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Comparing Fixed-Point and Revision Theories of Truth from the Perspective of Paradoxicality*

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Abstract. For the aim "to get a sense of the lay of the land amid a variety of options", Kremer (2009) defined three relations to compare ten fixed-point theories suggested by Kripke (1975) and three revision theories considered by Gupta and Belnap (1993). This paper extends Kremer's comparative work by comparing these theories from another important perspective, the perspective of paradoxicality. The notion of paradoxicality is very important for theories of truth, which has influenced philosophers' choice of specific theory of truth. We define a new relation in terms of this perspective, and establish the relationship among the thirteen theories of truth according to this relation.

1 Introduction

In 1975, Kripke put forward the fixed-point theory of truth in [7]. Martin and Woodruff independently proposed this theory at almost the same time in [9]. With the publication of Kripke ([7]) and Martin and Woodruff ([9]), "not only was it established once and for all that three-valued languages could contain T-predicates for themselves, but tools became available that could be used to construct systematic theories of truth." ([3], p. 58)

The basic idea of the fixed-point theory of truth is that paradoxical sentences such as the Liar sentence are *neither true nor false*. Interpreting the truth predicate as a *fixed point* can guarantee that the truth value of any sentence A is the same as the one of the sentence that "A" is true. In [7], Kripke provided an *inductive construction* according to which we can construct fixed points of the *jump operators* derived from certain kind of three-valued valuation schemes, and he pointed out that those jump operators not only have a *least fixed point*, but also have a *greatest intrinsic fixed point*. The inductive construction for constructing fixed points applies to the strong Kleene scheme κ, the weak Kleene scheme μ, the van Fraassen's supervaluation scheme, σ, and some variants of σ (including σ_1 and σ_2).

Kripke ([7]) did not make any particular recommendation among the three-valued valuation schemes to which the inductive construction applies. Neither did he make

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any firm recommendation among the fixed points of a given valuation scheme. Kripke did think the least fixed point as "the most natural" interpretation for the intuitive concept of truth, and the greatest intrinsic fixed point as "the unique 'largest' interpretation" consistent with our intuitive idea of truth.

In the fixed-point theory of truth, a sentence is defined to be *paradoxical* if it has no truth value in any fixed point.

The revision theory of truth is a competitive theory of the fixed-point theory of truth. In 1982, Herzberger suggested this theory in [4], and Gupta independently put forward this theory in [2]. They came up with the theory for different reasons. Since the inductive construction for constructing fixed points provided by Kripke does not apply to the classical valuation scheme, τ, and "it is reasoning in accordance with classical logic which in the first instance gives rise to the semantic paradoxes" ([4], p. 61), Herzberger adapted Kripke's inductive construction for admitting the classical valuation scheme. The revision theory of truth was thus suggested by Herzberger. Gupta proposed this theory to solve the descriptive problem about the truth concept[1], basing on the following idea: "When we learn the meaning of 'true' what we learn is a rule that enables us to improve on a proposed candidate for the extension of truth." ([2], p. 37) This rule is called "the *revision rule* for truth" in [2].

In the revision theory of truth, sentences are classified according to their behavior in *revision sequences*, which are ordinal-length sequences based on the revision rule. A *limit rule* is needed to determine the objects of a revision sequence at limit stages. Hence, different limit rule policies adopted lead to different resultant revision theories. The limit rule policy adopted by Herzberger ([4]) is different from the one adopted by Gupta ([2]). Belnap criticized Gupta's limit rule policy, and suggested another limit rule policy (and hence another resultant revision theory) in [1].

In Gupta and Belnap's monograph ([3]), they provided a clear philosophical basis for the following idea: "the signification of truth is a rule of revision." ([3], p. 139)[2] They viewed Tarski biconditionals as Tarski suggested, i.e., as partial definitions of the truth concept. Hence, they reckoned the truth concept is a circular concept and its signification is a revision rule determined by its definition, i.e., the totality of all

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[1]Gupta has distinguished two different problems that the Liar paradox raises about the concept of truth in his paper [2], the descriptive one and the normative one. The descriptive problem is described as follows: "The first is the descriptive problem of explaining our use of the word 'true', and, in particular, of giving the meaning of sentences containing 'true'." ([2], p. 1) The normative one is described as follows: "The second problem that the liar paradox raises about the concept of truth is the normative one of discovering the changes (if any) that the paradox dictates in our conception and use of 'true'." ([2], p. 1–2)

[2]Gupta and Belnap defined the notion of *signification* as follows: "Let the (extensional) signification of an expression (or a concept) in a word w be an abstract something that carries all the information about the expression's extensional relations in w." ([3], p. 30) They concluded that "Signification is a generalization of the notion of extension." ([3], p. 30)

Tarski biconditionals. Three revision theories have been presented in detail by Gupta and Belnap ([3]), denoted by $\mathbf{T}^*$, $\mathbf{T}^\#$ and $\mathbf{T}^C$ respectively. $\mathbf{T}^*$ is actually the theory proposed by Belnap in [1].

