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Privacy in Arrow Update Logic*

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Abstract. Arrow Update Logic is a theory of epistemic access elimination that can be used to reason about multi-agent belief change. In Arrow Update Logic, it is common knowledge among agents how each will process incoming information. This paper develops the basic theory of Arrow Update Logic to deal with private announcements. In this framework, the information is private for an agent group. Moreover, this paper proposes a labelled tableau calculus for this logic and also shows that this logic is decidable.

1 Introduction

Information plays an important role in several fields of scientific research, such as philosophy, game theory, and artificial intelligence. In this paper, the notion of information is confined to the kind of information in one's mind, which can also be called *belief* or *knowledge*. In real-life contexts, information is often communicated. This leads to a change of agents' information without any change in the bare facts of the world. One kind of these communicative events is *announcements*. This paper will focus on reasoning about information change due to announcements.

In a multi-agent system, there are at least three types of announcements: public, private and semi-private (cf. [3]). Imagine a scenario where two agents a_1 and a_2 are in a room, and in front of them, there is a coin in a closed box. Neither of them knows whether the coin is lying heads up or tails up. A public announcement occurs when the box is opened for both to see. This changes not only the agent's information about the bare facts (basic information) but also agents' information about each other (higher-order information). When a_1 secretly opens the box and a_2 does not suspect that anything happened, the effect is the same as the effect that the truth is privately announced to a_1. This changes only a_1's basic and higher-order information. A semi-private announcement occurs when a_1 opens the box and a_2 observes a_1's action but a_2 does not see the coin. This changes a_1's basic information and the higher-order information of both. Please note that a public announcement can be seen as a special private announcement, i.e., an announcement which is privately announced to the whole group.

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Plaza ([17]) develops a logic that is concerned with information change by public announcements, in which a public announcement of a statement eliminates all epistemic possibilities in which the statement does not hold. Gerbrandy and Groeneveld ([12]) propose a more general dynamic epistemic logic for information update, where a private announcement for an agent group G is an update that is conscious only for G. The dynamic epistemic logic with action models (DEL, [3, 4]) can formalize a wide range of information change. An action model is a Kripke model-like object that describes agents' beliefs about incoming information. The logic LCC ([5]) can model information communication that is partial observation, and a private announcement for G is a piece of information whispered to G ([7]). Logics introduced in [8, 6] can model information change based on attentions, and an announcement is private for agents who are paying attention when the announcement is made. Private announcements can also be modeled in [11]. The precise method by which private announcements are dealt with in these logics are different due to their different motivations, but they share the same feature that the basic model will be expanded when a private announcement happens.

Kooi and Renne's Arrow Update Logic (AUL, [13]) can also formalize reasoning about information change produced by public and semi-private announcements. Different from other logical frameworks, AUL models information change by updating the epistemic access relation, without changing the domain of the model. However, in AUL, it is common knowledge among agents how each will process incoming information. This assumption of common update policy is dropped in its extension, Generalized Arrow Update Logic (GAUL, [14]), which can capture the same information change that can be modeled in DEL.

Although DEL and GAUL are much more expressive than AUL, the great expressive power does not come for free. Their update operators are much more complex than the update operator of AUL. This paper presents a variation of AUL, Private Arrow Update Logic (PAUL), which also drops the common update-policy assumption of AUL (so that private announcements can be expressed) and keeps the update operator as simple and intuitive as in AUL. In our framework, each information update is visible only for a group of agents, so private announcements can be dealt with. This logic framework is also inspired by the context-indexed semantics developed in [18] and [20]. As we will see, PAUL can formalize reasoning about information change due to public, private and semi-private announcements. Our logic is more general than AUL in the sense that each information update in AUL can be seen as update that is visible for all agents in our framework. Our logic can also be seen as a fragment of DEL, and there are information change, such as lying, cannot be expressed in our framework.

The rest of this paper is organized as follows: Section 2 proposes the language and semantics of PAUL, and works out some examples; Section 3 presents the tableau

calculus for PAUL and show soundness and completeness; Section 4 shows PAUL is decidable; Section 5 concludes with some directions for further research.

2 The Logic PAUL

2.1 Syntax and semantics

In this section, we introduce the language and semantics of this logic.

Definition 1 (PAUL Language)**.** Let **Agt** be a nonempty finite set of agents, and let **P** be a countable set of atomic propositions. The PAUL language is generated by the following BNF:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid [U, G]\phi \mid \Box_a\phi$$
$$U ::= (\phi, a, \phi) \mid (\phi, a, \phi), U$$

where $p \in \mathbf{P}$, $a \in \mathbf{Agt}$ and $G \subseteq \mathbf{Agt}$ is a superset of the set of agents occurring in U.

We will often omit parentheses around expressions when doing so ought not cause confusion. The expression ϕ is called a PAUL-formula (or just formula). The expression $[U, G]$ occurring in a formula is called a PAUL-update (or just update), which consists of an update core U and an agent group G to which the update is visible. We let $\mathscr{L}_{\text{PAUL}}$ denote the set of formulas. Given formulas ϕ and ψ and an agent $a \in \mathbf{Agt}$, the syntactic object $(\phi, a, \psi) \in U$ is called an a-arrow specification. As usual, we use the following abbreviations: $\bot := \neg\top$, $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\phi \to \psi := \neg\phi \vee \psi$, $\Diamond_a\phi := \neg\Box_a\neg\phi$. Moreover, if there is only one arrow specification in U, we write $[\{(\phi, a, \psi)\}, G]$ as $[(\phi, a, \psi), G]$. Similarly, we write $[U, \{a\}]$ as $[U, a]$.

Intuitively, the formula $\Box_a\phi$ expresses that agent a believes ϕ. The formula $[U, G]\phi$ expresses that ϕ holds after the arrow update $[U, G]$. The update $[U, G]$ means that the update is visible only to agents in G. Please note that the update in AUL has only one part, that is $[U]$, which is visible for all agents. Therefore, the update $[U]$ in AUL is the same as the update $[U, \mathbf{Agt}]$ here.

Definition 2 (Kripke Model)**.** A Kripke model $\mathcal{M}$ is a tuple $\langle W^{\mathcal{M}}, R^{\mathcal{M}}, V^{\mathcal{M}}\rangle$, consisting of a nonempty set $W^{\mathcal{M}}$ of worlds, a function $R^{\mathcal{M}}$ assigning each agent $a \in \mathbf{Agt}$ a binary relation $R_a^{\mathcal{M}} \subseteq W^{\mathcal{M}} \times W^{\mathcal{M}}$, and a function $V^{\mathcal{M}} : \mathbf{P} \to \mathcal{P}(W^{\mathcal{M}})$. A pointed Kripke model is a pair $(\mathcal{M}, s)$ consisting of a Kripke model $\mathcal{M}$ and a world $s \in W^{\mathcal{M}}$; the world s is called the point of $(\mathcal{M}, s)$.

Given a Kripke model $\mathcal{M}$, we call $W^{\mathcal{M}}$ the domain of the model. For each agent $a \in \mathbf{Agt}$, we call $R_a^{\mathcal{M}}$ a's *possibility relation* since it defines what worlds agent a considers possible in any given world. Please note that updates considered in this paper do not change any bare facts but only the agent's beliefs. Therefore, when an

update happens, we do not have to change the domain of the model but only change the possibility relations (or 'arrows').

Definition 3. Let $\rho = [U_1, G_1] \cdots [U_n, G_n]$ be an update sequence (or just sequence), and $\rho = \epsilon$ if $n = 0$. The update sequence $\rho|_a$ is defined by the following induction on n.

$$\epsilon|_a = \epsilon$$
$$(\rho[U,G])|_a = \begin{cases} \rho|_a & a \notin G \\ \rho|_a[U,G] & a \in G \end{cases}$$

In the following text, we will always use symbols $\rho, \rho', \rho[U,G], \rho'[U,G]$ to denote sequences of updates. The sequence $\rho|_a$ means that the updates are visible to the agent a.

Definition 4 (PAUL Semantics)**.** Given a pointed Kripke model $(\mathcal{M}, s)$, an update sequence ρ and a formula ϕ, we write $\mathcal{M}, s \vDash_\rho \phi$ to mean that ϕ is true at $\mathcal{M}, s$ after updates ρ, and we write $\mathcal{M}, s \nvDash_\rho \phi$ for the negation of $\mathcal{M}, s \vDash_\rho \phi$. The relation (notation: $\vDash_\rho$) is defined by the following induction on formula construction.

$$\begin{array}{lll}
\mathcal{M}, s \vDash_\rho \top & & \\
\mathcal{M}, s \vDash_\rho p & \textsf{iff} & s \in V(p) \\
\mathcal{M}, s \vDash_\rho \neg\phi & \textsf{iff} & \mathcal{M}, s \nvDash_\rho \phi \\
\mathcal{M}, s \vDash_\rho (\phi \wedge \psi) & \textsf{iff} & \mathcal{M}, s \vDash_\rho \phi \text{ and } \mathcal{M}, s \vDash_\rho \psi \\
\mathcal{M}, s \vDash_\rho [U,G]\phi & \textsf{iff} & \mathcal{M}, s \vDash_{\rho[U,G]} \phi \\
\mathcal{M}, s \vDash_\rho \Box_a\phi & \textsf{iff} & \forall t \in W^{\mathcal{M}} : (s,t) \in R_a^{\mathcal{M}} * (\rho|_a) \text{ implies } \mathcal{M}, t \vDash_{\rho|_a} \phi \\
R_a^{\mathcal{M}} * \epsilon & \stackrel{\text{def}}{=} & R_a^{\mathcal{M}} \\
R_a^{\mathcal{M}} * (\rho'[U,G]) & \stackrel{\text{def}}{=} & \{(s,t) \in R_a^{\mathcal{M}} * \rho' \mid \text{there exists } (\phi, a, \psi) \in U : \\
 & & \mathcal{M}, s \vDash_{\rho'} \phi \text{ and } \mathcal{M}, t \vDash_{\rho'} \psi\}
\end{array}$$

We also write $\mathcal{M}, s \vDash_\epsilon \phi$ as $\mathcal{M}, s \vDash \phi$. To say that a formula ϕ is valid, written as $\vDash \phi$, means that $\mathcal{M}, s \vDash \phi$ for each pointed Kripke model $(\mathcal{M}, s)$. The negation of $\vDash \phi$ is written as $\nvDash \phi$. To say that a formula ϕ is satisfiable means there exists a pointed model $(\mathcal{M}, s)$ such that $\mathcal{M}, s \vDash \phi$.

