phi_2$ (Axiom Taut)
16. $\vdash_{\mathbb{DML}} (\phi_1 \vee \phi_2) \wedge \neg\phi_2 \leftrightarrow \phi_1 \wedge \neg\phi_2$ (Axiom Taut)
17. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}((\phi_1 \vee \phi_2) \wedge \phi_2) = \mathrm{P}(\phi_2)$ (15.Rule Dst)
18. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}((\phi_1 \vee \phi_2) \wedge \neg\phi_2) = \mathrm{P}(\phi_1 \wedge \neg\phi_2)$ (16.Rule Dst)
19. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1 \vee \phi_2) = \mathrm{P}(\phi_2) + \mathrm{P}(\phi_1 \wedge \neg\phi_2)$ (14.17.18.Axiom LinIne)
20. $\vdash_{\mathbb{PL}_{\mathbb{DML}}} \mathrm{P}(\phi_1 \vee \phi_2) = \mathrm{P}(\phi_1) + \mathrm{P}(\phi_2)$ (13.19.Axiom LinIne)

$\square$

Theorem 9 (Soundness). *The system $\mathbb{PL}_{\mathbb{DML}}$ is sound.*

Proof. The key is to show that each axiom is valid and that each rule preserves validity.

Firstly, we will show that each axiom is valid. We will only focus on the axioms of probabilities.

- The axiom NonNeg is valid, i.e. $\vDash \mathrm{P}\psi \geq 0$.
  Let $\mathcal{M} = \langle S, \Omega, V\rangle$ be an arbitrary model and μ be an arbitrary probability distribution over Ω. Since $[\![\psi]\!]^{\mathcal{M}}$ is a subset of Ω, it follows by Definition 11 that $\mu([\![\psi]\!]^{\mathcal{M}}) \geq 0$. Thus, we have that $\mathcal{M}, \mu \vDash \mathrm{P}\psi \geq 0$.
- The axiom Cert is valid, i.e. $\vDash \mathrm{P}\top = 1$.
  Due to $[\![\top]\!]^{\mathcal{M}} = \Omega$, it follows by Definition 11 that $\mu([\![\top]\!]^{\mathcal{M}}) = 1$. Thus, we have that $\mathcal{M}, \mu \vDash \mathrm{P}\top = 1$.
- The axiom Add is valid, i.e. $\vDash \mathrm{P}(\psi_1 \wedge \psi_2) + \mathrm{P}(\psi_1 \wedge \neg\psi_2) = \mathrm{P}\psi_1$.
  Please note that $[\![\phi_1 \wedge \phi_2]\!]^{\mathcal{M}} = [\![\phi_1]\!]^{\mathcal{M}} \cap [\![\phi_2]\!]^{\mathcal{M}}$, and $[\![\phi_1 \wedge \neg\phi_2]\!]^{\mathcal{M}} = [\![\phi_1]\!]^{\mathcal{M}} \cap [\![\neg\phi_2]\!]^{\mathcal{M}} = [\![\phi_1]\!]^{\mathcal{M}} \cap (\Omega \setminus [\![\phi_2]\!]^{\mathcal{M}}) = [\![\phi_1]\!]^{\mathcal{M}} \setminus ([\![\phi_1]\!]^{\mathcal{M}} \cap [\![\phi_2]\!]^{\mathcal{M}})$. This follows that $[\![\phi_1 \wedge \phi_2]\!]$ and $[\![\phi_1 \wedge \neg\phi_2]\!]$ are disjoint. By Definition 11, we have that $\mu([\![\phi_1 \wedge \phi_2]\!]^{\mathcal{M}} \cup [\![\phi_1 \wedge \neg\phi_2]\!]^{\mathcal{M}}) = \mu([\![\phi_1 \wedge \phi_2]\!]^{\mathcal{M}}) + \mu([\![\phi_1 \wedge \neg\phi_2]\!]^{\mathcal{M}})$. What is more, due to $[\![\phi_1 \wedge \phi_2]\!]^{\mathcal{M}} \cup [\![\phi_1 \wedge \neg\phi_2]\!]^{\mathcal{M}} = [\![\phi_1]\!]^{\mathcal{M}}$, it follows that $\mu([\![\phi_1]\!]^{\mathcal{M}}) = \mu([\![\phi_1 \wedge \phi_2]\!]^{\mathcal{M}}) + \mu([\![\phi_1 \wedge \neg\phi_2]\!]^{\mathcal{M}})$. Thus, we have that $\mathcal{M}, \mu \vDash \mathrm{P}(\phi_1 \wedge \phi_2) + \mathrm{P}(\phi_1 \wedge \neg\phi_2) = \mathrm{P}(\phi_1)$.