In the revision theory of truth, a sentence is defined to be paradoxical if, it is *unstable* or not *nearly stable* in all or certain kind of revision sequences.

Kremer introduced three binary relations, $\leq_1$, $\leq_2$ and $\leq_3$, to compare fixed-point and revision theories of truth, for the point "to get a sense of the lay of the land amid a variety of options". ([5], p. 363) According to the three relations, he established the relationships among ten fixed-point theories suggested by Kripke in [7], and three revision theories considered by Gupta and Belnap in [3]. The ten fixed-point theories are respectively based on the least fixed point or the greatest intrinsic fixed point of one of the five three-valued valuation schemes, κ, μ, σ, σ_1 and σ_2. The three revision theories compared by Kremer ([5]) are $\mathbf{T}^*$, $\mathbf{T}^\#$ and $\mathbf{T}^C$.

$\leq_1$, $\leq_2$ and $\leq_3$ represent three different perspectives to compare theories of truth. For example, $\leq_1$ is defined according to the perspective of validity.[3]

In [6], Kremer used the comparative results in [5] to critique on a claim of Gupta and Belnap in [3]: "An important feature of the revision theory, and one that prompted our interest in it, is its consequence that truth behaves like an ordinary classical concept under certain conditions—conditions that can roughly be characterized as those in which there is non-vicious reference in the language." ([3], p. 201) After considering notions of "truth behaving like an ordinary classical concept" and notions of "non-vicious reference" generated both by revision theories and by fixed-point theories, Kremer showed that some fixed-point theories have an advantage analogous to what Gupta and Belnap claimed for their approach, and $\mathbf{T}^\#$ does not have the advantage.

Kremer's comparative work is very meaningful, which helps us to know more about the thirteen theories of truth and the relationships among them. His work is also very useful: for example, it helps us to get a better understanding of Gupta and Belnap's claim introduced above.

In addition to the three natural perspectives in Kremer's comparative work, it is also meaningful to compare theories of truth from the perspective of paradoxicality. The notion of paradoxical is an important notion for theories of truth. On the one hand, it is often the case that in formal theories of truth, the definition of paradoxicality is provided as soon as the signification of the truth concept is presented. On the other hand, this notion influences philosophers' choice of concrete theories of truth. When Gupta proposed the revision theory of truth in [2], one of the two reasons for which Gupta adopted a limit rule policy different from the one adopted by Herzberger ([4]), is that he reckoned the revision theory resulting from the limit rule policy adopted by Herzberger misclassifies certain sentences, which should be paradoxical according

[3]Readers can consult [5, p. 372–377] for detailed motivations of $\leq_1$, $\leq_2$, and $\leq_3$.

to Gupta's intuition ([2], p. 53). In [10], Yaqūb provided four kinds of artifacts to criticize $\mathbf{T}^*$, $\mathbf{T}^\#$ and $\mathbf{T}^C$ (and raised his own revision theory), two of which are related to the assertion of paradoxical sentences.

In a word, it is significant to compare theories of truth from the perspective of paradoxicality. This paper define a new relation according to the perspective of paradoxicality, denoted by $\leq_4$, among the thirteen theories of truth considered in Kremer's comparative work ([5]), and establish the relationship among the theories according to this relation.

The rest of this paper is structured as follows. The next section introduces the ten fixed-point theories of truth. The third section introduces $\mathbf{T}^*$, $\mathbf{T}^\#$ and $\mathbf{T}^C$. In the fourth section, the thirteen theories of truth are compared according to the relation $\leq_4$. This paper is concluded in Section 5.