The binary relation $R_a^{\mathcal{M}} * \rho|_a$ is a's possibility relation after the announcement sequence ρ. Compared to product semantics, such as in DEL and GAUL, the context-indexed semantics here has the following characteristics. Firstly, we know that updates change only agents' beliefs but not bare facts. This feature is more clear in this semantics because only the possibility relation is updated when $\Box$-formulas are evaluated. Moreover, product semantics always update the domain of the model when an update happens, which means that the size of the model may grow rapidly along

with the length of update sequence ρ, but this is not the case here. This is because when $\Box_a$-formulas are evaluated, we do not change the domain of the model but only update a's possibility relation with respect to the update sequence visible to a, namely $\rho|_a$.

Kooi and Renne ([13]) present an axiomatic theory for AUL, in which the most important axiom states that an agent's belief after an update can be reduced to his (her) belief before the update. The following proposition shows that the PAUL version of this reduction axiom also holds.

Proposition 5. $\vDash [U,G]\Box_a\phi \leftrightarrow \bigwedge_{(\psi,a,\chi)\in U}(\psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi))$ *if* $a \in G$.

Proof Let $(\mathcal{M}, s)$ be a pointed Kripke model. Firstly, we show that if $\mathcal{M}, s \vDash [U,G]\Box_a\phi$ then $\mathcal{M}, s \vDash \bigwedge_{(\psi,a,\chi)\in U}(\psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi))$. Assume that $\mathcal{M}, s \vDash [U,G]\Box_a\phi$ and $(\psi, a, \chi) \in U$, we will show that $\mathcal{M}, s \vDash \psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi)$. Let $\mathcal{M}, s \vDash \psi$ and t be a state such that $(s,t) \in R_a^{\mathcal{M}}$. We only need to show that if $\mathcal{M}, t \vDash \chi$ then $\mathcal{M}, t \vDash [U,G]\phi$. If $\mathcal{M}, t \vDash \chi$, since $\mathcal{M}, s \vDash \psi$ and $(\psi, a, \chi) \in U$, this follows that $(s,t) \in (R_a^{\mathcal{M}} * [U,G])$. Since $a \in G$, we have that $[U,G]|_a = [U,G]$. Thus, we have $(s,t) \in (R_a^{\mathcal{M}} * [U,G]|_a)$. Moreover, since $\mathcal{M}, s \vDash [U,G]\Box_a\phi$, this follows that $\mathcal{M}, t \vDash_{[U,G]} \phi$. Therefore, we have that $\mathcal{M}, t \vDash [U,G]\phi$.

Secondly, we show that if $\mathcal{M}, s \vDash \bigwedge_{(\psi,a,\chi)\in U}(\psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi))$ then $\mathcal{M}, s \vDash [U,G]\Box_a\phi$. Assume that $\mathcal{M}, s \nvDash [U,G]\Box_a\phi$. It follows that there exists $t \in W^{\mathcal{M}}$ such that $(s,t) \in R_a^{\mathcal{M}} * [U,G]$ and $\mathcal{M}, t \nvDash_{[U,G]} \phi$. Since $(s,t) \in R_a^{\mathcal{M}} * [U,G]$, it follows that $(s,t) \in R_a^{\mathcal{M}}$ and there exists $(\psi, a, \chi) \in U$ such that $\mathcal{M}, s \vDash \psi$ and $\mathcal{M}, t \vDash \chi$. Moreover, since $\mathcal{M}, s \vDash \bigwedge_{(\psi,a,\chi)\in U}(\psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi))$, we then have $\mathcal{M}, t \vDash [U,G]\phi$, namely $\mathcal{M}, t \vDash_{[U,G]} \phi$. This is in contradiction with $\mathcal{M}, t \nvDash_{[U,G]} \phi$. Therefore, we have if $\mathcal{M}, s \vDash \bigwedge_{(\psi,a,\chi)\in U}(\psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi))$ then $\mathcal{M}, s \vDash [U,G]\Box_a\phi$. □

The following proposition shows that if an update is not visible for an agent, then her belief after the update is the same as her belief before the update.

Proposition 6. $\vDash [U,G]\Box_a\phi \leftrightarrow \Box_a\phi$ *if* $a \notin G$.

Proof We have the following:

$$
\begin{aligned}
&\mathcal{M}, s \vDash [U,G]\Box_a\phi \\
\Leftrightarrow &\mathcal{M}, s \vDash_{[U,G]} \Box_a\phi \\
\Leftrightarrow &\text{ for all } (s,t) \in R_a * ([U,G]|_a) : \mathcal{M}, t \vDash_{[U,G]|_a} \phi \\
\Leftrightarrow &\text{ for all } (s,t) \in R_a : \mathcal{M}, t \vDash \phi \qquad \text{due to } [U,G]|_a = \epsilon \\
\Leftrightarrow &\mathcal{M}, s \vDash \Box_a\phi
\end{aligned}
$$

□

We know that the replacement of equivalents plays an important role in equivalently reducing an AUL formula to a formula with out updates. Next, we will show that the replacement of equivalents also holds in PAUL. Before that, we first show the the following proposition.

Proposition 7. *Let ρ be a sequence of updates and $(\mathcal{M}, s)$ be a pointed model. We then have that $\mathcal{M}, s \vDash_\rho \phi$ iff $\mathcal{M}', s \vDash \phi$ for each $\phi \in \mathscr{L}_{PAUL}$, where $\mathcal{M}' = \langle W^{\mathcal{M}}, R', V^{\mathcal{M}}\rangle$ and $R' = \{R_a^{\mathcal{M}} * \rho \mid a \in \mathbf{Agt}\}$.*

Proof We prove it by induction on ϕ. We only focus on the case of $[U, G]\phi$; the other cases are straightforward by IH.

We need to show that $\mathcal{M}, s \vDash_\rho [U, G]\phi$ iff $\mathcal{M}', s \vDash [U, G]\phi$. Please note that $R_a^{\mathcal{M}} * (\rho[U, G]) = (R_a^{\mathcal{M}} * \rho) * [U, G]$. Let $\mathcal{M}'' = \langle W^{\mathcal{M}}, R'', V^{\mathcal{M}}\rangle$ where $R'' = \{R_a^{\mathcal{M}} * (\rho[U, G]) \mid a \in \mathbf{Agt}\}$. By IH, we have that $\mathcal{M}, s \vDash_{\rho[U,G]} \phi$ iff $\mathcal{M}'', s \vDash \phi$. What is more, by IH, we also have that $\mathcal{M}', s \vDash_{[U,G]} \phi$ iff $\mathcal{M}'', s \vDash \phi$. Therefore, we have the following:

$$
\begin{aligned}
&\mathcal{M}, s \vDash_\rho [U, G]\phi \\
\Leftrightarrow &\mathcal{M}, s \vDash_{\rho[U,G]} \phi \\
\Leftrightarrow &\mathcal{M}'', s \vDash \phi \qquad \text{by IH} \\
\Leftrightarrow &\mathcal{M}', s \vDash_{[U,G]} \phi \qquad \text{by IH} \\
\Leftrightarrow &\mathcal{M}', s \vDash [U, G]\phi
\end{aligned}
$$

□

Please note that, by the proposition above, we have that, for any update sequence ρ, $\vDash_\rho \phi$ if $\vDash \phi$. Therefore, we have the general version of Propositions 5 and 6:

$$
\begin{aligned}
&\vDash_\rho [U, G]\Box_a\phi \leftrightarrow \bigwedge_{(\psi,a,\chi)\in U} (\psi \to \Box_a(\chi \to [U, G]\phi)) && a \in G \\
&\vDash_\rho [U, G]\Box_a\phi \leftrightarrow \Box_a\phi && a \notin G
\end{aligned}
$$

Now we are ready to prove the replacement of equivalents.

Proposition 8. *Let ϕ' be the result of replacing some occurrences of ψ in ϕ by ψ'*[1]*. We then have that $\vDash \phi \leftrightarrow \phi'$ if $\vDash \psi \leftrightarrow \psi'$.*

Proof We prove it by induction on ϕ. We only focus on the case of $[U, G]\phi$; the other cases are straightforward by IH.

We need to show that $\mathcal{M}, s \vDash [U, G]\phi$ iff $\mathcal{M}, s \vDash [U, G]\phi'$. Let the model $\mathcal{M}'$ be $\langle W^{\mathcal{M}}, R', V^{\mathcal{M}}\rangle$ and $R' = \{R_a^{\mathcal{M}} * [U, G] \mid a \in \mathbf{Agt}\}$. We then have that

[1]Here we confine the occurrence of ψ in ϕ on the occurrence that ψ is not in an update. In the following, we will see that this confined version of replacement of equivalents is sufficient to show the reduction theorem.

$\mathcal{M}, s \vDash [U,G]\phi$ iff $\mathcal{M}, s \vDash_{[U,G]} \phi$, and by Proposition 7 we have that $\mathcal{M}, s \vDash_{[U,G]} \phi$ iff $\mathcal{M}', s \vDash \phi$. By IH, we then have that $\mathcal{M}', s \vDash \phi$ iff $\mathcal{M}', s \vDash \phi'$. Then, by Proposition 7 again, we have that $\mathcal{M}', s \vDash \phi'$ iff $\mathcal{M}, s \vDash [U,G]\phi'$. □

Next, we will show that each formula in PAUL can be equivalently reduced to be a formula with out updates.

Theorem 9. *For each formula ϕ, there is a formula ϕ' such that $\vDash \phi \leftrightarrow \phi'$ and there are no updates in ϕ'.*

Proof We first define the translation function t as follows:

$$
\begin{array}{llll}
t(p) = p & & t([U,G]p) = p & \\
t(\neg\phi) = \neg t(\phi) & & t([U,G]\neg\phi) = \neg t([U,G]\phi) & \\
t(\phi \wedge \psi) = t(\phi) \wedge t(\psi) & & t([U,G](\phi \wedge \psi)) = t([U,G]\phi) \wedge t([U,G]\psi) & \\
t(\Box_a \phi) = \Box_a t(\phi) & & & \\
t([U,G]\Box_a\phi) = \bigwedge_{(\psi,a,\chi)\in U}(t(\psi) \rightarrow \Box_a(t(\chi) \rightarrow t([U,G]\phi))) & & & a \in G \\
t([U,G]\Box_a\phi) = \Box_a t(\phi) & & & a \notin G \\
t([U,G][U',G']\phi) = t([U,G]t([U',G']\phi)). & & &
\end{array}
$$

By induction on ϕ, we will show that $\vDash \phi \leftrightarrow t(\phi)$ and that there are no updates occurring in $t(\phi)$. We then only focus on the case of $[U,G]\phi$; the other cases are straightforward by IH.

To show that $\vDash [U,G]\phi \leftrightarrow t([U,G]\phi)$ and that there are no updates occurring in $t([U,G]\phi)$, we continue to do induction on ϕ.