Next, we need to show each rule preserves validity. We will only focus on the rule Dst.

Suppose that $\vdash_{\mathbb{DML}} \psi_1 \leftrightarrow \psi_2$, by Theorem 7, it follows that the formula $\phi_1 \leftrightarrow \phi_2 \in \mathcal{L}_{\mathrm{PL}_{\mathrm{DML}}}$ is valid. We then have that $[\![\psi_1]\!]^{\mathcal{M}} = [\![\psi_2]\!]^{\mathcal{M}}$ for each DML-model $\mathcal{M}$. Thus, we have that $\mu([\![\phi_1]\!]^{\mathcal{M}}) = \mu([\![\phi_2]\!]^{\mathcal{M}})$. It follows that $\mathcal{M}, \mu \vDash \mathrm{P}(\phi_1) = \mathrm{P}(\phi_2)$.
$\square$

5 Completeness of $\mathbb{PL}_{\mathbb{DML}}$

Given a formula set Γ, we use $Sub(\Gamma)$ to denote the minimal extension of Γ such that $Sub(\Gamma)$ is subformula closed, and use $Sub^+(\phi)$ to denote the minimal extension of $Sub(\phi)$ such that if $\psi \in Sub(\phi)$ and ψ is not a negation formula then $\neg\psi \in Sub^+(\phi)$.

If Γ is finite, both $Sub(\Gamma)$ and $Sub^+(\Gamma)$ are finite.

Definition 15 ($\mathbb{DML}$/$\mathbb{PL}_{\mathbb{DML}}$-Atom)**.** Let Γ be a set of $\mathcal{L}_{\mathrm{DML}}$-formulas ($\mathcal{L}_{\mathrm{PL_{DML}}}$-formulas). A subset s of $Sub^+(\Gamma)$ is called a $\mathbb{DML}$-atom ($\mathbb{PL}_{\mathbb{DML}}$-atom) of $Sub^+(\Gamma)$ if and only if s is a maximal $\mathbb{DML}$-consistent ($\mathbb{PL}_{\mathbb{DML}}$-consistent) subset of $Sub^+(\Gamma)$.

We use $At_{\mathbb{DML}}(\Gamma)$ to denote the set of all $\mathbb{DML}$-atoms of $Sub^+(\Gamma)$, and we use $At_{\mathbb{PL}_{\mathbb{DML}}}(\Gamma)$ to denote the set of all $\mathbb{PL}_{\mathbb{DML}}$-atoms. We use ϕ_Θ to denote the conjunction of all formulas in Θ if Θ is finite.

Next, we will show that each $\mathbb{PL}_{\mathbb{DML}}$-atom is satisfiable. Before that, we need the following two auxiliary propositions. Due to space limitations, the proofs are omitted.

Proposition 10. *Let Γ be a finite set of $\mathcal{L}_{\mathrm{DML}}$-formulas. We have that the disjunction of all atoms is provable in $\mathbb{DML}$, namely $\vdash_{\mathbb{DML}} \bigvee_{\Theta \in At_{\mathbb{DML}}(\Gamma)} \psi_\Theta$.*

Proposition 11. *Let Γ be a finite set of $\mathcal{L}_{\mathrm{DML}}$-formulas and ψ be a formula in $Sub^+(\Gamma)$. We have that $\vdash_{\mathbb{DML}} \psi \leftrightarrow \bigvee_{\Theta \in At_{\mathbb{DML}}(\Gamma), \psi \in \Theta} \phi_\Theta$.*

Lemma 2. *Given a finite $\mathbb{PL}_{\mathbb{DML}}$-atom Θ, there is a $\mathrm{PL_{DML}}$-model $(\mathcal{M}, \mu)$ such that $\mathcal{M}, \mu \vDash \psi$ for each $\psi \in \Theta$.*

Proof. Let Γ be the set of $\mathcal{L}_{\mathrm{DML}}$-formulas that occurs in Θ. It is obvious that Γ is finite. So, $Sub^+(\Gamma)$ is finite as well. Therefore, there are finite many DML-atoms of $Sub^+(\Gamma)$. Let the set of all the DML-atoms of $Sub^+(\Gamma)$ be $At_{\mathbb{DML}}(\Gamma) = \{\Delta_1, \cdots, \Delta_n\}$.