2 Fixed-point Theories of Truth

Denote the set of all sentences of a first-order language $\mathcal{L}$ by *Sent*$(\mathcal{L})$. If $\mathcal{L}$ and $\mathcal{L}^+$ are two first-order languages such that $\mathcal{L}^+$ includes only all the symbols of $\mathcal{L}$ and an additional 1-place predicate symbol $\boldsymbol{T}$, and $\ulcorner A \urcorner$ is a constant symbol of $\mathcal{L}$ for any $A \in$ *Sent*$(\mathcal{L}^+)$, then say that $\mathcal{L}^+$ is a *truth language*, and $\mathcal{L}$ is the *ground language* of $\mathcal{L}^+$. $\ulcorner A \urcorner$ is called a *quote name* in $\mathcal{L}^+$. If $\mathcal{M} = \langle D, I \rangle$ is a classical model for $\mathcal{L}$ satisfying the condition that *Sent*$(\mathcal{L}^+) \subseteq D$ and $I(\ulcorner A \urcorner) = A$ for each $A \in$ *Sent*$(\mathcal{L}^+)$, then say that $\mathcal{M}$ is a *ground model* of $\mathcal{L}^+$.

Given a ground model $\mathcal{M} = \langle D, I \rangle$ of a truth language $\mathcal{L}^+$, if h is a function from D to the truth value set $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, then h is called a *hypothesis* (relative to $\mathcal{M}$). If the domain of h is a subset of $\{\mathbf{t}, \mathbf{f}\}$, then h is called a *classical hypothesis*. Denote the expanded model of $\mathcal{M}$ for $\mathcal{L}^+$ in which $\boldsymbol{T}$ is interpreted by h by $\mathcal{M} + h$.

Given a first-order language $\mathcal{L}$, call $\mathcal{M} = \langle D, I \rangle$ a *three-valued model* for $\mathcal{L}$ if it satisfies the following conditions: (1) D is a nonempty set; and (2) I is a function the domain of which is the set of the nonlogical symbols of $\mathcal{L}$ such that it maps every constant symbol to an element of D, every n-place function symbol to a function from D^n to D, and every n-place predicate symbol to a function from D^n to the truth value set $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$. Given a model $\mathcal{M}$ and a valuation scheme ρ that applies to $\mathcal{M}$, the truth value of sentence A in $\mathcal{M}$ according to ρ is denoted by $Val^{\rho}_{\mathcal{M}}(A)$.

In κ and μ, negation is treated in the same way: $\neg\mathbf{t} = \mathbf{f}$, $\neg\mathbf{f} = \mathbf{t}$, and $\neg\mathbf{n} = \mathbf{n}$. In κ, conjunction is treated as follows: $\wedge(\mathbf{x}, \mathbf{y}) = \mathbf{t}$ iff $\mathbf{x} = \mathbf{y} = \mathbf{t}$, and $\wedge(\mathbf{x}, \mathbf{y}) = \mathbf{f}$ iff $\mathbf{x} = \mathbf{f}$ or $\mathbf{y} = \mathbf{f}$. And κ treats universal quantifier analogously to conjunction. In μ, conjunction is treated as follows: $\wedge(\mathbf{x}, \mathbf{y}) = \mathbf{t}$ iff $\mathbf{x} = \mathbf{y} = \mathbf{t}$, and $\wedge(\mathbf{x}, \mathbf{y}) = \mathbf{n}$ iff $\mathbf{x} = \mathbf{n}$ or $\mathbf{y} = \mathbf{n}$. And universal quantifier is treated analogously to conjunction in μ too.

Order the truth values as follows: $\mathbf{n} \leq \mathbf{n} \leq \mathbf{t} \leq \mathbf{t}$, and $\mathbf{n} \leq \mathbf{f} \leq \mathbf{f}$. Given two three-valued models for $\mathcal{L}$, $\mathcal{M} = \langle D, I \rangle$ and $\mathcal{M}' = \langle D, I' \rangle$, and an n-place predicate symbol R, if for any $d_1, d_2, \dots, d_n \in D$, $I(R)(d_1, d_2, \dots, d_n) \leq I'(R)(d_1, d_2, \dots, d_n)$, then $I(R) \leq I'(R)$. $\mathcal{M} \leq \mathcal{M}'$ if I and I' agree on all constant symbols and function symbols, and $I(R) \leq I'(R)$ for each predicate symbol R. Given a three-valued model $\mathcal{M}$ and a sentence A, $\mathit{Val}^{\sigma}_{\mathcal{M}}(A)$ is defined as follows:

$$\mathit{Val}^{\sigma}_{\mathcal{M}}(A) = \begin{cases} \mathbf{t} & \text{if for any classical model } \mathcal{M}' \geq \mathcal{M},\ \mathit{Val}^{\tau}_{\mathcal{M}'}(A) = \mathbf{t}, \\ \mathbf{f} & \text{if for any classical model } \mathcal{M}' \geq \mathcal{M},\ \mathit{Val}^{\tau}_{\mathcal{M}'}(A) = \mathbf{f}, \\ \mathbf{n} & \text{otherwise.} \end{cases}$$

