On Dimensions, Standard Part Maps,
and p-Adically Closed Fields*

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Abstract.   The aim of this paper is to study the dimensions and standard part maps between the field of p-adic numbers $\mathbb{Q}_p$ and its elementary extension K in the language of rings L_r. We show that for any K-definable set $X \subseteq K^m$, $\dim_K(X) \geq \dim_{\mathbb{Q}_p}(X \cap \mathbb{Q}_p^m)$. Let $V \subseteq K$ be convex hull of K over $\mathbb{Q}_p$, and st $: V \to \mathbb{Q}_p$ be the standard part map. We show that for any K-definable function $f : K^m \to K$, there is definable subset $D \subseteq \mathbb{Q}_p^m$ such that $\mathbb{Q}_p^m \backslash D$ has no interior, and for all $x \in D$, either $f(x) \in V$ and $\mathrm{st}(f(\mathrm{st}^{-1}(x)))$ is constant, or $f(\mathrm{st}^{-1}(x)) \cap V = \emptyset$. We also prove that $\dim_K(X) \geq \dim_{\mathbb{Q}_p}(\mathrm{st}(X \cap V^m))$ for every definable $X \subseteq K^m$.

1   Introduction

In [5], L. van den Dries consider a pair (R, V), where R is an o-minimal extension of a real closed field, and V is a convex hull of an elementary submodel M of R. Let $\mu \subseteq R$ be the set infinitesimals over M and $\hat{V} = V/\mu$ be the reside field with residue class map $x \mapsto \hat{x}$. If M is Dedekind complete in R, then $\hat{V} = M$ and the residue class map coincide the standard part map st $: R \longrightarrow M$. In this context, van den Dries showed the follows:

Theorem 1 ([5]). *Let $S \subseteq R^n$ be R-definable and $\hat{S} = \{\hat{x} | \ x \in S \cap V^n\}$. Then*

  (i) *$S \cap M^n$ is definable in M and $\dim_M(S \cap M^n) \leq \dim_R(S)$;*
  (ii) *$\mathrm{st}(S)$ is definable in M and $\dim_M(\mathrm{st}(S)) \leq \dim_R(S)$.*

Theorem 2 ([5]). *Let $f : R^m \to R$ be an R-definable function. Then here is a finite partition $\mathcal{P}$ of M^m into definable sets, where each set in the partition is either open in M^m or lacks of interior. On each open set $C \in \mathcal{P}$ we have:*

  (i) *either $f(x) \notin V$ for all $x \in C^h$;*
  (ii) *or there is a continuous function $g : C \longrightarrow M$, definable in M, such that $f(x) \in V$ and $\mathrm{st}(f(x)) = g(\mathrm{st}(x))$, for all $x \in C^h$,*

where C^h is the hull of C defined by

$$C^h = \{\bar{x} \in R^m | \exists \bar{y} \in C\big(\bigwedge_{i=1}^{m}(x_i - y_i \in \mu)\big)\}.$$

Remark 1.1. *For any topological space Y, and $X \subseteq Y$, by $\mathrm{Int}(X)$ we mean the set of interiors in X. Namely, $x \in \mathrm{Int}(X)$ **iff** there is an open neighborhood $B \subseteq Y$ of x contained in X.*

There are fairly good analogies between the field of reals $\mathbb{R}$ and the field of p-adic numbers $\mathbb{Q}_p$, in both model-theoretic and field-theoretic view. For example, both of them are complete and locally compact topological fields, are distal and dp-minimal structures, have quantifier eliminations with adding the new predicates for n-th power, and have cell decompositions.

In this paper, we treat the p-adic analogue of above two Theorems, where M is replaced by $\mathbb{Q}_p$, and R is replaced by an arbitrary elementary extension K of $\mathbb{Q}_p$. In our case, the convex hull V is the set

$$\left\{x \in K \mid x = 0 \vee \exists n \in \mathbb{Z}\Big(v(x) > n\Big)\right\}$$

and μ, the infinitesimals of K over $\mathbb{Q}_p$, is the set

$$\left\{x \in K \mid x = 0 \vee \forall n \in \mathbb{Z}\Big(v(x) > n\Big)\right\}.$$

By [12, Lemma 2.1], for every $x \in V$, there is a unique element $\mathrm{st}(x)$ in $\mathbb{Q}_p$ such that $a - \mathrm{st}(a) \in \mu$, we call it the standard part of a and $\mathrm{st} : a \mapsto \mathrm{st}(a)$ the standard part map. It is easy to see that $\mathrm{st} : V \longrightarrow \mathbb{Q}_p$ is a surjective ring homomorphism and $\mathrm{st}^{-1}(0) = \mu$. So $\hat{V} = V/\mu$ is isomorphic to $\mathbb{Q}_p$ in our context. With the notations as above, we now highlight our main results.

Theorem 1.2. *Let $S \subseteq K^n$ be K-definable. Then*

  (i) *$S \cap \mathbb{Q}_p^n$ is definable in $\mathbb{Q}_p$ and $\dim_{\mathbb{Q}_p}(S \cap \mathbb{Q}_p^n) \leq \dim_K(S)$;*
  (ii) *$\mathrm{st}(S \cap V^n)$ is definable in $\mathbb{Q}_p$ and $\dim_{\mathbb{Q}_p}(\mathrm{st}(S \cap V^n)) \leq \dim_K(S)$.*

Theorem 1.3. *Let $f : K^m \to K$ be a K-definable function. Then here is a finite partition $\mathcal{P}$ of $\mathbb{Q}_p$ into definable sets, where each set in the partition is either open in $\mathbb{Q}_p^m$ or lacks of interior. On each open set $C \in \mathcal{P}$ we have:*

  (i) *either $f(x) \notin V$ for all $x \in C^h$;*
  (ii) *or there is a continuous function $g : C \longrightarrow \mathbb{Q}_p$, definable in $\mathbb{Q}_p$, such that $f(x) \in V$ and $\mathrm{st}(f(x)) = g(\mathrm{st}(x))$, for all $x \in C^h$.*

In the rest of this introduction we give more notations and model-theoretic approach.

1.1 Notations

Let p be a fixed prime number, $\mathbb{Q}_p$ the field the p-adic field, and $v : \mathbb{Q}_p\backslash\{0\} \to \mathbb{Z}$ the valuation map. Let K be a fixed elementary extension of $\mathbb{Q}_p$. Then valuation v extends to a valuation map from $K\backslash\{0\}$ to Γ_K, we also denote it by v, where $(\Gamma_K, +, <, 0)$ is the corresponding elementary extension of $(\mathbb{Z}, +, <, 0)$.

Fact 1.4. *Let $v : K\backslash\{0\} \longrightarrow \Gamma_K$ be as above. Then we have*

- $v(xy) = v(x) + v(y)$ *for all* $x, y \in K$;
- $v(x + y) \geq min\{v(x), v(y)\}$, *and* $v(x + y) = min\{v(x), v(y)\}$ *if* $v(x) \neq v(y)$;
- *For* $x \in \mathbb{Q}_p$, $|x| = p^{-v(x)}$ *if* $x \neq 0$ *and* $|x| = 0$ *if* $x = 0$ *defines a non-archimedean metric on* $\mathbb{Q}_p$.
- K *is a non–archimedean topological field.*

We will assume a basic knowledge of model theory. Good references are [13] and [10]. We will be referring a lot to the comprehensive survey [1] for the basic model theory of the p-adics. A key point is Macintyre's theorem [9] that $\mathrm{Th}(\mathbb{Q}_p, +, \times, 0, 1)$ has quantifier elimination in the language $L = L_r \cup \{P_n \mid n \in \mathbb{N}^+\}$, where L_r is the language of rings, and the predicate P_n is interpreted as the n-th powers

$$\{y \in \mathbb{Q}_p \mid y \neq 0 \wedge \exists x(y = x^n)\}$$

for each $n \in \mathbb{N}^+$. Note that P_n is definable in L_r. Moreover, the valuation is definable in L_r as follows.

Fact 1.5 ([3]). *Let $f, g \in K[x_1, \ldots, x_m]$. Then $\{\bar{a} \in K^m \mid v(f(\bar{a})) \leq v(g(\bar{a}))\}$ is definable.*

Remark 1.6. *It is easy to see from Fact 1.5 that $\{a \in K | v(a) = \gamma\}$ and $\{a \in K | v(a) < \gamma\}$ are definable for any fixed $\gamma \in \Gamma_K$.*

For A a subset of K, by an L_A-formula we mean a formula with parameters from A. By $\bar{x}, \bar{y}, \bar{z}$ we mean arbitrary n-variables and $\bar{a}, \bar{b}, \bar{c} \in K^n$ denote n-tuples in K^n with $n \in \mathbb{N}^+$. By $|\bar{x}|$, we mean the length of the tuple $\bar{x}$. We say that $X \subseteq K^m$ is A-definable if there is a L_A-formula $\phi(x_1, \ldots, x_m)$ such that

$$X = \{(a_1, \ldots, a_m) \in K^m \mid K \models \phi(a_1, \ldots, a_m)\}.$$

We also denote X by $\phi(K^m)$ and say that X is defined by $\phi(\bar{x})$. We say that X is definable in K if $X \subseteq K^m$ is K-definable. If $X \in \mathbb{Q}_p^m$ is defined by some $L_{\mathbb{Q}_p}$-formula $\psi(\bar{x})$. Then by $X(K)$ we mean $\psi(K^m)$, namely, the realizations of ψ in K, which is a definable subset of K^m.