- $[U,G]p$. It is obvious.
- $[U,G]\neg\phi$. By semantics, we have that $\vDash [U,G]\neg\phi \leftrightarrow \neg[U,G]\phi$. By IH, we have that $\vDash [U,G]\phi \leftrightarrow t([U,G]\phi)$ and that there are no updates occurring in $t([U,G]\phi)$. Thus, we have that $\vDash \neg[U,G]\phi \leftrightarrow \neg t([U,G]\phi)$ and that there are no updates in $\neg t([U,G]\phi)$.
- $[U,G](\phi \wedge \psi)$. Since $\vDash [U,G](\phi \wedge \psi) \leftrightarrow [U,G]\phi \wedge [U,G]\psi$ by semantics, it is straightforward by IH.
- $[U,G]\Box_a\phi$ and $a \in G$. By Proposition 5, we only need to show that, for each $(\psi,a,\chi) \in U$, $\vDash (\psi \rightarrow \Box_a(\chi \rightarrow [U,G]\phi)) \leftrightarrow (t(\psi) \rightarrow \Box_a(t(\chi) \rightarrow t([U,G]\phi)))$ and there are no updates in $t(\psi)$, $t(\chi)$, or $t([U,G]\phi)$. These are straightforward by IH.
- $[U,G]\Box_a\phi$ and $a \notin G$. By Proposition 6, we only need to show that $\Box_a\phi$ can be equivalently reduced to be a formula with out updates. By IH, we have that $\vDash \phi \leftrightarrow t(\phi)$ and there are no updates in $t(\phi)$. Thus, we have that $\vDash \Box_a\phi \leftrightarrow \Box_a t(\phi)$ by Proposition 8.
- $[U,G][U',G']\phi$.We need to show that $\vDash [U,G][U',G']\phi \leftrightarrow t([U,G]t([U',G']\phi))$ and that there are no updates in $t([U,G]t([U',G']\phi))$. By IH, we have that there

is some formula $\phi' = t([U', G']\phi)$ such that $\vDash [U', G']\phi \leftrightarrow \phi'$ and that there are no updates in ϕ'. By IH again, we have that $\vDash [U, G]\phi' \leftrightarrow t([U, G]\phi')$ and there are no updates in $t([U, G]\phi')$. Since $\vDash [U', G']\phi \leftrightarrow \phi'$, by Proposition 8, we then have that $\vDash [U, G][U', G']\phi \leftrightarrow t([U, G]\phi')$. □

2.2 Announcements in PAUL

In this section, we will show how public, private and semi-private announcements are captured in PAUL. Let us consider the following scenario of a concealed coin, which is a tweaked version of an example used in [3].

Example 10 (Basic scenario)**.** Two agents a_1 and a_2 enter a large room which contains a remote-controlled mechanical coin flipper. One of them presses a button, and the coin spins through the air and lands in a small box on a table with heads or tails lying up. The box is closed and they are too far away to see the coin.

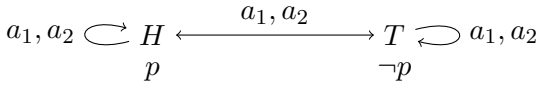


Figure 1: the basic model $\mathcal{M}$

Just as in [3], this can be modelled by a Kripke model $\mathcal{M}$, which is pictured in Figure 1. The possible world $H \in W^{\mathcal{M}}$ represents the possible fact that the coin is lying heads up, and $T \in W^{\mathcal{M}}$ represents tails up. The proposition p means that the coin is lying heads up, so it is only true in H. The possibility relations of a_1 and a_2 indicate that both of them do not know whether the coin is lying heads or tails up.

Example 11 (Public announcement)**.** After the basic scenario, one of them opens the box and puts the coin on the table for both to see. The effect of this event on their beliefs is the same as that of a *truthful* statement publicly announced to them that the coin is lying heads or tails up.

Figure 2: The possibility relations after the update $[U_1, G_1]$

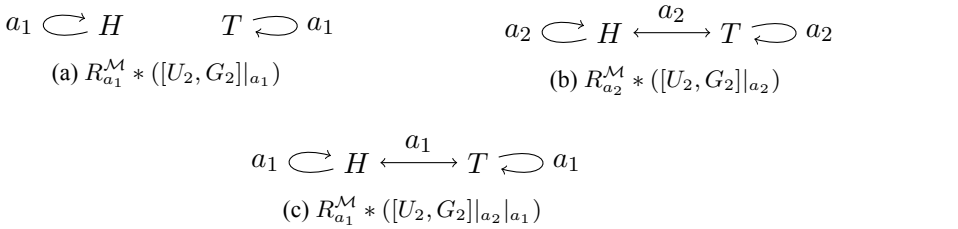
After the truthful announcement that the coin is lying heads or tails up, both of them think there is only one possibility in any given world. Thus only their epistemic

accesses to any given world should be preserved. This announcement is visible for both, since it is publicly announced. Therefore, this public announcement can be captured by $[U_1, G_1]$ where $U_1 = \{(p, a_1, p), (\neg p, a_1, \neg p), (p, a_2, p), (\neg p, a_2, \neg p)\}$ and $G_1 = \{a_1, a_2\}$. After the update $[U_1, G_1]$, the possibility relations of a_1 and a_2 turn out to be the same, as shown in Figure 2. Moreover, since the update is visible to both of them, a_1's possibility relation in a_2's opinion is the same as a_1's real possibility relation, namely $R_a^{\mathcal{M}} * ([U_1, G_1]|_{a_2}|_{a_1}) = R_{a_1}^{\mathcal{M}} * ([U_1, G_1]|_{a_1})$. If H is the actual world, after this public and truthful announcement, both of them believe that the coin is lying heads up and that the other also believes so. We can check the following formulas.

- $\mathcal{M}, H \vDash [U_1, G_1](\Box_{a_1} p \wedge \Box_{a_2} p)$
- $\mathcal{M}, H \vDash [U_1, G_1](\Box_{a_1} \Box_{a_2} p \wedge \Box_{a_2} \Box_{a_1} p)$

More generally, public announcements of the truth value of ϕ can be expressed in our framework by the update $[U, \mathbf{Agt}]$ where $U = \{(\phi, a, \phi), (\neg\phi, a\neg\phi) \mid a \in \mathbf{Agt}\}$. Therefore, public announcement logic is a fragment of PAUL.

Example 12 (Private announcement)**.** After the basic scenario of Example 10, agent a_1 secretly opens the box herself. Agent a_2 does not observe that a_1 opens the box, and indeed a_1 is certain that a_2 does not suspect that anything happened. The effect of this on their beliefs is the same as secretly and privately announcing the truth to a_1.



(a) $R_{a_1}^{\mathcal{M}} * ([U_2, G_2]|_{a_1})$ (b) $R_{a_2}^{\mathcal{M}} * ([U_2, G_2]|_{a_2})$ (c) $R_{a_1}^{\mathcal{M}} * ([U_2, G_2]|_{a_2}|_{a_1})$

Figure 3: The possibility relations after the update $[U_2, G_2]$

After the truth is announced to a_1, she thinks that there is only one possibility from any given world. Thus a_1's epistemic accesses to the world itself should be preserved after the announcement. Since the announcement is secret and private, it is visible only to a_1. This private and truthful announcement can be captured by the update $[U_2, G_2]$ which is defined as $U_2 = \{(p, a_1, p), (\neg p, a_1, \neg p)\}$ and $G_2 = \{a_1\}$.

After the update $[U_2, G_2]$, a_1's possibility relation (Figure 3a) will change, but a_2's possibility relation (Figure 3b) will remain the same as before. Moreover, since a_2 does not suspect that anything happened, a_1's possibility relation in a_2's opinion

(Figure 3c) does not change at all after the announcement. After this private and truthful announcement to a_1, only a_1 believes the truth while nothing happened to a_2's beliefs. We can check the following formulas.

- $\mathcal{M}, H \vDash [U_2, G_2](\Box_{a_1} p \wedge \neg \Box_{a_2} p)$
- $\mathcal{M}, H \vDash [U_2, G_2] \neg \Box_{a_2} (\Box_{a_1} p \vee \Box_{a_1} \neg p)$

More generally, a private announcement of the truth value of ϕ to a group of agents $G \subseteq$ **Agt** can be expressed in our framework by the update $[U, G]$ where $U = \{(\phi, a, \phi), (\neg\phi, a, \neg\phi) \mid a \in G\}$. Our logic is a fragment of DEL, since there is certain information change, such as lying, that can not be expressed in PAUL but can be modeled in DEL.

(a) $R^{\mathcal{M}}_{a_1} * ([U_3, G_3]|_{a_1})$ (b) $R^{\mathcal{M}}_{a_2} * ([U_3, G_3]|_{a_2})$

(c) $R^{\mathcal{M}}_{a_1} * ([U_3, G_3]|_{a_2}|_{a_1})$

Figure 4: The possibility relations after the update $[U_3, G_3]$

Example 13 (Semi-private announcement)**.** After the basic scenario of Example 10, agent a_1 opens the box herself. Agent a_2 observes that a_1 opens the box but does not see the coin. Agent a_1 also does not disclose whether it is heads or tails. The effect of this on their beliefs is the same as a semi-private announcement to a_1, which means that the truth is announced to a_1 only, but a_2 notices what happened.

Since the truth is announced to a_1, she will know the truth after the announcement. The situation of a_2 is a little complex. Firstly, a_2's possibility relation will remain the same as before since a_2 is not announced the truth. Secondly, a_1's possibility relation in a_2's opinion will change since he observed that a_1 is announced the truth. This semi-private announcement can be captured by the update $[U_3, G_3]$ which is defined as $U_3 = \{(p, a_1, p), (\neg p, a_1, \neg p), (\top, a_2, \top)\}$ and $G_3 = \{a_1, a_2\}$.

Agent a_1's possibility relation (Figure 4a) will change to the reflexive relation after the update. Since the announcement is not disclosed to a_2, a_2's possibility relation (Figure 4b) will not change after the update. However, after the update, a_1's possibility relation in a_2's opinion (Figure 4c) will change because a_2 observes the announcement. After the announcement, a_2 believes that a_1 believes the truth, but a_2 still could not distinguish between the fact that a_1 believes p and the fact that a_1 believes $\neg p$. We can check the following formulas.

- $\mathcal{M}, H \vDash [U_3, G_3](\Box_{a_1} p \wedge \neg \Box_{a_2} p)$
- $\mathcal{M}, H \vDash [U_3, G_3] \Box_{a_2} (\Box_{a_1} p \vee \Box_{a_1} \neg p)$

More generally, each semi-private announcement of the truth value of ϕ to a group of agents $G \subseteq$ **Agt** can be expressed in our framework by the update $[U, \mathbf{Agt}]$ where $U = \{(\phi, a, \phi), (\neg\phi, a, \neg\phi), (\top, b, \top) \mid a \in G, b \in \mathbf{Agt} \setminus G\}$. Each update in AUL is a semi-private announcement in nature, since all updates in AUL is visible to all agents. Therefore, each formula ϕ in AUL can be equivalently transformed to a formula in PAUL by replacing each update $[U]$ in ϕ with $[U, \mathbf{Agt}]$. So, AUL is a fragment of PAUL.