Given a ground model $\mathcal{M} = \langle D, I \rangle$ of a truth language $\mathcal{L}^+$, and given that $\rho = \kappa$, μ or σ, the *jump operator* (derived from ρ) $\rho_{\mathcal{M}}$ is a function from hypotheses to hypotheses such that, for any hypothesis h and any $d \in D$,

$$\rho_{\mathcal{M}}(h)(d) = \begin{cases} \mathit{Val}^{\rho}_{\mathcal{M}+h}(d) & \text{if } d \in \mathit{Sent}(\mathcal{L}^+), \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

Given a ground model $\mathcal{M} = \langle D, I \rangle$ of a truth language $\mathcal{L}^+$, and given a jump operator $\rho_{\mathcal{M}}$, if h is a hypothesis such that $\rho_{\mathcal{M}}(h) = h$, then call h a *fixed point* of $\rho_{\mathcal{M}}$. If h is a fixed point of $\rho_{\mathcal{M}}$, then for any sentence $A \in \mathit{Sent}(\mathcal{L}^+)$, $\mathit{Val}^{\rho}_{\mathcal{M}+h}(\boldsymbol{T}\ulcorner A \urcorner) = \mathit{Val}^{\rho}_{\mathcal{M}+h}(A)$, since $\mathit{Val}^{\rho}_{\mathcal{M}+h}(\boldsymbol{T}\ulcorner A \urcorner) = h(A) = \rho_{\mathcal{M}}(h)(A) = \mathit{Val}^{\rho}_{\mathcal{M}+h}(A)$.

Given any truth language $\mathcal{L}^+$ and any ground model $\mathcal{M} = \langle D, I \rangle$ of $\mathcal{L}^+$, Kripke ([7]) provided an inductive construction to construct fixed points of jump operators derived from certain kind of three-valued valuation schemes, in which the monotonicity of a jump operator plays a crucial role.

A jump operator $\rho_{\mathcal{M}}$ is *monotone* if, for all hypotheses h and h' in the domain of $\rho_{\mathcal{M}}$ such that $h \leq h'$, $\rho_{\mathcal{M}}(h) \leq \rho_{\mathcal{M}}(h')$. Taking κ as an example, Kripke ([7]) constructed a fixed point according to the inductive construction.

Suppose that $\mathcal{M} = \langle D, I \rangle$ is a ground model of a truth language $\mathcal{L}^+$, and that h is the *empty* hypothesis (relative to $\mathcal{M}$), i.e., $h(d) = \mathbf{n}$ for each $d \in D$. Suppose that α is a limit ordinal, and that $\langle h_\beta \mid \beta < \alpha \rangle$ is a sequence of hypotheses whose length is α. Let the *union* of $\langle h_\beta \mid \beta < \alpha \rangle$ be as follows (denoted by $\bigcup_{\beta<\alpha} h_\beta$): for every $d \in D$,

$$\bigcup_{\beta<\alpha} h_\beta(d) = \begin{cases} \mathbf{t} & \text{if there is an ordinal } \beta < \alpha \text{ such that } h_\beta(d) = \mathbf{t}, \\ \mathbf{f} & \text{if there is an ordinal } \beta < \alpha \text{ such that } h_\beta(d) = \mathbf{f}, \\ \mathbf{n} & \text{otherwise.} \end{cases}$$

When applied to $\kappa_{\mathcal{M}}$ and beginning with the empty hypothesis, Kripke's inductive construction can be represented as follows: for every ordinal α,

if $\alpha = 0$, then let $h_\alpha = h$;

if $\alpha = \beta + 1$, then let $h_\alpha = \kappa_{\mathcal{M}}(h_\beta)$; and

if α is a limit ordinal, then let $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$.

Since $\kappa_{\mathcal{M}}$ is a total function on the set of all hypotheses and is monotone, it can be proved that a unique ordinal-length sequence of hypotheses is obtained in this way (denoted by $\langle h_\alpha \mid \alpha \in On\rangle$, where On denotes the class of all ordinals), and that for all ordinals $\alpha \leq \alpha'$, $h_\alpha \leq h_{\alpha'}$. Since the cardinality of the set of all hypotheses is bounded, there must be ordinals $\alpha < \alpha'$ such that $h_\alpha = h_{\alpha'}$. So $h_\alpha = h_{\alpha+1} = \kappa_{\mathcal{M}}(h_\alpha)$, i.e., h_α is a fixed point of $\kappa_{\mathcal{M}}$.