For any subset A of K, by $\mathrm{acl}(A)$ we mean the algebraic closure of A. Namely, $b \in \mathrm{acl}(A)$ if and only if there is a formula $\phi(x)$ with parameters from A such that

$b \in \phi(K)$ and $\phi(K)$ is finite. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in K^m$, we denote $\mathrm{acl}(A \cup \{\alpha_1, \ldots, \alpha_n\})$ by $\mathrm{acl}(A, \alpha)$.

By a saturated extension $\mathbb{K}$ of K, we mean that $|\mathbb{K}|$ is a sufficiently large cardinality, and every type over $A \subseteq \mathbb{K}$ is realized in $\mathbb{K}$ whenever $|A| < |\mathbb{K}|$.

1.2   Preliminaries

The p-adic field $\mathbb{Q}_p$ is a complete, locally compact topological field, with basis given by the sets

$$B_{(a,n)} = \{x \in \mathbb{Q}_p \mid x = a \wedge v(x - a) \geq n\}$$

for $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. The elementary extension $K \succ \mathbb{Q}_p$ is also a topological field but not need to be complete or locally compact. Let $X \subseteq K^m$, we say that $\bar{a} \in X$ is an interior if there is $\gamma \in \Gamma_k$ such that

$$B_{(\bar{b},\gamma)} = \{(b_1, \ldots, b_m) \in \mathbb{K} \mid \bigwedge_{i=1}^{n} v(a_i - b_i) > \gamma\} \subseteq X.$$

Let $V = \{x \in K \mid x = 0 \vee \exists n \in \mathbb{Z}(v(x) > n)\}$. We call V the convex hull of $\mathbb{Q}_p$. It is easy to see that for any $a, b \in K$, if $b \in V$ and $v(a) > v(b)$, then $a \in V$. As we said before, for every $a \in V$, there is a unique $a_0 \in \mathbb{Q}_p$ such that $v(a - a_0) > n$ for all $n \in \mathbb{Z}$. This gives a map $a \mapsto a_0$ from V onto $\mathbb{Q}_p$. We call this map the standard part map, denoted by $\mathrm{st} : V \longrightarrow \mathbb{Q}_p$. For any $\bar{a} = (a_1, \ldots, a_m) \in V^m$, by $\mathrm{st}(\bar{a})$ we mean $(\mathrm{st}(a_1), \ldots, \mathrm{st}(a_m))$. Let $f(\bar{x}) \in K[\bar{x}]$ be a polynomial with every coefficient contained in V. Then by $\mathrm{st}(f)$, we mean the polynomial over $\mathbb{Q}_p$ obtained by replace each coefficient of f by its standard part. Let

$$\mu = \{a \in K \mid \mathrm{st}(a) = 0\},$$

which is the collection of all infinitesimals of K over $\mathbb{Q}_p$. It is easy to see that for any $a \in K \setminus \{0\}$, $a \notin V$ iff $a^{-1} \in \mu$.

Any definable subset $X \subseteq K^n$ has a topological dimension which is defined as follows:

Definition 1.7. Let $X \subseteq K^n$. By $\dim_K(X)$, we mean the maximal $k \leq n$ such that the image of the projection

$$\pi : X \longrightarrow K^k; \ (x_1, \ldots, x_n) \mapsto (x_{r_1}, \ldots, x_{r_k})$$

has interiors, for suitable $1 \leq r_1 < \ldots < r_k \leq n$. We call $\dim_K(X)$ the topological dimension of X.

Recall that $\mathbb{Q}_p$ is a geometry structure (see [8, Def. 2.1, Prop. 2.11]), so any $K \models$ $\mathrm{Th}(\mathbb{Q}_p)$ is a geometry structure. The fields has geometric structure are certain fields in which model-theoretic algebraic closure equals field-theoretic algebraic closure.

Every geometry structure is a pregeometry structure, which means that for any $\bar{a} = (a_1, \dots, a_n) \in K^n$ and $A \subseteq K$, $\dim(\bar{a}/A)$ makes sense, which by definition is the maximal k such that $a_{r_1} \notin \mathrm{acl}(A)$ and $a_{r_{i+1}} \notin \mathrm{acl}(A, a_{r_1}, \dots, a_{r_i})$ for some subtuple $(a_{r_1}, \dots, a_{r_k})$ of $\bar{a}$. We call $\dim(\bar{a}/A)$ the algebraic dimension of $\bar{a}$ over A.

Fact 1.8 ([8]). *Let A be a subset of K and X an A-definable subset of K^m.*

(i) *If $\bar{a} \in K^m$ and $\bar{b} \in K^n$. Then we have*

$$\dim(\bar{a}, \bar{b}/A) = \dim(\bar{a}/A, \bar{b}) + \dim(\bar{b}/A) = \dim(\bar{b}, \bar{a}/A).$$

(ii) *Let $\mathbb{K} \succ K$ be a saturated model. Then $\dim_K(X) = \max\{\dim(\bar{a}/A) \mid \bar{a} \in X(\mathbb{K})\}$.*

(iii) *Let $\phi(x_1, \dots, x_m, y_1, \dots, y_n)$ be any L_A-formula and $r \in \mathbb{N}$. Then the set*

$$\{\bar{b} \in K^n \mid \dim_K(\phi(K^m, \bar{b})) \leq r\}$$

is A-definable.

(iv) *If $X \subseteq K$ is K-definable. Then X is infinite iff $\dim_K(X) \geq 1$.*

(v) *Let A_0 be a countable subset of $\mathbb{Q}_p$, and let Y be an A_0-definable subset of $\mathbb{Q}_p^n$. Then there is $\bar{a}_0 \in Y$ such that $\dim(\bar{a}_0/A_0) = \dim_{\mathbb{Q}_p}(Y)$.*

It is easy to see from Fact 1.8 that for any L_K-formula $\phi(x_1, \dots, x_n)$ and $K' \succ K$, we have

$$\dim_K(\phi(K^n)) = \dim_{K'}(\phi(K'^n)).$$

We will write $\dim_K(X)$ by $\dim(X)$ if there is no ambiguity. If the function $f : X \longrightarrow K$ is definable in K, and $Y \subseteq X \times K$ is the graph of f. Then we conclude directly that $\dim(X) = \dim(Y)$ by Fact 1.8 (ii).

For later use, we recall some well-known facts and terminology.

Hensel's Lemma. *Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p | x = 0 \vee v(x) \geq 0\}$ be the valuation ring of $\mathbb{Q}_p$. Let $f(x)$ be a polynomial over $\mathbb{Z}_p$ in one variable x, and let $a \in \mathbb{Z}_p$ such that $v(f(a)) > 2n + 1$ and $v(f'(a)) \leq n$, where f' denotes the derivative of f. Then there exists a unique $\hat{a} \in \mathbb{Z}_p$ such that*

$$f(\hat{a}) = 0 \text{ and } v(\hat{a} - a) \geq n + 1.$$

We say a field E is a Henselian field if Hensel's Lemma holds in E. Note that to be a henselian field is a first-order property of a field in the language of rings. Namely, there is a L_r-sentence σ such that $E \models \sigma$ iff E is a henselian field. So any $K \succ \mathbb{Q}_p$ is henselian.

2   Main Results

2.1   Some properties of Henselian fields

Since $\mathbb{Q}_p$ is complete and local compact, it is easy to see that:

Fact 2.1. *Suppose that E is a finite (or algebraic) field extension of $\mathbb{Q}_p$. Then for any $\alpha \in E \backslash \mathbb{Q}_p$, there is $n \in \mathbb{Z}$ such that $v(\alpha - a) < n$ ($|\alpha - a| > p^{-n}$) for all $a \in \mathbb{Q}_p$. Namely, $\mathbb{Q}_p$ is closed in E.*

We now show that Fact 2.1 holds for any $K \models \mathrm{Th}(\mathbb{Q}_p)$.