3 Tableau Method

This section will present a proof method for PAUL that uses analytic tableaux. As a typical tableau method, given a formula ϕ, it systematically tries to construct a model for it. When it fails, ϕ is inconsistent and thus its negation is valid.

The tableau method in this paper will manipulate tableau terms, which consist of two parts: the first part is an update sequence; the second part is a formula, or a check mark, or a cross mark. In addition, each term is prefixed by a label which stands for a possible world in the model under construction. A similar method is used in [1, 2, 10, 16].

Definition 14 (Term)**.** A term (or tableau term) is a pair (ρ, x) where ρ is a finite update sequence $[U_1, G_1] \cdots [U_n, G_n]$ ($\rho = \epsilon$ if $n = 0$) and x is a check mark ✓, a cross mark ✗ or a formula $\phi \in \mathscr{L}_{\text{PAUL}}$.

Definition 15 (Labelled term)**.** A *label* is an alternating sequence of integers and agents, namely $\sigma ::= n \mid \sigma a n$ where $n \in \mathbb{N}$ and $a \in$ **Agt**. A *labelled term* is a pair consisting of a label and a term, and we also write it as a triple $\langle \sigma, \rho, x \rangle$.

In the following text, we will always use symbols $\sigma, \sigma', \sigma a n$ to denote labels. Each label represents a possible world in a Kripke model. Moreover, a label $\sigma a n$ occurring on a branch of a tableau also indicates that there is an a-arrow from the possible world σ to the possible world $\sigma a n$. A labelled term $\langle \sigma, \rho, \phi \rangle$ means ϕ is true at the possible world σ after the announcement sequence ρ. A labelled term $\langle \sigma a n, \rho, ✓ \rangle$ means the a-arrow from σ to $\sigma a n$ is preserved after the update sequence ρ. Conversely, a labelled term $\langle \sigma a n, \rho, ✗ \rangle$ means the a-arrow is not preserved.

Definition 16 (Branch)**.** A branch is a set of labelled terms. A label σ is new in a branch b if there is no term in b that is labelled with σ.

In the following text, we will always use $b, b', b_1, \cdots$ to denote branches and B to denote a set of branches.

Definition 17 (Tableau)**.** A tableau T for $\phi_0 \in \mathscr{L}_{\text{PAUL}}$ is a set of branches inductively defined as follows.

- The set $\{\{\langle 0, \epsilon, \phi_0\rangle\}\}$ is a tableau for ϕ_0, which is called the initial tableau for ϕ_0.
- Let T' be a tableau for ϕ_0 and b be a branch in T'. If B is a finite set of branches generated by applying one of the tableau rules in Table 1 on b (for instance, let $b = \{\langle \sigma, \rho, \neg(\phi \wedge \psi)\rangle\}$ then $B = \{b \cup \{\langle \sigma, \rho, \neg\phi\rangle\}, b \cup \{\langle \sigma, \rho, \neg\psi\rangle\}\}$), then the set $(T' \setminus \{b\}) \cup B$ is a tableau for ϕ_0.

Rules $(\neg\neg)$, $(\neg\wedge)$ and $(\wedge)$ are exactly as for propositional logic. Rules $(\neg\Box_a)$ and $\Box_a$ are different from their counterparts commonly used in tableau calculi for normal modal logic. The intuition behind Rule $(\neg\Box_a)$ is that if the possible world that σ stands for satisfied $\neg\Box_a\phi$ after the update sequence ρ then it needs to satisfy the following conditions: there exists a possible world that is represented by σan (the form of σan indicates that there is an a-arrow from σ to σan); $\langle \sigma an, \rho|_a, \checkmark\rangle$ means the a-arrow from σ to σan will be preserved after the update sequence $\rho|_a$; $\langle \sigma an, \rho|_a, \neg\phi\rangle$ means $\neg\phi$ is true in σan after the update sequence $\rho|_a$. Similarly, Rule $(\Box_a)$ means that $\Box_a\phi$ is true in σ after ρ if and only if for each possible world that is accessible by a-arrow from σ: either the a-arrow is removed after $\rho|_a$, or ϕ is true in it after $\rho|_a$.

Rule $(\neg[U, G])$ and Rule $([U, G])$ reflect the feature of the semantics that the updates are just remembered and they are used to update the possibility relation only when $\Box_a$ formulas are evaluated. Rule $(\checkmark_1)$ means that the a-arrow is preserved after $\rho[U, G]$ if and only if it is firstly preserved after ρ and then preserved by some a-arrow specification in U. Rule $(\checkmark_2)$ says it is not possible that the a-arrow from σ to σan is preserved after $\rho[U, G]$ if there are no a-arrow specifications in U. Rule (✗_1) and Rule (✗_2) specify the conditions under which the a-arrow from σ to σan will be removed. It is removed after $\rho[(\psi, a, \chi), G]$ if either it is already removed after ρ, or it cannot be preserved by the specification (ψ, a, χ). Rule (✗_2) corresponds to the semantics that $R_a * (\rho[U, G]) = \bigcup_{(\psi,a,\chi)\in U} R_a * (\rho[(\psi, a, \chi), G])$. Please note that it is always true that any a-arrow will be remove after $\rho[(\psi, b, \chi), G]$ if $b \neq a$. Therefore, we do not need a rule for $\langle \sigma an, \rho[(\psi, b, \chi), G], \text{✗}\rangle$ if $b \neq a$.

The following proposition is obvious according to the tableau rules.

Proposition 18. *Given a tableau T and a branch $b \in T$, if $\langle \sigma, \rho, x\rangle \in b$ and $x = \checkmark/\text{✗}$ then $\sigma = \sigma' an$ for some label σ', $a \in A$ and $n \in \mathbb{N}$.*

Definition 19 (Closed tableau)**.** A branch b is *closed* if and only if we have either $\{\langle \sigma, \rho, p\rangle, \langle \sigma, \rho', \neg p\rangle\} \subseteq b$ for some σ, ρ, ρ' and p, or $\{\langle \sigma an, \epsilon, \checkmark\rangle, \langle \sigma an, \epsilon, \text{✗}\rangle\} \subseteq b$ for some σan, otherwise it is *open*. A tableau is *closed* if and only if all its branches are closed, otherwise it is *open*.

$(\neg\neg)\ \dfrac{\sigma, \rho, \neg\neg\phi}{\sigma, \rho, \phi}$

$(\neg\wedge)\ \dfrac{\sigma, \rho, \neg(\phi \wedge \psi)}{\sigma, \rho, \neg\phi \mid \sigma, \rho, \neg\psi}$ $\qquad$ $(\wedge)\ \dfrac{\sigma, \rho, \phi \wedge \psi}{\begin{array}{l}\sigma, \rho, \phi \\ \sigma, \rho, \psi\end{array}}$

$(\neg\Box_a)\ \dfrac{\sigma, \rho, \neg\Box_a\phi}{\begin{array}{l}\sigma an, \rho|_a, \checkmark \\ \sigma an, \rho|_a, \neg\phi\end{array}}$ n is new

$(\Box_a)\ \dfrac{\sigma, \rho, \Box_a\phi}{\sigma an, \rho|_a, \phi \mid \sigma an, \rho|_a, \text{✗}}$ n is used

$(\neg[U, G])\ \dfrac{\sigma, \rho, \neg[U, G]\phi}{\sigma, \rho[U, G], \neg\phi}$ $\qquad$ $([U, G])\ \dfrac{\sigma, \rho, [U, G]\phi}{\sigma, \rho[U, G], \phi}$

$(\checkmark_1)\ \dfrac{\sigma an, \rho[U, G], \checkmark}{\begin{array}{l|c|l}\sigma an, \rho, \checkmark & & \sigma an, \rho, \checkmark \\ \sigma, \rho, \psi_1 & \ldots & \sigma, \rho, \psi_k \\ \sigma an, \rho, \chi_1 & & \sigma an, \rho, \chi_k\end{array}}$ let the set of all a-arrow specifications in U be $\{(\psi_1, a, \chi_1), \cdots, (\psi_k, a, \chi_k)\}$

$(\checkmark_2)\ \dfrac{\sigma an, \rho[U, G], \checkmark}{\begin{array}{l}\sigma an, \epsilon, \checkmark \\ \sigma an, \epsilon, \text{✗}\end{array}}$ there are no a-arrow speicifications in U

$(\text{✗}_1)\ \dfrac{\sigma an, \rho[(\psi, a, \chi), G], \text{✗}}{\sigma an, \rho, \text{✗} \mid \sigma, \rho, \neg\psi \mid \sigma an, \rho, \neg\chi}$

$(\text{✗}_2)\ \dfrac{\sigma an, \rho[U, G], \text{✗}}{\begin{array}{l}\sigma an, \rho[(\psi_1, a_1, \chi_1), G], \text{✗} \\ \qquad\vdots \\ \sigma an, \rho[(\psi_k, a_k, \chi_k), G], \text{✗}\end{array}}$ $U = \{(\psi_1, a_1, \chi_1), \cdots, (\psi_k, a_k, \chi_k)\}$ and $k \geq 2$

Table 1: Tableau rules

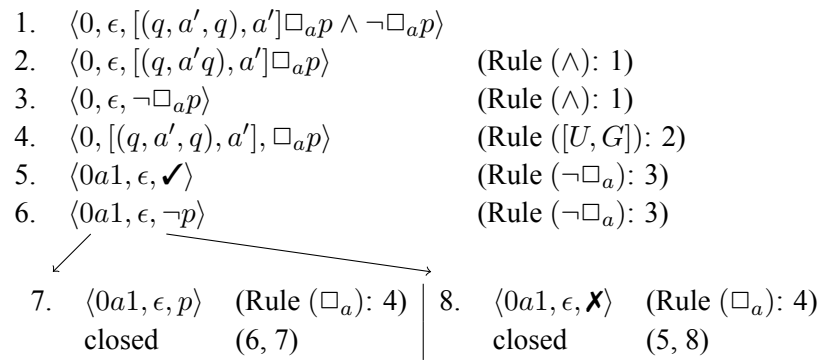


Figure 5: Closed tableau for the formula $[(q, a', q), a']\Box_a p \wedge \neg\Box_a p$

Example 20. In Figure 5, the tableau method is used to show the validity of one instance of the formula of Proposition 6. The rightmost column shows which tableau rule is applied in each line.

Next, we will show the soundness, but first we need another definition.

Definition 21 (Interpretation)**.** Given a Kripke model $\mathcal{M}$ and a branch b, let f be a function from the labels used in b to $W^{\mathcal{M}}$. We say that f is an *interpretation* of b in $\mathcal{M}$ if the following hold.