For each $1 \leq i \leq n$, since Δ_i is DML-consistent, it follows by Theorem 7 that Δ_i is satisfiable. Let $(\mathcal{M}_i, \rho)$ be a pointed model such that $\mathcal{M}_i, \rho \vDash \Delta_i$. To indicate this fact, we will write the sequence ρ as ρ_{Δ_i}.

Firstly, we generate the submodel $(\mathcal{M}_i)|_{\rho_{\Delta_i}}$ of $\mathcal{M}_i$ for each $1 \leq i \leq n$. By Proposition 2, it follows that $(\mathcal{M}_i)|_{\rho_{\Delta_i}}, \rho_{\Delta_i} \vDash \Delta_i$ for each $1 \leq i \leq n$.

Secondly, we do the disjoint union of $\{(\mathcal{M}_1)|_{\rho_{\Delta_1}}, \cdots, (\mathcal{M}_n)|_{\rho_{\Delta_n}}\}$. We use $\mathcal{M} = \langle S, \Omega, V\rangle$ to denote the disjoint union model. By the definition of disjoint union, it follows that $\Omega = \{\rho_{\Delta_1}, \cdots, \rho_{\Delta_n}\}$. By Proposition 3, it follows that $\mathcal{M}, \rho_{\Delta_i} \vDash \Delta_i$ for each $1 \leq i \leq n$. Moreover, since each Δ_i is a maximal DML-consistent subset of $Sub^+(\Gamma)$, it follows that any two atoms Δ_i and Δ_j where $1 \leq i, j \leq n$ are

DML-inconsistent. Therefore, any two atoms are not satisfiable. Thus, each Δ_i is only satisfied by $\mathcal{M}, \rho_{\Delta_i}$. So, we have that $[\![\Delta_i]\!]^{\mathcal{M}} = \{\rho_{\Delta_i}\}$.

Next, we need to show that there is a probability distribution μ over Ω such that $\mathcal{M}, \mu \vDash \Theta$.

By a similar process presented in [4], it can be proved that there indeed is a probability distribution μ over Ω such that

- If $(a_1 \mathrm{P}\psi_1 + \cdots + a_j \mathrm{P}\psi_j \geq a) \in \Theta$, then $\mathcal{M}, \mu \vDash a_1 \mathrm{P}\psi_1 + \cdots + a_j \mathrm{P}\psi_j \geq a$.
- If $(a_1 \mathrm{P}\psi_1 + \cdots + a_j \mathrm{P}\psi_j \geq a) \notin \Theta$, then $\mathcal{M}, \mu \nvDash a_1 \mathrm{P}\psi_1 + \cdots + a_j \mathrm{P}\psi_j \geq a$.

Due to space limitations, the proof details of this result are omitted. With this result, it can be shown by induction on ϕ that for each $\psi \in Sub^+(\Theta)$,

$$\mathcal{M}, \mu \vDash \psi \text{ if and only if } \psi \in \Theta.$$

It follows that $\mathcal{M}, \mu \vDash \psi$ for all $\psi \in \Theta$. □

Theorem 12 (Weak completeness)**.** *The system $\mathbb{PL}_{\mathbb{DML}}$ is weak complete, that is, if a formula $\phi \in \mathcal{L}_{\mathrm{PL}_{\mathrm{DML}}}$ is $\mathbb{PL}_{\mathbb{DML}}$-consistent then it is satisfiable.*

Proof. Since ϕ is $\mathbb{PL}_{\mathbb{DML}}$-consistent, it follows by Lindenbaum's lemma that there is a $\mathbb{PL}_{\mathbb{DML}}$-atom Θ of $Sub^+(\phi)$ such that $\phi \in \Theta$. By Lemma 2, we then have that there is a $\mathrm{PL}_{\mathrm{DML}}$-model $\mathcal{M}, \mu$ such that $\mathcal{M}, \mu \vDash \psi$ for all $\psi \in \Theta$. Thus, ϕ is satisfied by $\mathcal{M}, \mu$, then it is satisfiable. □

6 Conclusion

In this paper, we propose semantics for a modal language to capture the reasoning about successive events in probability theory. We prove that this logic (called DML) is equivalent to the normal modal logic on deterministic Kripke models. We then construct a probability logic on DML and show the completeness of its deductive system $\mathbb{PL}_{\mathbb{DML}}$.