Lemma 2.2. *Let K be a henselian field, $R = \{x \in K \mid x = 0 \vee v(x) \geq 0\}$ be the valuation ring of K, and $f(x) \in R[x]$ a polynomial, $D \subseteq \Gamma_K$ a cofinal subset, and $X = \{x_d \mid d \in D\} \subseteq R$. If*

$$\lim_{d \in D, d \to +\infty} f(x_d) = 0,$$

Then there exist a cofinal subset $I \subseteq D$ and $a \in K$ such that

$$\lim_{i \in I, i \to +\infty} x_i = a \text{ and } f(a) = 0.$$

Proof   Induction on $\deg(f)$. Suppose that f has degree 1, say, $f(x) = \alpha x + \beta$. Then for any $\gamma \in \Gamma_K$, there is $d_0 \in D$ such that $v(f(x_d)) > \gamma$ for all $d_0 < d \in D$. Now $v(\alpha x_d + \beta) > \gamma$ implies that $v(x_d - (-\frac{\beta}{\alpha})) > \gamma - v(\alpha)$. So

$$\lim_{d \in D, d \to +\infty} |x_d - (-\frac{\beta}{\alpha})| = 0$$

and hence

$$\lim_{d \in D, d \to +\infty} x_d = -\frac{\beta}{\alpha} \text{ and } f(-\frac{\beta}{\alpha}) = 0$$

as required.

Now suppose that $\deg(f) = n+1 > 1$. We see that the derivative f' has degree n.

If there are $\gamma_0 \in \Gamma_K$ and $\varepsilon_0 \in D$ such that $v(f'(x_\varepsilon)) \leq \gamma_0$ for all $\varepsilon_0 < \varepsilon \in D$. Take ε_0 sufficiently large such that

$$v(f(x_\varepsilon)) > 4\gamma_0 + 1$$

for all $\varepsilon_0 < \varepsilon \in D$. Then, by Hensel's Lemma, we see that for all $\varepsilon > \varepsilon_0$, there is $\hat{x}_\varepsilon$ such that

$$v(\hat{x}_\varepsilon - x_\varepsilon) \geq \frac{v(f(x_\varepsilon)) - 1}{2} \text{ and } f(\hat{x}_\varepsilon) = 0$$

As f has at most finitely many roots, there is a cofinal subset $I \subseteq D$ and some $\hat{x}_\varepsilon \in K$ such that

$$v(\hat{x}_\varepsilon - x_i) > \frac{v(f(x_i)) - 1}{2}$$

for all $i \in I$. Since $v(f(x_i)) \to +\infty$, we see that $v(\hat{x}_\varepsilon - x_i) \to +\infty$. Thus we have

$$\lim_{i \in I, i \to +\infty} x_i = \hat{x}_\varepsilon \text{ and } f(\hat{x}_\varepsilon) = 0,$$

as required.

Otherwise, if for every $\gamma \in \Gamma_K$, there is $\gamma < d_\gamma \in D$ such that $v(f'(x_{d_\gamma})) > \gamma$. Then there is a cofinal subset $I = \{d_\gamma \mid \gamma \in \Gamma_K\} \subseteq D$ such that

$$\lim_{i \in I, i \to +\infty} f'(x_i) = 0,$$

Then, by induction hypothesis, there exist a cofinal subset $J \subseteq I$ and $b \in K$ such that

$$\lim_{j \in J, j \to +\infty} x_j = b \text{ and } f'(b) = 0.$$

Since f is continuous, $\lim_{j \in J, j \to +\infty} f(x_j) = f(b)$. Now J is cofinal in I, and I is cofinal in D, we conclude that J is cofinal in D. This complete the proof. $\qquad\square$

Proposition 2.3. *If K is a henselian field, and E is a finite extension of K. Then for any $\alpha \in E\backslash K$, there is $\gamma_0 \in \Gamma_K$ such that $v(\alpha - a) < \gamma_0$ for all $a \in K$. Namely, K is closed in E.*

Proof By [6, Lem. 4.1.1], the valuation of K extends uniquely to E. For each $\beta \in E$, Let $g(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be the minimal polynomial of β over K, then the valuation of β is exactly $\frac{v(a_0)}{n}$ (See [7, Prop. 5.3.4]).

Let $\alpha \in E\backslash K$, and $d(x)$ be the minimal polynomial of α over K with degree k. Then $d(x + a)$ is the minimal polynomial of $(\alpha - a)$ over K for any $a \in K$. Since $d(x + a) = xf(x) + d(a)$ for some $f(x) \in K[x]$, we see that $v(\alpha - a) = \frac{v(d(a))}{k}$. We claim that there is $\gamma_0 \in \Gamma_K$ such that $v(d(a)) < \gamma_0$ for all $a \in K$. Otherwise, we will find a sequence $\{a_\gamma \mid \gamma \in \Gamma_K\}$ such that $v(d(a_\gamma)) > \gamma$. Replace $d(x)$ by $\epsilon d(x)$ with some ϵ sufficiently close to 0, we may assume that $d \in R[x]$. Moreover, fix $\gamma_0 \in \Gamma$, if $v(\alpha - a) > \gamma_0$, and $v(\alpha - b) > 2\gamma_0$, then $v(a - b) \geq \gamma_0$. So

$$\{b \in K \mid v(\alpha - b) > \gamma_0\} \subseteq \delta_0 R$$

for some $\delta_0 \in K$, and hence

$$\{b \in K \mid v(d(b)) > \gamma_0\} = \{b \in K \mid kv(\alpha - b) > \gamma_0\} \subseteq k\delta_0 R.$$

Let $\delta = k\delta_0$. If $\delta \in R$, then, by Lemma 2.2, there is $b \in K$ such that $d(b) = 0$. However d is minimal polynomial of degree > 1, so has no roots in K. A contradiction.

If $\delta \notin R$, then $\delta^{-1} \in R$. Suppose that

$$d(x) = d_k x^k + \ldots + d_1 x + d_0.$$

Let

$$h(x) = d_k x^k + \ldots + d_1 \delta^{-k+1} x + d_0 \delta^{-k}.$$

We see that $h(x) \in R[x]$ and

$$h(\delta^{-1}x) = d_k(\delta^{-1}x)^k + \ldots + d_1\delta^{-k+1}(\delta^{-1}x) + d_0\delta^{-k} = \delta^{-k}d(x).$$

Now we have

$$v(h(\delta^{-1}a_\gamma)) = v(\delta^{-k}d(a_\gamma)) > \gamma - kv(\delta).$$

For $\gamma > \gamma_0$, we have $a_\gamma \in \delta R$. Therefore $\delta^{-1}a_\gamma \in R$ for all $\gamma > \gamma_0$. Applying Lemma 2.2 to $h(x)$, we can find $c \in K$ such that

$$h(c) = h(\delta^{-1}\delta c) = 0 = \delta^{-k}d(\delta c).$$

So $d(\delta c) = 0$. A contradiction.   □

Now we assume that K is an elementary extension of $\mathbb{Q}_p$ in the language of rings L_r. This follow result was proven by [14] in the case of $K = \mathbb{Q}_p$.

Lemma 2.4. *Let $\bar{x} = (x_1, \ldots, x_m)$ and $f(\bar{x}, y) = \sum_{i=0}^{n} p_i(\bar{x})y^i \in K[\bar{x}, y]$. Then there is a partition of*

$$R = \{\bar{x} \in K^m \mid \bigvee_{i=0}^{n} p_i(\bar{x}) \neq 0 \wedge \exists y(f(\bar{x}, y) = 0) \}$$

into finitely many definable subsets S, over each of which f has some fixed number $k \geq 1$ of distinct roots in K with fixed multiplicities $m_1, \ldots, m_k$. For any fixed $\bar{x}_0 \in S$, let the roots of $f(\bar{x}_0, y)$ be $r_1, \ldots, r_k$, and $e = \max\{v(r_i - r_j) \mid 1 \leq i < j \leq k\}$. Then $\bar{x}_0$ has a neighborhood $N \subseteq K^m$, $\gamma \in \Gamma_K$, and continuous, definable functions $F_1, \ldots, F_k : S \cap N \longrightarrow K$ such that for each $\bar{x} \in S \cap N$, $F_1(\bar{x}), \ldots, F_k(\bar{x})$ are roots of $f(\bar{x}, y)$ of multiplicities $m_1, \ldots, m_k$ and $v(F_i(\bar{x}) - r_i) > 2e$.