- $\mathcal{M}, f(\sigma) \vDash_\rho \phi$ for each $\langle \sigma, \rho, \phi\rangle \in b$;
- $(f(\sigma), f(\sigma an)) \in R_a^{\mathcal{M}} * \rho$ for each $\langle \sigma an, \rho, \checkmark\rangle \in b$;
- $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho$ for each $\langle \sigma an, \rho, \boldsymbol{\times}\rangle \in b$.

Proposition 22. *Let f be an interpretation of b in $\mathcal{M}$. If b contains the premise of a rule in Table 1, then f can be extended to be an interpretation of b' for some $b' \in B$ where B is the set of branches generated by applying the rule on b.*

Proof If the rule is $(\neg\neg)$, $(\neg\wedge)$ or $(\wedge)$, it is straightforward. We restrict our attention to the other rules.

1. Rule $(\neg\Box_a)$: The premise of the rule $(\neg\Box_a)$ is $\langle \sigma, \rho, \neg\Box_a\phi\rangle$. Since f in an interpretation of b in $\mathcal{M}$ and $\langle \sigma, \rho, \neg\Box_a\phi\rangle \in b$, we have that $\mathcal{M}, f(\sigma) \vDash_\rho \neg\Box_a\phi$. Let $b' = b \cup \{\langle \sigma an, \rho|_a, \checkmark\rangle, \langle \sigma an, \rho|_a, \neg\phi\rangle\}$ where n is new in b, then we know that $B = \{b'\}$. Since $\mathcal{M}, f(\sigma) \vDash_\rho \neg\Box_a\phi$, it follows that there exists $t \in W^{\mathcal{M}}$ such that $(f(\sigma), t) \in R_a^{\mathcal{M}} * \rho|_a$ and $\mathcal{M}, t \vDash_{\rho|_a} \neg\phi$. Now let the function f' be $f \cup \{\sigma an \mapsto t\}$. We then have $(f'(\sigma), f'(\sigma an)) \in R_a^{\mathcal{M}} * \rho|_a$ and $\mathcal{M}, f'(\sigma an) \vDash_{\rho|_a} \neg\phi$. Therefore, f' is an interpretation of b' in $\mathcal{M}$.
2. Rule $(\Box_a)$: Please note that $B = \{b \cup \{\langle \sigma an, \rho|_a, \phi\rangle\}, b \cup \{\langle \sigma an, \rho|_a, \boldsymbol{\times}\rangle\}\}$. For each label σan which is used in b, we have either $(f(\sigma), f(\sigma an)) \in R_a^{\mathcal{M}} * \rho|_a$

or $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho|_a$. It follows by $\langle \sigma, \rho, \Box_a \phi\rangle \in b$ that $\mathcal{M}, f(\sigma) \vDash_\rho \Box_a \phi$. If $(f(\sigma), f(\sigma an)) \in R_a^{\mathcal{M}} * \rho|_a$ then we have $\mathcal{M}, f(\sigma an) \vDash_{\rho|_a} \phi$. Thus, f is an interpretation of the branch $b \cup \{\langle \sigma an, \rho|_a, \phi\rangle\}$. If $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho|_a$, f is an interpretation of the branch $b \cup \{\langle \sigma an, \rho|_a, \text{✗}\rangle\}$.

3. Rule $(\neg[U, G])$: We have that $B = \{b \cup \{\langle \sigma, \rho, [U, G]\neg\phi\rangle\}\}$. Since f is an interpretation and $\langle \sigma, \rho, \neg[U, G]\phi\rangle \in b$, these follow that $\mathcal{M}, f(\sigma) \vDash_\rho \neg[U, G]\phi$. By semantics, we have that $\mathcal{M}, f(\sigma) \vDash_{\rho[U,G]} \neg\phi$. Therefore, f is an interpretation of $b \cup \{\langle \sigma, \rho[U, G], \neg\phi\rangle\}$.
4. Rule $([U, G])$: We have that $B = \{b \cup \{\langle \sigma, \rho[U, G], \phi\rangle\}\}$.Since $\langle \sigma, \rho, [U, G]\phi\rangle \in b$, this follows that $\mathcal{M}, f(\sigma) \vDash_\rho [U, G]\phi$. By semantics, we have $\mathcal{M}, f(\sigma) \vDash_{\rho[U,G]} \phi$. Therefore, f is an interpretation of $b \cup \{\langle \sigma, \rho[U, G], \phi\rangle\}$.
5. Rule (✓_1): Since the premise of this rule $\langle \sigma an, \rho[U, G], \text{✓}\rangle$ is in b, this follows that $(f(\sigma), f(\sigma an)) \in R_a^{\mathcal{M}} * \rho[U, G]$. By the semantics, we have that $(f(\sigma), f(\sigma an)) \in R_a^{\mathcal{M}} * \rho$ and that there exists $(\psi, a, \chi) \in U$ such that $\mathcal{M}, f(\sigma) \vDash_\rho \psi$ and $\mathcal{M}, f(\sigma an) \vDash_\rho \chi$. Therefore, f is an interpretation of $b \cup \{\langle \sigma an, \rho, \text{✓}\rangle, \langle \sigma, \rho, \psi\rangle, \langle \sigma an, \rho, \chi\rangle\}$, and it is in B.
6. Rule (✓_2): Since f is an interpretation of b, the rule whose premise is in b cannot be (✓_2). If so, we should have $(f(\sigma), f(\sigma an)) \in R_a^{\mathcal{M}} * (\rho[U, G])$. However, since there are no a-arrow specifications in U, by semantics, we have that $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * (\rho[U, G])$.
7. Rule (✗_1): Since $\langle \sigma an, \rho[(\psi, a, \chi), G], \text{✗}\rangle \in b$, this follows that $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho[(\psi, a, \chi), G]$. By semantics, we have either $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho$ or $\mathcal{M}, f(\sigma) \nvDash_\rho \psi$ or $\mathcal{M}, f(\sigma an) \nvDash_\rho \chi$. Therefore, f is an interpretation of at least one branch in $B = \{b \cup \{\langle \sigma an, \rho, \text{✗}\rangle\}, b \cup \{\langle \sigma, \rho, \neg\psi\rangle\}, b \cup \{\langle \sigma an, \rho, \neg\chi\rangle\}\}$.
8. Rule (✗_2): Since $\langle \sigma an, \rho[U, G], \text{✗}\rangle \in b$, this follows that $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho[U, G]$. By semantics, we know that $R_a^{\mathcal{M}} * \rho[U, G] = \bigcup_{(\psi,a,\chi)\in U} R_a^{\mathcal{M}} * \rho[(\psi, a, \chi), G]$. Therefore, we have $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho[(\psi, a, \chi), G]$ for each $(\psi, a, \chi) \in U$. For the specification $(\psi, a', \chi) \in U$ and $a' \neq a$, it follows by semantics that $R_a^{\mathcal{M}} * (\rho[(\psi, a', \chi), G]) = \emptyset$. Therefore, we have $(f(\sigma), f(\sigma an)) \notin R_a^{\mathcal{M}} * \rho[(\psi, a', \chi), G]$ for each $(\psi, a', \chi) \in U$. Thus f is an interpretation of $b \cup \{\langle \sigma an, \rho[(\psi, a', \chi), G], \text{✗}\rangle \mid (\psi, a', \chi) \in U\}$. □

Theorem 23 (Soundness)**.** *If there is a closed tableau for $\neg\phi_0$, then ϕ_0 is valid.*

Proof Let T be the closed tableau for $\neg\phi_0$. Assuming that ϕ_0 is not valid, this follows that $\neg\phi_0$ is satisfiable. Let $\mathcal{M}, s \vDash \neg\phi_0$. Please note that the branch in the initial tableau for $\neg\phi_0$ is the branch $b = \{\langle 0, \epsilon, \neg\phi_0\rangle\}$. Define the function f as $f(0) = s$. This follows that $\mathcal{M}, f(0) \vDash \neg\phi_0$. Therefore, f is an interpretation of b in $\mathcal{M}$. Please note that each branch in T is generated by extended b by applying rules in Table 1. By Proposition 22, this follows that there is some branch $b' \in T$ such that f

can be extended to be an interpretation of b'. This is contradictory with the fact that b' is closed. Therefore, ϕ_0 is valid. □

In the rest of the section, we prove completeness. First, we need another auxiliary definition.

Definition 24 (Saturated tableau)**.** A branch b is saturated iff it is saturated under all tableau rules, as defined below:

1. b is saturated under Rule $(\neg\neg)$ iff $\langle\sigma, \rho, \neg\neg\phi\rangle \in b$ implies $\langle\sigma, \rho, \phi\rangle \in b$;
2. b is saturated under Rule $(\neg\wedge)$ iff $\langle\sigma, \rho, \neg(\phi \wedge \psi)\rangle \in b$ implies $\langle\sigma, \rho, \neg\phi\rangle \in b$ or $\langle\sigma, \rho, \neg\psi\rangle \in b$;
3. b is saturated under Rule $(\wedge)$ iff $\langle\sigma, \rho, (\phi \wedge \psi)\rangle \in b$ implies $\langle\sigma, \rho, \phi\rangle \in b$ and $\langle\sigma, \rho, \psi\rangle \in b$;
4. b is saturated under Rule $(\neg\Box_a)$ iff $\langle\sigma, \rho, \neg\Box_a\phi\rangle \in b$ implies that $\{\langle\sigma an, \rho|_a, \checkmark\rangle, \langle\sigma an, \rho|_a, \neg\phi\rangle\} \subseteq b$ for some $n \in \mathbb{N}$;
5. b is saturated under Rule $(\Box_a)$ iff $\langle\sigma, \rho, \Box_a\phi\rangle \in b$ implies that for each σan occurring in b we have $\langle\sigma an, \rho|_a, \text{✗}\rangle \in b$ or $\langle\sigma an, \rho|_a, \phi\rangle \in b$;
6. b is saturated under Rule $(\neg[U, G])$ iff $\langle\sigma, \rho, \neg[U, G]\phi\rangle \in b$ implies $\langle\sigma, \rho[U, G], \neg\phi\rangle \in b$;
7. b is saturated under Rule $([U, G])$ iff $\langle\sigma, \rho, [U, G]\phi\rangle \in b$ implies $\langle\sigma, \rho[U, G], \phi\rangle \in b$;
8. b is saturated under Rule $(\checkmark_1)$ iff $\langle\sigma an, \rho[U, G], \checkmark\rangle \in b$ implies $\{\langle\sigma an, \rho, \checkmark\rangle, \langle\sigma, \rho, \psi\rangle, \langle\sigma an, \rho, \chi\rangle\} \subseteq b$ for some $(\psi, a, \chi) \in U$;
9. b is saturated under Rule $(\checkmark_2)$ iff $\langle\sigma an, \rho[U, G], \checkmark\rangle \in b$ implies $\{\langle\sigma an, \epsilon, \checkmark\rangle, \langle\sigma, \epsilon, \text{✗}\rangle, \} \subseteq b$;
10. b is saturated under Rule (✗_1) iff $\langle\sigma an, \rho[(\psi, a, \chi), G], \text{✗}\rangle \in b$ implies $\langle\sigma an, \rho, \text{✗}\rangle \in b$ or $\langle\sigma, \rho, \neg\psi\rangle \in b$ or $\langle\sigma an, \rho, \neg\chi\rangle \in b$.
11. b is saturated under Rule (✗_2) iff $\langle\sigma an, \rho[U, G], \text{✗}\rangle \in b$ implies that $\{\langle\sigma an, \rho[(\psi, a', \chi), G], \text{✗}\rangle \mid (\psi, a', \chi) \in U\} \subseteq b$, where there are at least two specifications in U.