There is no nesting of probabilistic operators and modal operators in the probability logic $\mathrm{PL}_{\mathrm{DML}}$. Hence, one of the natural future directions is to extend $\mathrm{PL}_{\mathrm{DML}}$ to allow nesting probabilistic operators and modal operators. This will allow us to express formulas like "$\phi; (\mathrm{P}\psi = a)$", which could help us to reconsider the notion of independence in probability theory. Traditionally, the independence of two events A and B is defined as $\mu(AB) = \mu(A) \times \mu(B)$. This definition is challenged in various aspects (see [2]). We suggest that independence might be defined as $\phi; \mathrm{P}\psi = a \leftrightarrow \mathrm{P}((\phi \vee \neg\psi); \psi) = a$, which means that the occurrence of ϕ has no effect on the probability of ψ (see [2]). We will investigate this definition of independence in future work.

References

[1] P. Blackburn, M. d. Rijke and Y. Venema, 2002, *Modal Logic*, Cambridge University Press.

[2] David Miller, 2018, "Independence (Probabilistic) and Independence (Logical)", *Proceedings of the Conference held in Klagenfurt*, pp. 36–40.

[3] R. Fagin and J. Y. Halpern, 1994, "Reasoning about knowledge and probability", *Journal of the ACM*, **41(2):** 340–367.

[4] R. Fagin, J. Y. Halpern and N. Megiddo, 1990, "A logic for reasoning about probabilities", *Information and Computation*, **87(1-2):** 78–128.

[5] M. Fattorosi-Barnaba and G. Amati, 1987, "Modal operators with probabilistic interpretations", *Studia Logica*, **46(4):** 383–393.

[6] J. Gerbrandy and W. Groeneveld, 1997, "Reasoning about Information Change", *Journal of Logic, Language and Information*, **6(2):** 147–169.

[7] W. van der Hoeck, 1990, "Some considerations on the logic PfD", *Proceedings of the First Russian Conference on Logic Programming*, pp. 474–485.

[8] W. van der Hoeck, 1997, "Some considerations on the logics PFD A logic combining modality and probability", *Journal of Applied Non-Classical Logics*, **7(3):** 287–307.

[9] G. Mirkowska, 1981, "PAL - Propositional Algorithmic Logic", *Fundamenta Informaticae*, 23–101.

[10] Z. Ognjanović, 2006, "Discrete Linear-time Probabilistic Logics: Completeness, Decidability and Complexity", *Journal of Logic and Computation*, **16(2):** 257–285.

[11] Z. Ognjanović and N. Ikodinovic, 2007, "A logic with higher order conditional probabilities", *Publications de l'Institut Mathematique*, **(96):** 141–154.

[12] Z. Ognjanović, Z. Markovic and M. Raskovic, 2005, "Completeness theorem for a logic with imprecise and conditional probabilities", *Publications de l'Institut Mathematique*, **78(92):** 35–49.

[13] Z. Ognjanović, M. Rašković and Z. Marković, 2016, *Probability Logics: Probability-Based Formalization of Uncertain Reasoning*, Springer Link.

[14] J. Plaza, 2007, "Logics of public communications", *Synthese*, **158(2):** 165–179.

[15] A. Pnueli, 1977, "The temporal logic of programs", *18th Annual Symposium on Foundations of Computer Science (sfcs 1977)*, pp. 46–57.

[16] C. Zhou, 2007, *Complete Deductive Systems for Probability Logic with Application in Harsanyi Type Spaces*, PhD, Indiana University.

[17] C. Zhou, 2009, "A Complete Deductive System for Probability Logic", *Journal of Logic and Computation*, **19(6):** 1427–1454.

关于概率中序列事件的逻辑

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摘　要

在概率论的集合语言中，除了补、交、并等运算之外，还有一个重要的运算：乘积。两个基本事件的乘积表示这些事件连续发生。然而，关于概率的逻辑文献中对序列事件的研究却比较少。在本文中，我们提出了一个模态逻辑（记为 DML）来刻画概率论中关于序列事件的推理，然后我们在 DML 逻辑之上构造了一个概率逻辑（记为 $\mathrm{PL}_{\mathrm{DML}}$）。我们将 DML 与克里普克语义上的标准模态逻辑进行了比较，并证明了 DML 等价于确定性模型类上的正规模态逻辑。最后，我们还给出了 $\mathrm{PL}_{\mathrm{DML}}$ 的演绎系统并证明了其完备性。

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