Proof The proof of [14, Lem 1.1] applies almost word for word to the present context. The only problem is that the authors used Fact 2.1 in their proof. But the Proposition 2.3 saying that we could replace $\mathbb{Q}_p$ by arbitrary $K \models \mathrm{Th}(\mathbb{Q}_p)$ in our argument.
□

Remark 2.5. *Lemma 1.1 of [14] saying that definable functions $F_1, \ldots, F_k$ are not only continuous but analytic. However we can't prove it in arbitrary $K \models \mathrm{Th}(\mathbb{Q}_p)$ as K might not be complete as a topological field.*

Similarly, Lem. 1.3 in [14] could be generalized to arbitrary $K \models \mathrm{Th}(\mathbb{Q}_p)$ as follows:

Lemma 2.6. *If $A \subseteq K^m$ and $f : A \longrightarrow K$ is definable. Then there is a definable set $B \subseteq A$, open in K^m such that $A\backslash B$ has no interior and f is continuous on B.*

Proof The proof of Lem. 1.3 in [14] applies almost word for word to the present context.   □

2.2　Dimensions

We now assume that K is an elementary extension of $\mathbb{Q}_p$.

Lemma 2.7. *Suppose that $A \subseteq K$, X, Y are A-definable in K, $f : X \longrightarrow Y$ is an A-definable function. If f is a finite-to-one map, $\dim(X) = \dim(f(X))$.*

Proof　Let $\mathbb{K}$ be a saturated elementary extension of K. By Fact 1.8 (iii), there is $r \in \mathbb{N}$ such that $|f^{-1}(y)| \leq r$ for all $y \in Y(\mathbb{K})$. For any $a \in X(\mathbb{K})$, since

$$|\{b \in X \mid f(b) = f(a)\}| \leq r,$$

we see that $a \in \mathrm{acl}(A, f(a))$. So $\dim(a/A, f(a)) = 0$. By Fact 1.8 (i) we have

$$\dim(a/A) = \dim(a, f(a)/A) = \dim(a/A, f(a)) + \dim(f(a)/A) = 0 + \dim(f(a)/A).$$

So $\dim(a/A) = \dim(f(a)/A)$. By Fact1.8 (ii), we conclude that $\dim(X) = \dim(Y)$.　□

Lemma 2.8. *Suppose that $A \subseteq K$, $f : X \longrightarrow Y$ is an A-definable function in K. Then*

$$\dim(X) \geq \dim(f(X)).$$

Proof　Generally, we have

$$\dim(a/A) = \dim(a, f(a)/A) = \dim(a/A, f(a)) + \dim(f(a)/A) \geq \dim(f(a)/A).$$

By Fact1.8 (ii), we conclude that $\dim(X) \geq \dim(Y)$.　□

Corollary 2.9. *Suppose that $A \subseteq K$, $f : X \longrightarrow Y$ is an A-definable bijection function in K. Then*

$$\dim(X) = \dim(f(X)).$$

Proof　f^{-1} is a definable function as f is bijection. So we conclude that

$$\dim(X) \geq \dim(f(X)) = \dim(Y) \geq \dim(f^{-1}(Y)) = \dim(X).$$

□

Lemma 2.10. *Suppose that X, Y are A-definable in K. Then*

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}.$$

Proof　By Fact1.8 (ii).　□

Lemma 2.11. *Let $X \subseteq K^n$. Then $\dim(X)$ is the minimal $k \leq n$ such that there is definable $Y \subseteq X$ with $\dim(Y) = \dim(X)$ and projection*

$$\pi : X \longrightarrow K^k; \ \ (x_1, \ldots, x_n) \mapsto (x_{r_1}, \ldots, x_{r_k})$$

is a finite-to-one map on Y, for suitable $1 \leq r_1 < \ldots < r_k \leq n$.

Proof   Let k be as above and $\pi : X \longrightarrow K^k$ be a projection with $\pi(x_1, \ldots, x_n) = (x_{r_1}, \ldots, x_{r_k})$. If $Y \subseteq X$ such that the restriction $\pi \restriction Y : Y \longrightarrow K^k$ is a finite-to-one map. Then by Lemma 2.7 we have $\dim(Y) = \dim(\pi(Y))$ and hence

$$\dim(X) = \dim(Y) = \dim(\pi(Y)) \leq k.$$

Now suppose that $\dim(X) = l \leq k$. Without loss of generality, we assume that $f : X \longrightarrow K^l; (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_l)$ is a projection such that $f(X)$ has nonempty interior. Then we prove a claim:

Claim. Let $Z_0 = \{b \in K^l \mid f^{-1}(b) \text{ is finite }\}$ and $Z_1 = K^l \backslash Z_0$. Then $\dim(Z_1) < l$.

Clearly,

$$Z_1 = \{b \in K^l \mid \dim(f^{-1}(b)) \geq 1\}$$

is definable in K. If $\dim(Z_1) = l$. Then, there is $\beta \in Z_1(\mathbb{K})$ such that $\dim(\beta/A) = l$, where is $\mathbb{K} \succ K$ is saturated. Since $\dim(f^{-1}(\beta)) \geq 1$, by Fact 1.8 (ii), there is $\alpha \in \dim(f^{-1}(\beta))$ such that $\dim(\alpha/A, \beta) \geq 1$. By Fact 1.8 (i), we conclude that

$$\dim(\alpha/A) = \dim(\alpha, f(\alpha)/A) = \dim(a/A, f(\alpha)) + \dim(f(\alpha)/A) \geq l + 1.$$

But $\dim(\alpha/A) \leq \dim(X) = l$. A contradiction.

Since $\dim(Z_1) < l$, by Lemma 2.10, $\dim(Z_0) = l$. The restriction of f on $f^{-1}(Z_0)$ is a finite-to-one map, we conclude that

$$\dim(f^{-1}(Z_0)) = \dim(Z_0) = l = \dim(X)$$

by Lemma 2.7. Now $\dim(f^{-1}(Z_0)) = \dim(X)$ and the restriction of f on $f^{-1}(Z_0)$ is a finite-to-one map. So $k \leq l$ as k is minimal. We conclude that $k = l = \dim(X)$ as required.   □

Corollary 2.12. *Let $X \subseteq K^n$ be definable with $\dim(X) = k$. Then there exists a partition of X into finitely many K-definable subsets S such that whenever $\dim(S) = \dim(X)$, there is a projection $\pi_S : S \longrightarrow K^k$ on k suitable coordinate axes which is finite-to-one.*

Proof Let $X_0 = X$ and $[n]^k$ be the set of all subset of $\{1, \ldots, n\}$ of cardinality k. By Lemma 2.11, there exist $D_0 = \{r_1, \ldots, r_k\} \in [n]^k$, $S_0 \subseteq X$ with $\dim(S_0) = \dim(X_0)$, such that the projection

$$\pi : (x_1, \ldots, x_n) \mapsto (x_{r_1}, \ldots, x_{r_k})$$

is finite-to-one on S_0 and infinite-to-one on $X_0 \backslash S_0$. If $\dim(X_0 \backslash S_0) < \dim(X_0)$, then the partition $\{X_0 \backslash S_0, S_0\}$ meets our requirements.

Otherwise, let $X_1 = X_0 \backslash S_0$, we could find $D_1 \in [n]^k \backslash \{D_0\}$ and $S_1 \in X_1$ such that the projection on coordinate axes from D_1 is finite-to-one over S_1. Repeating the above steps, we obtained sequences X_i and S_i such that $X_{i+1} = X_i \backslash S_i$. As $[n]^k$ is finite, there is a minimal $t \in \mathbb{N}$ such that $\dim(X_t) < \dim(X_0)$ and $\dim(S_i) = \dim(X_0)$ for all $i < t$. It is easy to see that $\{S_0, \ldots, S_{t-1}, X_t\}$ meets our requirements. $\square$

Recall that by [4], $\mathrm{Th}(\mathbb{Q}_p)$ admits definable Skolem functions. Namely, we have

Fact 2.13 (([4])). *Let $A \subseteq K$ and $\phi(\bar{x}, y)$ be a L_A-formula such that*

$$K \models \forall \bar{x} \exists y \phi(\bar{x}, y).$$

Then there A-definable function $f : K^m \to K$ such that $K \models \forall \bar{x} \phi(\bar{x}, f(\bar{x}))$.