We say a tableau is *saturated* iff all its branches are saturated.

The following two propositions are obvious by the tableau rules.

Proposition 25. *Given a saturated tableau T and a branch $b \in T$, if $\langle\sigma an, \rho, \checkmark\rangle \in b$ then $\langle\sigma an, \epsilon, \checkmark\rangle \in b$.*

Proposition 26. *Given a saturated tableau T and a branch $b \in T$, if a label σan occurs in b then $\langle\sigma an, \epsilon, \checkmark\rangle \in b$.*

Definition 27 (Length of term)**.** The length of a formula is defined as follows:

$$
\begin{aligned}
l(p) &= 1 \\
l(\neg\phi) &= l(\phi) + 1 \\
l(\phi \wedge \psi) &= l(\phi) + l(\psi) + 1 \\
l(\Box_a \phi) &= l(\phi) + 1 \\
l([U, G]\phi) &= l(U) + |G| + l(\phi) + 1 \\
l(U) &= \sum\nolimits_{(\psi, a, \chi) \in U} (l(\psi) + l(\chi))
\end{aligned}
$$

The length of an update sequence is defined as follows:

$$l(\epsilon) = 0;\ l(\rho[U, G]) = l(\rho) + l(U) + |G|.$$

The length of a term is defined as follows:

$$l(\rho, ✗) = l(\rho, ✓) = l(\rho);\ l(\rho, \phi) = l(\rho) + l(\phi).$$

Please note that the length of the term (ϵ, ϕ) is the same as the length of ϕ.

Now, we are ready to prove the completeness.

Theorem 28 (completeness)**.** *If ϕ_0 is valid, there is a closed tableau for $\neg\phi_0$.*

Proof We only need to show that if all tableaux for ϕ_0 are open then ϕ_0 is satisfiable. Since each tableau for ϕ_0 can be extended to be saturated and there is at least one tableau for ϕ_0, i.e., the initial tableau, there exists an open and saturated tableau for ϕ_0 if all its tableaux are open.

Let T be an open and saturated tableau for ϕ_0 and b be an open and saturated branch of T. In order to show ϕ_0 is satisfiable, we only need to show that there is an interpretation of b (recall Definition 21). Next we will construct a model $\mathcal{M}^c$ and we will show that there is an interpretation of b in $\mathcal{M}^c$. The model $\mathcal{M}^c = \langle W, R, V \rangle$ is defined as follows.

$$
\begin{aligned}
W &= \{\sigma \mid \sigma \text{ is a label that is used in } b\} \\
R_a &= \{(\sigma, \sigma an) \mid (\sigma an, \epsilon, ✓) \in b\} \text{ for each } a \in \mathbf{Agt} \\
V(p) &= \{\sigma \mid (\sigma, \rho, p) \in b \text{ for some } \rho\}
\end{aligned}
$$

Please note that if σan is used in b then so is σ.

Let I be the function $I(\sigma) = \sigma$. By induction on the length of terms, we will show that I is an interpretation of b in $\mathcal{M}^c$. For abbreviation, we will write $I(\sigma)$ as σ. For the case of $l(\rho, x) = 0$, the term (ρ, x) can only be of the form $(\epsilon, ✗)$ or $(\epsilon, ✓)$. Furthermore, it cannot be of the form $(\epsilon, ✗)$. Assuming $(\sigma an, \epsilon, ✗) \in b$ for some label σan, it follows by Proposition 26 that $(\sigma an, \epsilon, ✓) \in b$, this is in contradiction with that b is open. Therefore, in this case, we only need to show that $(\sigma, \sigma an) \in R_a$ for each σan with $(\sigma an, \epsilon, ✓) \in b$, which is obvious by the definition of the model $\mathcal{M}^c$.

With the inductive hypothesis that each labelled term $(\sigma, \rho, x) \in b$ with $l(\rho, x) < n$ satisfies the conditions declared in Definition 21, we will show that each labelled term $(\sigma, \rho, x) \in b$ with $l(\rho, x) = n$ also satisfies the conditions, where $n \geq 1$.

If x is a formula, there are different cases according to the form of the formula, as below:

1. $(\sigma, \rho, p) \in b$: It is obvious that $\mathcal{M}^c, \sigma \vDash_\rho p$.
2. $(\sigma, \rho, \neg p) \in b$: Assuming $\sigma \in V(p)$, it follows that $(\sigma, \rho', p) \in b$ for some ρ'. This is in contradiction with the assumption that b is open. Therefore, we have $\sigma \notin V(p)$, namely $\mathcal{M}^c, \sigma \vDash_\rho \neg p$.
3. $(\sigma, \rho, \neg\neg\phi) \in b$: Since b is saturated, it follows that $(\sigma, \rho, \phi) \in b$. Since $l(\rho, \phi) < l(\rho, \neg\neg\phi)$, it follows by IH that $\mathcal{M}^c, \sigma \vDash_\rho \phi$. Therefore, we have $\mathcal{M}^c, \sigma \vDash_\rho \neg\neg\phi$.
4. $(\sigma, \rho, \phi \wedge \psi) \in b$: Since b is saturated, it follows that $(\sigma, \rho, \phi) \in b$ and $(\sigma, \rho, \psi) \in b$. Since $l(\rho, \phi), l(\rho, \psi) < l(\rho, \phi \wedge \psi)$, it follows by IH that $\mathcal{M}^c, \sigma \vDash_\rho \phi$ and $\mathcal{M}^c, \sigma \vDash_\rho \psi$. Therefore, we have $\mathcal{M}^c, \sigma \vDash_\rho \phi \wedge \psi$.
5. $(\sigma, \rho, \neg(\phi \wedge \psi)) \in b$: Since b is saturated, it follows that $(\sigma, \rho, \neg\phi) \in b$ or $(\sigma, \rho, \neg\psi) \in b$. Since $l(\rho, \neg\phi), l(\rho, \neg\psi) < l(\rho, \neg(\phi \wedge \psi))$, it follows by IH that $\mathcal{M}^c, \sigma \vDash_\rho \neg\phi$ or $\mathcal{M}^c, \sigma \vDash_\rho \neg\psi$. Therefore, we have $\mathcal{M}^c, \sigma \vDash_\rho \neg(\phi \wedge \psi)$.
6. $(\sigma, \rho, \neg\Box_a\phi) \in b$: Since b is saturated, it follows that $(\sigma an, \rho|_a, \neg\phi) \in b$ and $(\sigma an, \rho|_a, \checkmark) \in b$ for some $n \in \mathbb{N}$. Since $l(\rho|_a, \neg\phi), l(\rho|_a, \checkmark) < l(\rho, \neg\Box_a\phi)$, it follows by IH that $(\sigma, \sigma an) \in R_a * (\rho|_a)$ and $\mathcal{M}^c, \sigma \vDash_{\rho|_a} \neg\phi$. Therefore, we have $\mathcal{M}^c, \sigma \vDash_\rho \neg\Box_a\phi$.
7. $(\sigma, \rho, \Box_a\phi) \in b$: Let $\sigma' \in W$ be a state with $(\sigma, \sigma') \in R_a * (\rho|_a)$. In order to show $\mathcal{M}^c, \sigma \vDash_\rho \Box_a\phi$, we need to show that $\mathcal{M}^c, \sigma' \vDash_{\rho|_a} \phi$. Since $R_a * (\rho|_a) \subseteq R_a$, it follows that $\sigma' = \sigma an$ for some $n \in \mathbb{N}$. Assuming $(\sigma an, \rho|_a, \text{✗}) \in b$, it follows by IH that $(\sigma, \sigma an) \notin R_a * (\rho|_a)$. This is in contradiction with the assumption that $(\sigma, \sigma') \in R_a * (\rho|_a)$. Therefore, we have $(\sigma an, \rho|_a, \text{✗}) \notin b$. Since b is saturated, it follows that $(\sigma an, \rho|_a, \phi) \in b$. It follows by IH that $\mathcal{M}^c, \sigma an \vDash_{\rho|_a} \phi$.
8. $(\sigma, \rho, \neg[U, G]\phi) \in b$: Since b is saturated, it follows that $(\sigma, \rho[U, G], \neg\phi) \in b$. Since $l(\rho[U, G], \neg\phi) < l(\rho, \neg[U, G]\phi)$, it follows by IH that $\mathcal{M}^c, \sigma \vDash_{\rho[U,G]} \neg\phi$. Therefore, we have $\mathcal{M}^c, \sigma \vDash_\rho \neg[U, G]\phi$.
9. $(\sigma, \rho, [U, G]\phi) \in b$: Since b is saturated, it follows that $(\sigma, \rho[U, G], \phi) \in b$. Since $l(\rho[U, G], \phi) < l(\rho, [U, G]\phi)$, it follows by IH that $\mathcal{M}^c, \sigma \vDash_{\rho[U,G]} \phi$. Therefore, we have $\mathcal{M}^c, \sigma \vDash_\rho [U, G]\phi$.

If x in the term (ρ, x) is of the form ✓ or ✗, we have ρ is not ϵ because $l(\rho, x) \geq 1$. There are different cases, as below:

1. $(\sigma an, \rho[U, G], \checkmark) \in b$ and there exists an a-arrow specification in U: Since b is saturated, it follows that $\{\langle \sigma an, \rho, \checkmark\rangle, \langle \sigma, \rho, \psi\rangle, \langle \sigma an, \rho, \chi\rangle\} \subset b$ for some

$(\psi, a, \chi) \in U$. Since $l(\rho, \checkmark), l(\rho, \psi), l(\rho, \chi) < l(\rho[U, G], \checkmark)$, it follows by IH that $(\sigma, \sigma an) \in R_a * \rho$, $\mathcal{M}^c, \sigma \vDash_\rho \psi$ and $\mathcal{M}^c, \sigma an \vDash_\rho \chi$. It follows that $(\sigma, \sigma an) \in R_a * (\rho[U, G])$.