With the above Fact, we could refine Corollary 2.12 as follows:

Corollary 2.14. *Let $X \subseteq K^n$ be definable with $\dim(X) = k$. Then there exists a partition of X into finitely many K-definable subsets S such that whenever $\dim(S) = \dim(X)$, there is a projection $\pi_S : S \longrightarrow K^k$ on k suitable coordinate axes which is injective.*

Proof Let $X_0 = X$. By Corollary 2.12, we may assume that the projection $\pi : X_0 \longrightarrow K^k$ given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$ is finite-to-one. By compactness, there is $r \in \mathbb{N}$ such that

$$|\pi^{-1}(\bar{y}) \cap X_0| \leq r$$

for all $\bar{y} \in K^k$. Induction on r. If $r = 1$, then π is injective on X_0. Otherwise, by Fact 2.13, there is a definable function

$$f : \pi(X) \longrightarrow X$$

such that $\pi(f(\bar{y})) = x$ for all $\bar{y} \in \pi(X_0)$. It is easy to see that f is injective and hence, by Corollary 2.7, $S_0 = f(\pi(X_0))$ is a definable subset of X of dimension k. Moreover $\pi : S_0 \longrightarrow K^k$ is exactly the inverse of f, hence injective. If $\dim(X_0 \backslash S_0)$ then the partition $\{X_0 \backslash S_0, S_1\}$ satisfies our require. Otherwise, $X_1 = X_0 \backslash S_0$ has dimension k and

$$|\pi^{-1}(\bar{y}) \cap X_1| \leq r - 1$$

By our induction hypothesis, there is a partition of X_1 into finitely may definable subsets meets our requirements. This completes the proof. $\square$

Theorem 2.15. *Let* $B \subseteq K^m$ *be definable in* K. *Then* $\dim_K(B) \geq \dim_{\mathbb{Q}_p}(B \cap \mathbb{Q}_p^m)$.

Proof Suppose that $\dim_K(B) = k$. By Lemma 2.10 and Corollary 2.14, we may assume that $\pi : B \longrightarrow K^k$ is injective. The restriction of π to $B \cap \mathbb{Q}_p^m$ is a injective projection from $B \cap \mathbb{Q}_p^m$ to $\mathbb{Q}_p^k$. By Lemma 2.11, $\dim_{\mathbb{Q}_p}(B \cap \mathbb{Q}_p^m) \leq k$. $\square$

Note that $P_n(K) = \{a \in K \mid a \neq 0 \wedge \exists b \in K(a = b^n)\}$ is an open subset of K whenever K is a hensilian field. For any polynomial $f(x_1, \ldots, x_m) \in K[x_1, \ldots, x_m]$,

$$P_n(f(K^m)) = \{a \in K^m \mid f(a) \neq 0 \wedge \exists b \in K(f(a) = b^n)\}$$

is an open subset of K^m since f is continuous.

2.3 Standard part map and definable functions

The following Facts will be used later.

Fact 2.16 (([2])). *Every complete* n-*type over* $\mathbb{Q}_p$ *is definable. Equivalently, for any* $K \succ \mathbb{Q}_p$, *any* L_r-*formula* $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, *and any* $\bar{b} \in K^m$, *the set*

$$\{\bar{a} \in \mathbb{Q}_p^n \mid K \models \phi(\bar{a}, \bar{b})\}$$

is definable in $\mathbb{Q}_p$.

Fact 2.17 (([11])). *Let* $X \subseteq K^m$ *be a* $\mathbb{Q}_p$-*definable open set, let* $Y \subseteq X$ *be a* K-*definable subset of* X. *Then either* Y *or* $X \backslash Y$ *contains a* $\mathbb{Q}_p$-*definable open set.*

Fact 2.18 (([11])). *Let* $X \subseteq K^m$ *be a* K-*definable set. Then* $\mathrm{st}(X) \cap \mathrm{st}(K^m \backslash X)$ *has no interior.*

Recall that μ is the collection of all infinitesimals of K over $\mathbb{Q}_p$, which induces a equivalence relation $\backsim_\mu$ on K, which is defined by

$$a \backsim_\mu b \iff a - b \in \mu.$$

Definition 2.19. Let $f(\bar{x}, y), g(\bar{x}, y) \in K[\bar{x}, y]$ be polynomials. By $f \backsim_\mu g$ we mean that

(i) if $|\bar{x}| = 0$, $f(y) = \sum_{i=1}^n a_i y^i$, and $g(y) = \sum_{i=1}^n b_i y^i$, then $f \backsim_\mu g$ iff $a_i \backsim_\mu b_i$ for each $i \leq n$.

(ii) if $|\bar{x}| > 0$, $f(\bar{x}, y) = \sum_{i=1}^n a_i(\bar{x}) y^i$, and $g(y) = \sum_{i=1}^n b_i(\bar{x}) y^i$, where a's and b's are polynomials with variables from $\bar{x}$. Then $f \backsim_\mu g$ iff $a_i \backsim_\mu b_i$ for each $i \leq n$.

Lemma 2.20. *Let $\bar{x} = (x_1,\ldots,x_n)$, $f(\bar{x})$ and $g(\bar{x})$ be polynomials over K with $f \backsim_\mu g$. If $\bar{a} = (a_1,\ldots,a_n)$ and $\bar{b} = (b_1,\ldots,b_n)$ are tuples from K with $a_i \backsim_\mu b_i$ for each $i \leq n$. Then $f(\bar{a}) \backsim_\mu g(\bar{b})$.*

Proof   We see that $\alpha \in \mu$ iff $v(\alpha) > \mathbb{Z}$. Since $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$ and $v(\alpha\beta) = v(\alpha) + v(\beta)$, we see that μ is closed under addition and multiplication. As polynomials are functions obtained by compositions of addition and multiplication, we conclude that $f(\bar{a}) \backsim_\mu g(\bar{b})$. $\square$

Since $V \subseteq K$ is also closed under addition and multiplication. We conclude directly that:

Corollary 2.21. *Let $\bar{x} = (x_1,\ldots,x_n)$, and $f(\bar{x}) \in K[\bar{x}]$ be a polynomial with every coefficient contained in V. If $a = (a_1,\ldots,a_n) \in V^n$, then $\mathrm{st}(f)(\mathrm{st}(a)) = \mathrm{st}(f(a))$.*

Corollary 2.22. *Let $f(x) = a_n x^n + \ldots + a_1 x^1 + a_0$ be a polynomial over K with every coefficient contained in V and $a_n \notin \mu$. If $b \in K$ such that $f(b) = 0$. Then $b \in V$.*

Proof   Suppose for a contradiction that $b \notin V$. Then $\mathrm{st}(b^{-1}) = 0$. Clearly, we have

$$b^{-n} f(b) = a_n + \ldots + a_1 b^{-n+1} + a_0 b^{-n} = 0.$$

Let

$$g(y) = a_0 y^n + \ldots + a_{n-1} y + a_n.$$

Then $g(b^{-1}) = 0$. By Corollary 2.21, we have $\mathrm{st}(g)(\mathrm{st}(b^{-1})) = 0$. As $\mathrm{st}(b^{-1}) = 0$, we see that $\mathrm{st}(a_n) = 0$, which contradicts to $a_n \notin \mu$. $\square$

Fact 2.23 ([11]). *Let $S \subseteq K^m$ be definable in K. Then $\mathrm{st}(S \cap V^m) \subseteq \mathbb{Q}_p^m$ is definable in $\mathbb{Q}_p$.*

Lemma 2.24. *Let $f : K^k \longrightarrow K$ be definable in K. Then*

 (i) *$X_\infty = \{a \in \mathbb{Q}_p^k \mid f(a) \notin V\}$ is definable in $\mathbb{Q}_p$.*
 (ii) *Let $X = \mathbb{Q}_p^k \backslash X_\infty$. Then $g : X \longrightarrow \mathbb{Q}_p$ given by $a \mapsto \mathrm{st}(f(a))$ is definable in $\mathbb{Q}_p$*

Proof   By Fact 2.16, there is a $L_{\mathbb{Q}_p}$-formula $\phi(x,y)$ such that for all $a \in \mathbb{Q}_p^k$ and $b \in \mathbb{Q}_p$, we have

$$\mathbb{Q}_p \models \phi(a,b) \iff v(f(a)) < v(b).$$

Hence

$$a \in X_\infty \iff \mathbb{Q}_p \models \forall y \phi(a,y),$$

which shows that X_∞ is definable in $\mathbb{Q}_p$. Again by Fact 2.16, there is $L_{\mathbb{Q}_p}$-formula $\psi(x, y_1, y_2)$ such that for all $a \in \mathbb{Q}_p^k$, $b_1, b_2 \in \mathbb{Q}_p$,

$$M \models \psi(a, b_1, b_2) \iff v(f(a) - b_1) > v(b_2 - b_1).$$

Therefore

$$b = \text{st}(f(a)) \iff \mathbb{Q}_p \models \forall y_1 \forall y_2 (v(b - y_1) > v(y_1 - y_2) \to \psi(a, y_1, y_2)).$$

for all $a \in \mathbb{Q}_p^k$ and $b \in \mathbb{Q}_p$. We conclude that $g : X \longrightarrow \mathbb{Q}_p$, $a \mapsto \text{st}(f(a))$ is definable in $\mathbb{Q}_p$ $\qquad \square$

Lemma 2.25. *Let $X \subseteq \mathbb{Q}_p^m$ be a clopen subset of $\mathbb{Q}_p^m$. If $\bar{a} \in V^m$ and $\text{st}(\bar{a}) \notin X$, then $\bar{a} \notin X(K)$.*

Proof   As X is clopen and $\text{st}(\bar{a}) \notin X$, there is $N \in \mathbb{Z}$ such that

$$B_{(\bar{a},N)} = \{\bar{b} \in \mathbb{Q}_p^m \mid \bigwedge_{i=1}^m v(b_i - \text{st}(a_i)) > N\} \cap X = \emptyset.$$

So $B_{(\bar{a},N)}(K) \cap X(K) = \emptyset$. But $v(a_i - \text{st}(a_i)) > \mathbb{Z}$, hence

$$\bar{a} \in \{\bar{b} \in K^m \mid \bigwedge_{i+1}^m v(b_i - \text{st}(a_i)) > N\} = B_{(\bar{a},N)}(K).$$

So $\bar{a} \notin X(K)$ as required. $\qquad \square$

Lemma 2.26. *If $X \subseteq K^m$ and $f : X \longrightarrow K$ are definable in K, then there is a polynomial $q(x_1, \ldots, x_m, y)$ such that the graph of f is contained in the variety*

$$\{(a_1, \ldots, a_m, b) \in K^{m+1} \mid q(a_1, \ldots, a_m, b) = 0\}.$$

Proof   Let Y be the graph of f. Since $\text{Th}(\mathbb{Q}_p)$ has quantifier elimination, Y is defined by a disjunction $\bigvee_{i=1}^s \phi_i(\bar{x})$, where each $\phi_i(\bar{x})$ is a conjunction

$$(\bigwedge_{j=1}^{l_i} g_{i_j}(\bar{x}, y) = 0) \wedge \bigwedge_{j=1}^{l_i} P_{n_{i_j}}(h_{i_j}(\bar{x}, y)),$$

where g's and h's belong to $K[\bar{x}, y]$. Now each $P_{n_{i_j}}(h_{i_j}(\bar{x}, y))$ defines an open subset of K^{m+1}. Since $\dim(Y) \leq m$, we see that for each $i \leq s$, there is $f(i) \leq l_i$ such that $g_{i_{f(i)}} \neq 0$. Let $q(\bar{x}, y) = \Pi_{i=1}^s g_{i_{f(i)}}(\bar{x}, y)$. Then

$$Y \subseteq \{(a_1, \ldots, a_m, b) \in K^{m+1} \mid q(a_1, \ldots, a_m, b) = 0\}$$

as required. $\qquad \square$

Proposition 2.27. *If $f : K^m \longrightarrow K$ is definable in K. Let $X = \{\bar{a} \in \mathbb{Q}_p^m \mid f(\bar{a}) \in V\}$. Then*

$$D_X = \{\bar{a} \in X \mid \exists \bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a}) \left(f(\bar{b}) - f(\bar{c}) \notin \mu \right)\}.$$

has no interiors.

Proof   By Lemma 2.26, there is a polynomial

$$g(x_1, \ldots, x_m, y) \in K[x_1, \ldots, x_m, y]$$

such that the graph of f is contained in the variety of g. Without loss of generality, we may assume that each coefficient of g is in V, otherwise, we could replace g by g/c, where c is a coefficient of g with minimal valuation. Moreover, we could assume that at least one coefficient of g is not in μ.

Suppose for a contradiction that D_X contains a open subset of $\mathbb{Q}_p^m$. Shrink D_X if necessary, we may assume that $D_X \subseteq X$ is a $\mathbb{Q}_p$-definable open set in $\mathbb{Q}_p^m$. By Lemma 2.4, there is a partition $\mathcal{P}$ of $D_X(K) \subseteq K^m$ into finitely many definable subsets S, over each of which g has some fixed number $k \geq 1$ of distinct roots in K with fixed multiplicities $m_1, \ldots, m_k$. For any fixed $\bar{x}_0 \in S$, let the roots of $g(\bar{x}_0, y)$ be $r_1, \ldots, r_k$, and $e = \max\{v(r_i - r_j) \mid 1 \leq i < j \leq k\}$. Then $\bar{x}_0$ has a neighborhood $N \subseteq K^m$, $\gamma \in \Gamma_K$, and continuous, definable functions $F_1, \ldots, F_k : S \cap N \longrightarrow K$ such that for each $\bar{x} \in S \cap N$, $F_1(\bar{x}), \ldots, F_k(\bar{x})$ are roots of $g(\bar{x}, y)$ of multiplicities $m_1, \ldots, m_k$ and $v(F_i(\bar{x}) - r_i) > 2e$.

Since $D_X(K)$ is a $\mathbb{Q}_p$-definable open subset of $X(K)$. By Fact 2.17, some $S \in \mathcal{P}$ contains a $\mathbb{Q}_p$-definable open subset $\psi(K^m)$ of $X(K)$. Where ψ is an $L_{\mathbb{Q}_p}$-formula. Let $A_0 = \phi(\mathbb{Q}_p^m)$. Then $A_0 \subseteq A$ is an open subset of $\mathbb{Q}_p^m$, and over $A_0(K)$ we have

(i) g has some fixed number $k \geq 1$ of distinct roots in K with fixed multiplicities $m_1, \ldots, m_k$.

(ii)   For any fixed $\bar{x}_0 \in A_0(K)$, let the roots of $g(\bar{x}_0, y)$ be $r_1, \ldots, r_k$, and $e = \max\{v(r_i - r_j) \mid 1 \leq i < j \leq k\}$. Then $\bar{x}_0$ has a neighborhood $N \subseteq K^m$, $\gamma \in \Gamma_K$, and continuous, definable functions $F_1, \ldots, F_k : A_0(K) \cap N \longrightarrow K$ such that for each $\bar{x} \in A_0(K) \cap N$, $F_1(\bar{x}), \ldots, F_k(\bar{x})$ are roots of $g(\bar{x}, y)$ of multiplicities $m_1, \ldots, m_k$ and $v(F_i(\bar{x}) - r_i) > 2e$.

(iii)   for any $\bar{a} \in A_0$, there exist $\bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a})$ such that $\mathrm{st}(f(\bar{b})) \neq \mathrm{st}(f(\bar{c}))$.

Suppose that

$$g(x_1, \ldots, x_m, y) = \sum_{i=0}^{n} g_i(x_1, \ldots, x_m)y^i,$$

where each $g_i(\bar{x}) \in K[\bar{x}]$. Since the variety $\{\bar{a} \in \mathbb{Q}_p^m \mid \mathrm{st}(g_n)(\bar{a}) = 0\}$ has dimension $m - 1$, it has no interior. By Fact 2.17,

$$A_0 \backslash \{\bar{a} \in \mathbb{Q}_p^m \mid \mathrm{st}(g_n)(\bar{a}) = 0\}$$

contains an open subset of $\mathbb{Q}_p^m$. Without loss of generality, we may assume that

$$\{\bar{a} \in \mathbb{Q}_p^m \mid \operatorname{st}(g_n)(\bar{a}) = 0\} \cap A_0 = \emptyset.$$

Since the family of clopen subsets forms a base for topology on $\mathbb{Q}_p^m$, we may assume that A_0 is clopen. We now claim that

Claim 1. *For every $\bar{a} \in A_0(K)$, $g_n(\bar{a}) \notin \mu$.*

Proof Otherwise, by Corollary 2.21, we have $\operatorname{st}(g_n)(\operatorname{st}(\bar{a})) = \operatorname{st}(g_n(\bar{a})) = 0$. So $\operatorname{st}(\bar{a}) \notin A_0$. By Lemma 2.25, we see that $\bar{a} \notin A_0(K)$. A contradiction. □

By Claim 1 and Corollary 2.22, we see that for every $\bar{a} \in A_0(K)$ and $b \in K$, if $g(\bar{a}, b) = 0$, then $b \in V$. By Corollary 2.21, we conclude the following claim

Claim 2. *For every $\bar{a} \in A_0(K)$ and $b \in K$, if $g(\bar{a}, b) = 0$, then $b \in V$ and*

$$\operatorname{st}(g)(\operatorname{st}(\bar{a}), \operatorname{st}(b)) = 0.$$

Now $\operatorname{st}(g)$ is a polynomial over $\mathbb{Q}_p$. Applying Lemma 2.4 to $\operatorname{st}(g)$ and Fact 2.17, and shrink A_0 if necessary, we may assume that

- $\operatorname{st}(g)$ has some fixed number $d \geq 1$ of distinct roots in K with fixed multiplicities $n_1, \ldots, n_d$ over A_0.
- Fix some $\bar{x}_0 \in A_0$, let the roots of $\operatorname{st}(g)(\bar{x}_0, y)$ (in $\mathbb{Q}_p$) be $s_1, \ldots, s_d$, and

$$\Delta = \max\{v(s_i - s_j) \mid 1 \leq i < j \leq d\}.$$

  Then there are definable continuous functions $H_1, \ldots, H_d : A_0 \longrightarrow \mathbb{Q}_p$ such that for each $\bar{x} \in A_0$, $H_1(\bar{x}), \ldots, H_d(\bar{x})$ are roots of $\operatorname{st}(g)(\bar{x}, y)$ of multiplicities $n_1, \ldots, n_d$ and $v(H_i(\bar{x}) - s_i) > 2\Delta$.
- for any $\bar{a} \in A_0$, there exist $\bar{b}, \bar{c} \in \operatorname{st}^{-1}(\bar{a})$ such that $\operatorname{st}(f(\bar{b})) \neq \operatorname{st}(f(\bar{c}))$.