2. $(\sigma an, \rho[U, G], \checkmark) \in b$ and there are no a-arrow specifications in U: Due to Rule ($\checkmark_2$) and the fact that b is open and saturated, this case is impossible.
3. $(\sigma an, \rho[(\psi, a', \chi), G], \text{✗}) \in b$: If $a' \neq a$, it follows that $R_a * (\rho[(\psi, a', \chi), G]) = \emptyset$. It is obvious $(\sigma, \sigma an) \notin R_a * (\rho[(\psi, a', \chi), G])$. If $a' = a$, it follows by Rule (✗_1) that $\langle \sigma an, \rho, \text{✗} \rangle \in b$, or $\langle \sigma, \rho, \neg\psi \rangle \in b$, or $\langle \sigma an, \rho, \neg\chi \rangle \in b$. Since $l(\rho, \text{✗}), l(\rho, \neg\psi), l(\rho, \neg\chi) < l(\rho[(\psi, a, \chi), G], \text{✗})$, it follows by IH that $(\sigma, \sigma an) \notin R_a * \rho$, or $\mathcal{M}^c, \sigma \vDash_\rho \neg\psi$, or $\mathcal{M}^c, \sigma an \vDash_\rho \neg\chi$. Each of them can derive that $(\sigma, \sigma an) \notin R_a * (\rho[(\psi, a, \chi), G])$.
4. $(\sigma an, \rho[U, G], \text{✗}) \in b$ and $|U| \geq 2$: If there are no a-arrow specifications in U, it is obvious that $(\sigma, \sigma an) \notin R_a * (\rho[U, G])$ since $R_a * (\rho[U, G]) = \emptyset$. Otherwise, let $(\psi_1, a, \chi_1), \cdots, (\psi_k, a, \chi_k)$ be all the a-arrow specifications in U. Since b is saturated, it follows by Rule (✗_2) that $\langle \sigma an, \rho[(\psi_i, a, \chi_i), G], \text{✗} \rangle \in b$ for all $1 \leq i \leq k$. Since $l(\rho[(\psi_i, a, \chi_i), G], \text{✗}) < l([U, G], \text{✗})$ for all $1 \leq i \leq k$ due to $|U| \geq 2$, it follows by IH that $(\sigma, \sigma an) \notin R_a * (\rho[(\psi_i, a, \chi_i), G])$ for all $1 \leq i \leq k$. Since $R_a * (\rho[U, G]) = \bigcup_{1 \leq i \leq k} R_a * (\rho[(\psi_i, a, \chi_i), G])$, we have $(\sigma, \sigma an) \notin R_a * (\rho[U, G])$.

We have shown that all labelled terms in b satisfy the conditions declared in Definition 21. Since $\langle 0, \epsilon, \phi_0 \rangle \in b$, thus we have $\mathcal{M}^c, 0 \vDash \phi_0$. □

4 Decidability

In this section, we will show that PAUL is decidable, that is, the problem whether an PAUL formula ϕ is satisfiable can be answered in a finite number of steps. Please note that Theorem 9 already tells us that each formula ϕ in PAUL can be equivalently reduced to be a formula with out updates, i.e., a formula in normal modal logic K. We can see that PAUL is the same with K if it is confined on formulas with out updates. Since K is decidable, this follows that PAUL is decidable. However, as it is shown in [15], the equivalent translation from PAL to K might be exponential. Since PAL can be polynomially translated into PAUL by replacing the PAL operator $[\phi]$ with $[U, \mathbf{Agt}]$ where $U = \{(\phi, a, \phi) \mid a \in \mathbf{Agt} \text{ occurs in } \phi\}$, the equivalent translation from PAUL to K might be exponential too. In this section, we will directly prove the decidability of PAUL, based on the tableau system presented in the previous section.

Our method is to show that PAUL has small model property. We will show that each satisfiable PAUL formula ϕ has a bounded small model in which ϕ is true. From the proof of Theorem 28, we have seen that we can construct a model for ϕ based on a saturated open branch if ϕ is satisfiable, and each state in the model is exactly a label used in the branch. Therefore, the key is to show that there are only finitely

many labels used in the tableau branch.

For the commonly used tableau calculus for normal modal logic, each formula occurring in the tableau is a subformula of the destination formula, and this feature plays an important role to show the decidability of normal modal logic through tableau method. Similarly, we will define the notation of subterm here, and we will show that all terms occurring in the tableau are subterms.

Definition 29 (Subterm)**.** Given a term (ρ, x), the set of subterm of (ρ, x), denoted as $sub(\rho, x)$, is defined as below.

$$\begin{aligned}
sub(\epsilon, \text{✗/✓}) &= \{(\epsilon, \text{✗/✓})\} \\
sub(\rho[(\psi, a, \chi), G], \text{✗/✓}) &= \{(\rho[(\psi, a, \chi), G], \text{✗/✓})\} \cup sub(\rho, \psi) \cup sub(\rho, \chi) \\
sub(\rho[U, G], \text{✗/✓}) &= \{(\rho[U, G], \text{✗/✓})\} \\
&\quad \cup \bigcup_{(\psi,a,\chi)\in U} sub(\rho[(\psi, a, \chi), G], \text{✗/✓}), \text{where } |U| \geq 2 \\
sub(\rho, p) &= \{(\rho, p)\} \cup sub(\rho, \text{✗}) \cup sub(\rho, \text{✓}) \\
sub(\rho, \neg\phi) &= \{(\rho, \neg\phi)\} \cup sub(\rho, \phi) \\
sub(\rho, \phi \wedge \psi) &= \{(\rho, \phi \wedge \psi)\} \cup sub(\rho, \phi) \cup sub(\rho, \psi) \\
sub(\rho, \Box_a\phi) &= \{(\rho, \Box_a\phi)\} \cup sub(\rho|_a, \phi) \\
sub(\rho, [U, G]\phi) &= \{(\rho, [U, G]\phi)\} \cup sub(\rho[U, G], \phi)
\end{aligned}$$

Let $sub^+(\rho, x)$ be the set $\{(\rho, \neg\phi) \mid (\rho, \phi) \in sub(\rho, x)\} \cup sub(\rho, x)$.

The following proposition states some properties of the subterm set.

Proposition 30. *We have the following results.*

- $sub(\rho, x)$ *is finite;*
- $(\rho, \text{✗/✓}) \in sub(\rho, \phi)$*;*
- $(\rho, x) \in sub(\rho', x')$ *implies* $sub(\rho, x) \subseteq sub(\rho', x')$.

Proposition 31. *Let T be a tableau for ϕ_0 and b be a branch of T. If $(\sigma, \rho, x) \in b$ then $(\rho, x) \in sub^+(\epsilon, \phi_0)$.*

Proof According to Definition 17, we prove this by induction on the process of construction of T. For the initial tableau $\{\{(0, \epsilon, \phi_0)\}\}$, it is obvious. Next, we only need to show that all the tableau rules in Table 1 preserve the subterm property. The cases of the rules $(\neg\neg)$, $(\neg\wedge)$, $(\wedge)$, $([U, G])$ and (✗_2) are obvious; we will restrict our attention to the other rules.

1. Rule $(\neg\Box_a)$: If $(\rho, \neg\Box_a\phi) \in sub^+(\epsilon, \phi_0)$, then we have $(\rho, \Box_a\phi) \in sub(\epsilon, \phi_0)$. Because $(\rho|_a, \phi) \in sub(\rho, \Box_a\phi)$, it follows by Proposition 30 that $(\rho|_a, \text{✓})$, $(\rho|_a, \phi) \in sub^+(\epsilon, \phi_0)$.

2. Rule $(\Box_a)$: If $(\rho, \Box_a\phi) \in sub^+(\epsilon, \phi_0)$, then we have $(\rho, \Box_a\phi) \in sub(\epsilon, \phi_0)$. Because $(\rho|_a, \phi) \in sub(\rho, \Box_a\phi)$, it follows by Proposition 30 that $(\rho|_a, \text{✗})$, $(\rho|_a, \phi) \in sub^+(\epsilon, \phi_0)$.
3. Rule $(\neg[U, G])$: If $(\rho, \neg[U, G]\phi) \in sub^+(\epsilon, \phi_0)$, then we have $(\rho, [U, G]\phi) \in sub(\epsilon, \phi_0)$. Since $(\rho[U, G], \phi) \in sub(\rho, [U, G]\phi)$, it follows by Proposition 30 that $(\rho[U, G], \phi) \in sub(\rho, \phi_0)$. Therefore, we have $(\rho[U, G], \neg\phi) \in sub^+(\rho, \phi_0)$.
4. Rule (✓_1): If $(\rho[U, G], \text{✓}) \in sub^+(\epsilon, \phi_0)$ then $(\rho[U, G], \text{✓}) \in sub(\epsilon, \phi_0)$. Let $(\psi, a, \chi) \in U$. We have $(\rho[(\psi, a, \chi), G], \text{✓}) \in sub(\epsilon, \phi_0)$. Since (ρ, ψ), $(\rho, \chi) \in sub(\rho[(\psi, a, \chi), G], \text{✓})$, we have (ρ, ψ), $(\rho, \chi) \in sub(\epsilon, \phi_0)$. It follows by Proposition 30 that $(\rho, \text{✓}) \in sub(\rho, \psi)$, thus we also have $(\rho, \text{✓}) \in sub(\rho, \phi_0)$.
5. Rule (✓_2): It follows by Proposition 30 that $(\epsilon, \text{✓}), (\epsilon, \text{✗}) \in sub(\epsilon, \phi_0)$.
6. Rule (✗_1): If $(\rho[(\psi, a, \chi), G], \text{✗}) \in sub^+(\epsilon, \phi_0)$, then $(\rho[(\psi, a, \chi), G], \text{✗}) \in sub(\epsilon, \phi_0)$. Since (ρ, ψ), $(\rho, \chi) \in sub(\rho[(\psi, a, \chi), G], \text{✗})$, we have (ρ, ψ), $(\rho, \chi) \in sub(\epsilon, \phi_0)$. Therefore, we have $(\rho, \neg\psi)$, $(\rho, \neg\chi) \in sub^+(\epsilon, \phi_0)$. It follows by Proposition 30 that $(\rho, \text{✗}) \in sub(\rho, \psi)$, thus we also have $(\rho, \text{✗}) \in sub(\rho, \phi_0)$. □

Proposition 32. *Let T be a tableau for ϕ_0, and let b be a branch of T. If σ is a label present in b, then there are at most k labels present in b with the form of σan for some $n \in \mathbb{N}$, where $k = |sub^+(\epsilon, \phi_0)|$.*

Proof It follows by Definition 17 that each label σan present in b is generated by applying the rule $(\neg\Box_a)$ to a labelled term $(\sigma, \rho, \neg\Box_a\phi) \in b$. According to Proposition 31, there are at most k terms labelled with σ in b. Therefore, there are at most k labels present in b with the form of σan for some $n \in \mathbb{N}$. □

Definition 33 (Length of label)**.** The length of a label σ, denoted by $|\sigma|$, is defined by induction on σ: $|n| = 0$; $|\sigma an| = |\sigma| + 1$.