By Claim 2, we see that for any $\bar{x} \in A_0(K)$, and $b \in K$, if $g(\bar{x}, b) = 0$, then $b \frown_\mu H_i(\operatorname{st}(\bar{x}))$ for some $i \leq d$. As $g(\bar{x}, f(\bar{x})) = 0$ for all $\bar{x} \in K^m$, we see that

Claim 3. *For each $\bar{x} \in A_0(K)$, $f(\bar{x}) \frown_\mu H_i(\operatorname{st}(\bar{x}))$ for some $i \leq d$.*

Let $D_i = \{\bar{x} \in A(K) \mid v(f(\bar{x}) - s_i) > 2\Delta\}$. We claim that

Claim 4. $A_0(K) = \bigcup_{i=1}^{d} D_i$ *and* $D_i \cap D_j = \emptyset$ *for each* $i \neq j$. *Namely,* $\{D_1, \ldots, D_d\}$ *is a partition of* $A_0(K)$.

Proof Let $\bar{x} \in A_0(K)$. By Claim 3, there is some $i \leq d$ such that $f(\bar{x}) \frown_\mu H_i(\operatorname{st}(\bar{x}))$. It is easy to see that

$$v(f(\bar{x}) - s_i) = v(f(\bar{x}) - H_i(\operatorname{st}(\bar{x})) + H_i(\operatorname{st}(\bar{x})) - s_i) = v(H_i(\operatorname{st}(\bar{x})) - s_i) > 2\Delta.$$

So $\bar{x} \in D_i$ and this implies that $A_0(K) = \bigcup_{i=1}^{d} D_i$.

On the other side, if $\bar{x} \in D_i \cap D_j$ for some $1 \leq i < j \leq d$, we have $v(f(\bar{x}) - s_i) > 2\Delta$ and $v(f(\bar{x}) - s_j) > 2\Delta$, which implies that

$$v(s_i - s_j) = v(s_i - f(\bar{x}) + f(\bar{x}) - s_j) \geq \min\{v(s_i - f(\bar{x})), v(f(\bar{x}) - s_j)\} \geq 2\Delta.$$

But $v(s_i - s_j) \leq \Delta$. A contradiction. $\qquad\square$

Claim 5. *Let $i \leq d$. For any $\bar{a}, \bar{b} \in D_i$, if $\bar{a} \frown_\mu \bar{b}$ then $\mathrm{st}(f(\bar{a})) = \mathrm{st}(f(\bar{b}))$.*

Proof   Let $\bar{x} \in D_i$. By Claim 3, there is $j \leq d$ such that $\mathrm{st}(f(\bar{x})) = H_j(\mathrm{st}(\bar{x}))$. We see that

$$v(f(\bar{x}) - s_j) = v(f(\bar{x}) - H_j(\mathrm{st}(\bar{x})) + H_j \mathrm{st}(\bar{x})) - s_j) = v(H_j(\mathrm{st}(\bar{x})) - s_j) > 2\Delta.$$

So $\bar{x} \in D_j$. By Claim 4, $i = j$. We conclude that $\mathrm{st}(f(\bar{x})) = H_i(\mathrm{st}(\bar{x}))$ whenever $\bar{x} \in D_i$. This complete the proof of Claim 5. $\qquad\square$

Recall that for any $\bar{a} \in A_0$, there exist $\bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a})$ such that $\mathrm{st}(f(\bar{b})) \neq \mathrm{st}(f(\bar{c}))$. By Claim 4 and Claim 5, we see that for each $\bar{a} \in A_0$, there is $1 \leq i \neq j \leq d$ such that $\bar{a} \in \mathrm{st}(D_i) \cap \mathrm{st}(D_j)$. This means that

$$A_0 \subseteq \bigcup_{1 \leq i \neq j \leq d} \mathrm{st}(D_i) \cap \mathrm{st}(D_j)$$

By Fact 2.18, each $\mathrm{st}(D_i) \cap \mathrm{st}(D_j)$ has no interior. By Fact 2.17, A_0 has no interiors. A contradiction. $\qquad\square$

Corollary 2.28. *If $f : K^m \longrightarrow K$ is definable in K. Let $X_\infty = \{\bar{a} \in \mathbb{Q}_p^m \mid f(\bar{a}) \notin V\}$. Then*

$$U = \{\bar{a} \in X_\infty \mid \exists \bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a}) \left(f(\bar{b}) \in V \wedge f(\bar{c}) \notin V \right)\}.$$

has no interior.

Proof   Otherwise, suppose that $U \subseteq K^m$ is open. Applying Proposition 2.27 to $g(x) = (f(x))^{-1}$, we see that $g(U) \subseteq V$, and for all $\bar{a} \in U$ there are $\bar{b}, \bar{c} \in \mathrm{st}^{-1}(a)$ such that $\mathrm{st}(g(\bar{b})) \neq 0$ and $\mathrm{st}(g(\bar{c})) = 0$. A contradiction. $\qquad\square$

Lemma 2.29. *Let $f : K^k \longrightarrow K$ be definable in K, $X = \{a \in \mathbb{Q}_p^k \mid f(a) \in V\}$, and $X_\infty = \{a \in \mathbb{Q}_p^k \mid f(a) \notin V\}$. Then both*

$$D_X = \{\bar{a} \in X \mid \exists \bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a}) \left(f(\bar{b}) - f(\bar{c}) \notin \mu \right)\}.$$

and

$$U = \{\bar{a} \in X_\infty \mid \exists \bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a}) \left(f(\bar{b}) \in V \wedge f(\bar{c}) \notin V \right)\}.$$

are definable sets over $\mathbb{Q}_p$

Proof   Let $X_0 = \{\bar{a} \in X \mid \mathrm{st}(f(\bar{a})) = 0\}$ and $X_1 = \{\bar{a} \in X \mid \mathrm{st}(f(\bar{a})) \neq 0\}$. As we showed in Lemma 2.24, both X_0 and X_1 are $\mathbb{Q}_p$-definable sets. Let

$$g : K^k \backslash f^{-1}(0) \longrightarrow K$$

be the K-definable function given by $\bar{x} \mapsto 1/f(\bar{x})$.

$$S_1 = \{\bar{a} \in X \mid |\mathrm{st}(Y)_{\bar{a}}| > 1\}$$
$$S_2 = \{\bar{a} \in X_0 \mid |\mathrm{st}(Z)_{\bar{a}}| \geq 1\}$$
$$S_3 = \{\bar{a} \in X_1 \mid |\mathrm{st}(Z)_{\bar{a}}| > 1\}$$

Let $Y \subseteq K^{k+1}$ be the graph of f and $Z \subseteq K^{k+1}$ be the graph of g. For each $\bar{a} \in \mathbb{Q}_p^k$, let

$$\mathrm{st}(Y)_{\bar{a}} = \{b \in \mathbb{Q}_p \mid (\bar{a}, b) \in \mathrm{st}(Y)\} \ \text{ and } \ \mathrm{st}(Z)_{\bar{a}} = \{b \in \mathbb{Q}_p \mid (\bar{a}, b) \in \mathrm{st}(Z)\}.$$

We now show that $D_X = S_1 \cup S_2 \cup S_3$.

     Clearly, S_1 and S_3 are subsets of D_X. If $\bar{a} \in S_2$, then $\mathrm{st}(f(\bar{a})) = 0$ and there is $\bar{b} \in \mathrm{st}^{-1}(\bar{a})$ such that $1/f(\bar{b}) \in V$, so $f(\bar{a}) - f(\bar{b}) \notin \mu$, which implies that $\bar{a} \in D_X$. Therefore, we conclude that $S_1 \cup S_2 \cup S_3 \subseteq D_X$.