Proposition 34. *Let T be a tableau for ϕ_0 and b be a branch of T. If $(\sigma, \rho, x) \in b$ then $|\sigma| \leq l(\phi_0) - l(\rho, x)$.*

Proof Following Definition 17, the proof is by induction on the process of construction of T. For the initial tableau $\{\{(0, \epsilon, \phi_0)\}\}$, it is obvious. Next we will show that this property is preserved by all the tableau rules. The cases of the rules $(\neg\neg)$, $(\neg\wedge)$, $(\wedge)$ and (✓_2) are obvious; we will restrict our attention to the other rules.

1. Rule $(\neg\Box_a)$: If $|\sigma| \leq l(\phi_0) - l(\rho, \neg\Box_a\phi)$, we have $l(\phi_0) - l(\rho, \neg\Box_a\phi) \leq l(\phi_0) - l(\rho|_a, \neg\phi) - 1$ because $l(\rho, \neg\Box_a\phi) \geq l(\rho|_a, \neg\phi) + 1$. Thus we have $|\sigma| \leq l(\phi_0) - l(\rho|_a, \neg\phi) - 1$. It follows that $|\sigma an| \leq l(\phi_0) - l(\rho|_a, \neg\phi)$. What is more, since $l(\rho, \neg\Box_a\phi) \geq l(\rho|_a, \text{✓}) + 1$, we have $l(\phi_0) - l(\rho, \neg\Box_a\phi) \leq l(\phi_0) - l(\rho|_a, \text{✓}) - 1$. It follows $|\sigma| \leq l(\phi_0) - l(\rho|_a, \text{✓}) - 1$. Thus we have $|\sigma an| \leq l(\phi_0) - l(\rho|_a, \text{✓})$.

2. Rule $(\Box_a)$: Suppose $|\sigma| \leq l(\phi_0) - l(\rho, \Box_a\phi)$, we have $l(\phi_0) - l(\rho, \Box_a\phi) \leq l(\phi_0) - l(\rho|_a, \phi) - 1$ because $l(\rho, \Box_a\phi) \geq l(\rho|_a, \phi) + 1$. Therefore, we have $|\sigma an| \leq l(\phi_0) - l(\rho|_a, \phi)$. What is more, since $l(\rho|_a, \phi) \geq l(\rho|_a, \text{✗})$, we have $|\sigma an| \leq l(\phi_0) - l(\rho|_a, \text{✗})$.
3. Rule $(\neg[U, G])$: If $|\sigma| \leq l(\phi_0) - l(\rho, \neg[U, G]\phi)$, we have $l(\phi_0) - l(\rho, \neg[U, G]\phi) \leq l(\phi_0) - l(\rho[U, G], \neg\phi)$ because $l(\rho, \neg[U, G]\phi) = l(\rho[U, G], \neg\phi) + 1$. Therefore, we have $|\sigma| \leq l(\phi_0) - l(\rho[U, G], \neg\phi)$.
4. Rule $([U, G])$: Since $l(\rho, [U, G]\phi) = l(\rho[U, G], \phi) + 1$, if $|\sigma| \leq l(\phi_0) - l(\rho, [U, G]\phi)$, we have $|\sigma| \leq l(\phi_0) - l(\rho[U, G], \phi)$.
5. Rule (✓_1): Assume $|\sigma an| \leq l(\phi_0) - l(\rho[U, G], \text{✓})$. Since $l(\rho[U, G], \text{✓}) \geq l(\rho, \text{✓})$, it follows that $|\sigma an| \leq l(\phi_0) - l(\rho, \text{✓})$. Let $(\psi, a, \chi) \in U$. We have $l(\rho[U, G], \text{✓}) \geq l(\rho, \psi) - 1$ because $l(\rho[U, G], \text{✓}) \geq l(\rho, \psi)$. It follows that $l(\phi_0) - l(\rho[U, G], \text{✓}) \leq l(\phi_0) - l(\rho, \psi) + 1$. Thus we have $|\sigma an| \leq l(\phi_0) - l(\rho, \psi) + 1$. It follows that $|\sigma| \leq l(\phi_0) - l(\rho, \psi)$.
   What is more, since $l(\rho[U, G], \text{✓}) \geq l(\rho, \chi)$, it follows that $l(\phi_0) - l(\rho[U, G], \text{✓}) \leq l(\phi_0) - l(\rho, \chi)$. Thus we have $|\sigma an| \leq l(\phi_0) - l(\rho, \chi)$.
6. Rule (✗_1): Assume that $|\sigma an| \leq l(\phi_0) - l(\rho[(\psi, a, \chi), G], \text{✗})$. Since we have $l(\rho[(\psi, a, \chi), G], \text{✗}) \geq l(\rho, \text{✗})$, it follows that $|\sigma an| \leq l(\phi_0) - l(\rho, \text{✗})$.
   What is more, since $l(\rho[(\psi, a, \chi), G], \text{✗}) \geq l(\rho, \neg\psi)$ and $l(\rho[(\psi, a, \chi), G], \text{✗}) \geq l(\rho, \neg\chi)$, it follows that $l(\phi_0) - l(\rho[(\psi, a, \chi), G], \text{✗}) \leq l(\phi_0) - l(\rho, \neg\psi)$ and $l(\phi_0) - l(\rho[(\psi, a, \chi), G], \text{✗}) \leq l(\phi_0) - l(\rho, \neg\chi)$. Therefore, we have $|\sigma an| \leq l(\phi_0) - l(\rho, \neg\psi)$ and $|\sigma an| \leq l(\phi_0) - l(\rho, \neg\psi)$. Since $|\sigma| \leq |\sigma an|$, it is obvious $|\sigma| \leq l(\phi_0) - l(\rho, \neg\psi)$.
7. Rule (✗_2): Assume $|\sigma an| \leq l(\phi_0) - l(\rho[U, G], \text{✗})$ and $|U| \geq 2$. Suppose that $(\psi, a', \chi) \in U$, we have $l(\rho[U, G], \text{✗}) \geq l(\rho[(\psi, a', \chi), G], \text{✗})$. It follows that $l(\phi_0) - l(\rho[U, G], \text{✗}) \leq l(\phi_0) - l(\rho[(\psi, a', \chi), G], \text{✗})$. Thus we have $|\sigma an| \leq l(\phi_0) - l(\rho[(\psi, a', \chi), G], \text{✗})$. □

Lemma 1 (Small model property)**.** *If ϕ_0 is satisfiable then ϕ_0 is satisfiable in a model which is bounded by $k^{O(m)}$, where $k = |sub^+(\epsilon, \phi_0)|$ and $m = l(\phi_0)$.*

Proof It follows by Theorem 23 that all tableaux for ϕ_0 are open. According to the proof of Theorem 28, we can construct a model $\mathcal{M}^c$ from a saturated branch b such that ϕ_0 is satisfied in $\mathcal{M}^c$. By the definition of $\mathcal{M}^c$, we know that each state in $\mathcal{M}^c$ is a label present in b. Please note that all labels present in b form a tree. It follows by Proposition 32 that each label in the tree has at most k children. It follows by Proposition 34 that the depth of the tree is bounded by m. Therefore, there are at most $k^{O(m)}$ labels used in b. □

Theorem 35 (Decidability)**.** *The problem whether ϕ_0 is satisfiable is decidable.*

Proof It follows by Lemma 1 that we only need to check all the models no bigger than $k^{O(m)}$ where $k = |sub^+(\epsilon, \phi_0)|$ and $m = l(\phi_0)$, and this procedure can terminate in finitely many steps. □

5 Conclusion

This paper presented the framework of Private Arrow Update Logic (PAUL), which extends the arrow update of AUL with a relativized subgroup of agents. Public, private and semi-private announcements can be modeled in this framework. PAUL still is a particular case of GAUL, since some information change, like cheating, cannot be modeled in PAUL. This paper also provided a sound and complete tableau method of PAUL and showed that PAUL is decidable.

For future research, we can try to give an optimal algorithm for the satisfiability problem of PAUL by taking a depth-first search strategy on the tableau method. Since the normal modal logic K is a fragment of PAUL and K is PSPACE-complete, PAUL is at least PSPACE-hard. With an optimal search algorithm on the tableau, we conjecture that there might be a PSPACE upper bound for PAUL. What is more, since each AUL formula can be equivalently translated into a PAUL formula by replacing the update $[U]$ by $[U, \mathbf{Agt}]$, the tableau method presented in this paper can apply to AUL. Therefore, the optimal algorithm for PAUL (if there is one) will also be an algorithm for AUL and might also be optimal.

One direction for future study is to see how frame conditions are handled in arrow update logic. For example, if the original model is based on a symmetric frame, we might ask the symmetry is preserved after update. This could be done by asking the update U to satisfy some conditions. If the update U satisfies that $(\phi, a, \psi) \in U$ implies $(\psi, a, \phi) \in U$, then symmetry would be preserved. If $(\phi, a, \psi), (\psi, a, \chi) \in U$ implies $(\phi, a, \chi) \in U$, then transitivity would be preserved. The difficulty lies in how to preserve reflexivity. We doubt it might not be solved if we only confine the form of the update U.

Another direction for future research is to use PAUL to model the information change in logics of knowing how (cf.[19, 9]). The main feature of Arrow Update Logic is that it updates information but does not eliminate states. This makes it more suitable for modeling information update in knowing how. For example, a doctor may not know how to treat a patient since the only two available medicines a_1 and a_2 may cause some very bad side-effect. That is, there is an a_1-arrow and an a_2-arrow from the current state to the bad side-effect state. If the information is updated, for example, a new scientific discovery shows that a_1 will not cause the bad effect, then the doctor should know how to treat the patient. This kind of information update will eliminate arrows but not states.

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动态逻辑 AUL 中的秘密宣告

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摘　要

AUL (Arrow Update Logic) 是一个刻画多主体的信念变化的动态逻辑。AUL 通过更新模型中的可及关系来刻画行为对主体信念造成的影响。但是，在 AUL 中信息对所有主体都是公开的，因此 AUL 无法刻画在秘密宣告的情况下主体信念状态的变化。本论文在 AUL 的基础上进行扩充，得到一个新的动态逻辑系统 PAUL (Private Arrow Update Logic)。所有基于事实的宣告都可以被 PAUL 刻画，无论该宣告是公开的还是秘密的。同时，本论文还给出了 PAUL 的语义图，并证明了 PAUL 是可判定的。

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