     Conversely, take any $\bar{a} \in D_X$ and suppose that $\bar{b}, \bar{c} \in \mathrm{st}^{-1}(\bar{a})$ such that $f(\bar{b}) - f(\bar{c}) \notin \mu$. If both $f(\bar{b})$ and $f(\bar{c})$ are in V, then $\bar{a} \in S_1$; If $f(\bar{b}) \notin V$ and $\mathrm{st}(f(\bar{a})) = 0$, then $\bar{b} \in \mathrm{dom}(g)$. We see that $(\bar{a}, 0) \in \mathrm{st}(Z)$, so $\bar{a} \in S_2$; If $f(\bar{b}) \notin V$ and $\mathrm{st}(f(\bar{a})) \neq 0$, then $\bar{a}, \bar{b} \in \mathrm{dom}(g)$, $\mathrm{st}(g(\bar{b})) = 0$ and $\mathrm{st}(g(\bar{a})) \neq 0$, which implies that $|\mathrm{st}(Z)_{\bar{a}}| > 1$, and thus $\bar{a} \in S_3$. So we conclude that $D_X \subseteq S_1 \cup S_2 \cup S_3$ as required.

     As S_1, S_2, and S_3 are definable sets over $\mathbb{Q}_p$, D_X is definable over $\mathbb{Q}_p$. Similarly, U is definable over $\mathbb{Q}_p$. $\square$

     Suppose that $C \subseteq \mathbb{Q}_p^m$, we define the hull C^h by

$$C^h = \{\bar{x} \in K^m \mid \mathrm{st}(\bar{x}) \in C\}.$$

Theorem 2.30. *Let $f : K^m \to K$ be an K-definable function. Then here is a finite partition $\mathcal{P}$ of $\mathbb{Q}_p$ into definable sets, where each set in the partition is either open in $\mathbb{Q}_p^m$ or lacks of interior. On each open set $C \in \mathcal{P}$ we have:*

   (i) *either $f(x) \notin V$ for all $x \in C^h$;*
   (ii) *or there is a continuous function $g : C \longrightarrow \mathbb{Q}_p$, definable in $\mathbb{Q}_p$, such that $f(x) \in V$ and $\mathrm{st}(f(x)) = g(\mathrm{st}(x))$, for all $x \in C^h$.*

Proof   Let X, X_∞ be as in Lemma 2.24, D_X as in Proposition 2.27, and U as in Corollary 2.28, then D_X and U have no interior, and by Lemma 2.29, they are definable. Now $\{D_X, X \backslash D_X, \ U, X_\infty \backslash U\}$ is a partition of $\mathbb{Q}_p^m$. Clearly, $\{\mathrm{Int}(X_\infty \backslash U),$

$(X_\infty \backslash U) \setminus \mathrm{Int}(X_\infty \backslash U)\}$ is a partition of $X_\infty \backslash U$ where $\mathrm{Int}(X_\infty \backslash U)$ is open and $(X_\infty \backslash U) \setminus \mathrm{Int}(X_\infty \backslash U)$ lacks of interior.

Let $h : X \backslash D_X \longrightarrow \mathbb{Q}_p$ be a definable function defined by $x \mapsto \mathrm{st}(f(x))$. By Theorem 1.1 of [14], there is a finite partition $\mathcal{P}^*$ of $X \backslash D_X$ into definable sets, on each of which h is analytic. Each set in the partition is either open in $\mathbb{Q}_p^m$ or lacks of interior.

Clearly, the partition

$$\mathcal{P} = \{D_X, U, \mathrm{Int}(X_\infty \backslash U), \ (X_\infty \backslash U) \setminus \mathrm{Int}(X_\infty \backslash U)\} \cup \mathcal{P}^*$$

satisfies our condition. $\qquad\square$

We now prove our last result.

Lemma 2.31. *Let $Z \subseteq K^n$ be definable in K of dimension $k < n$, and the projection*

$$\pi : (x_1, \ldots x_n) \mapsto (x_1, \ldots, x_k)$$

is injective on Z. Then $\dim_{\mathbb{Q}_p}(\mathrm{st}(Z \cap V^n)) \leq k$.

Proof As π is injective on X, there is a definable function

$$f = (f_1, \ldots, f_n) : K^k \longrightarrow K^n$$

such that

- $f(\pi(\bar{x})) = (f_1(\pi(\bar{x})), \ldots, f_n(\pi(\bar{x}))) = \bar{x}$ for all $\bar{x} \in Z$;
- $f(\bar{y}) = (0, \ldots, 0)$ for all $\bar{y} \in K^k \backslash \pi(X)$.

By Lemma 2.26, for each $i \leq n$, there is a polynomial $F_i(\bar{y}, u)$ such that the graph of f_i is contained in the variety

$$V(F_i) = \{(\bar{y}, u) \in K^{k+1} \mid F_i(\bar{y}, u) = 0\}$$

of F_i. We assume that each coefficient belongs to V. It is easy to see that for each

$$(a_1, \ldots, a_n) \in Z \cap V^n,$$

we have $f_i(\pi(a_1, \ldots, a_n)) = a_i$. So $F_i(a_1, \ldots, a_k, a_i) = 0$. By Corollary 2.21,

$$\mathrm{st}(F_i)(\mathrm{st}(a_1), \ldots, \mathrm{st}(a_k), \mathrm{st}(a_i))) = 0.$$

So $\mathrm{st}(Z \cap V^n)$ is contained in the variety

$$V(\mathrm{st}(F_1), \ldots, \mathrm{st}(F_n))$$
$$= \left\{ (a_1, \ldots, a_n) \in \mathbb{Q}_p^n \mid \bigwedge_{i=1}^n \Big(\mathrm{st}(F_i)(\mathrm{st}(a_1), \ldots, \mathrm{st}(a_k), \mathrm{st}(a_i))) = 0 \Big) \right\}.$$

Let $A \subseteq \mathbb{Q}_p$ be the collection of all coefficients from $\mathrm{st}(F_i)$'s. Then for each

$$(a_1, \ldots, a_n) \in V(\mathrm{st}(F_1), \ldots, \mathrm{st}(F_n)),$$

we see that a_i is a root of $F_i(a_1, \ldots, a_k, u)$, and hence $a_i \in \mathrm{acl}(A, a_1, \ldots, a_k)$, where $i \leq n$. This implies that

$$\dim(a_1, \ldots, a_n/A) = \dim(a_1, \ldots, a_k/A) \leq k$$

for all $(a_1, \ldots, a_n) \in V(\mathrm{st}(F_1), \ldots, \mathrm{st}(F_n))$. By Fact 1.8 (v), we see that

$$\dim_{\mathbb{Q}_p}(V(\mathrm{st}(F_1), \ldots, \mathrm{st}(F_n)))$$
$$= \max \left\{ \dim(a_1, \ldots, a_n/A) \mid (a_1, \ldots, a_n) \in V(\mathrm{st}(F_1), \ldots, \mathrm{st}(F_n)) \right\} \leq k.$$

So $\mathrm{st}(Z \cap V^n) \leq k$ as required.   □

Theorem 2.32. *Let $Z \subseteq K^n$ be definable in K. Then $\dim_{\mathbb{Q}_p}(\mathrm{st}(Z \cap V^n)) \leq \dim_K(Z)$.*

Proof   Since $\mathrm{st}(X \cup Y) = \mathrm{st}(X) \cup \mathrm{st}(Y)$ and $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$ hold for all definable $X, Y \subseteq K^n$. Applying Corollary 2.14, we many assume that $\dim(Z) = k$ and $\pi : (x_1, \ldots x_n) \mapsto (x_1, \ldots, x_k)$ is injective on Z. If $k = n$, then

$$\dim_{\mathbb{Q}_p}(\mathrm{st}(Z \cap V^n)) \leq n$$

as $\mathrm{st}(Z \cap V^n) \subseteq \mathbb{Q}_p^n$. If $k < n$, then by Lemma 2.31,

$$\dim_{\mathbb{Q}_p}(\mathrm{st}(Z \cap V^n)) \leq k$$

as required.   □

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关于 p-进闭域的非标准模型上的维数和标准映射

姚宁远

摘　　要

我们研究了 p-进数域 $\mathbb{Q}_p$ 的初等扩张 K 上的可定义函数及其相关的维数理论。证明了 K^m 的可定义子集 X 在标准模型 $\mathbb{Q}_p$ 中的维数 $\dim_{\mathbb{Q}_p}(X \cap \mathbb{Q}_p^m)$ 不超过 X 在 K 中的维数。我们还证明了：对 K 中的可定义函数 $f : K^m \to K$，存在一个可定义集合 $D \subseteq \mathbb{Q}_p^m$ 以及 $\mathbb{Q}_p$ 上的可定义函数 $g : D \to \mathbb{Q}_p \cup \{\infty\}$ 使得 $\mathbb{Q}_p^m \backslash D$ 无内点，且对每个 $x \in D$ 有 $g(x) = \mathrm{st}(f(\mathrm{st}^{-1}(x)))$。进一步，我们证明了对 K^m 的可定义子集 X，有 $\dim_{\mathbb{Q}_p}(\mathrm{st}(X))$ 也不超过 X 在 K 中的维数。

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