Kripke pointed out that the fixed point obtained above is the least fixed point of $\kappa_{\mathcal{M}}$ and that the same way can be used to construct the least fixed point of $\mu_{\mathcal{M}}$ and that of $\sigma_{\mathcal{M}}$.

Given a ground model $\mathcal{M} = \langle D, I\rangle$ of a truth language $\mathcal{L}^+$, a valuation scheme ρ that applies to $\mathcal{M} + h$, and a hypothesis h, if $h \leq \rho_{\mathcal{M}}(h)$, then say that h is *sound* (relative to $\rho_{\mathcal{M}}$). Given that $\rho = \kappa$, μ, or σ, by [3, Thm. 5C.16], it can be seen that starting with any sound hypothesis (relative to $\rho_{\mathcal{M}}$) h, a fixed point of $\rho_{\mathcal{M}}$ can be constructed according to Kripke's inductive construction, which is the least fixed point larger than or equal to h.

Suppose that $\mathcal{M} = \langle D, I\rangle$ is a ground model of a truth language $\mathcal{L}^+$, and that $\rho_{\mathcal{M}}$ is a jump operator. Given two hypotheses h and h' (relative to $\mathcal{M}$), if there exists a hypothesis h'' (relative to $\mathcal{M}$) such that $h \leq h''$ and $h' \leq h''$, then say that h and h' are *compatible*. Given a hypothesis h, if h is compatible with all fixed points of $\rho_{\mathcal{M}}$, then say that h is *intrinsic* (relative to $\rho_{\mathcal{M}}$).

Let $\rho = \kappa$, μ, σ. Kripke ([7]) showed that for any truth language $\mathcal{L}^+$ and any ground model $\mathcal{M} = \langle D, I\rangle$ of $\mathcal{L}^+$, $\rho_{\mathcal{M}}$ has a greatest intrinsic fixed point. Following [5], the least fixed point of $\rho_{\mathcal{M}}$ is denoted by $\mathit{lfp}(\rho_{\mathcal{M}})$, and the greatest intrinsic fixed point of $\rho_{\mathcal{M}}$ is denoted by $\mathit{gifp}(\rho_{\mathcal{M}})$.

For any truth language $\mathcal{L}^+$ and any ground model $\mathcal{M}$ of $\mathcal{L}^+$, Kripke's inductive construction for constructing fixed points applies to two variants of $\sigma_{\mathcal{M}}$ that are denoted by $\sigma 1_{\mathcal{M}}$ and $\sigma 2_{\mathcal{M}}$ respectively in [5]. Given that Γ is a set of sentences, if there is no sentence A such that both A and $\neg A$ are classical first-order consequences of Γ, then say that Γ is *consistent*. Given a hypothesis h, if the set $\{A \in \mathit{Sent}(\mathcal{L}^+) \mid h(A) = \mathbf{t}\}$ is consistent, then say that h is *weakly consistent*; and if the set $\{A \in \mathit{Sent}(\mathcal{L}^+) \mid h(A) = \mathbf{t}\} \cup \{\neg A \mid A \in \mathit{Sent}(\mathcal{L}^+), h(A) = \mathbf{f}\}$ is consistent, then say that h is *strongly consistent*. $\sigma 1_{\mathcal{M}}$ is a function from weakly consistent hypotheses to weakly consistent hypotheses such that for every weakly consistent hypothesis h, every $d \in D$,

$$\sigma 1_{\mathcal{M}}(h)(d) = \begin{cases} \mathbf{t} & \text{if } d \text{ is a sentence, and } \mathit{Val}^{\tau}_{\mathcal{M}+h'}(d) = \mathbf{t} \\ & \text{for every weakly consistent classical hypothesis } h' \geq h, \\ \mathbf{f} & \text{if } d \text{ is a nonsentence, or } d \text{ is a sentence and } \mathit{Val}^{\tau}_{\mathcal{M}+h'}(d) = \mathbf{f} \\ & \text{for every weakly consistent classical hypothesis } h' \geq h, \\ \mathbf{n} & \text{otherwise.} \end{cases}$$

$\sigma 2_{\mathcal{M}}$ is a function from strongly consistent hypotheses to strongly consistent hypotheses such that for every strongly consistent hypothesis h, every $d \in D$,

$$\sigma_{2\mathcal{M}}(h)(d) = \begin{cases} \mathbf{t} & \text{if } d \text{ is a sentence, and } \mathit{Val}^{\tau}_{\mathcal{M}+h'}(d) = \mathbf{t} \\ & \text{for every strongly consistent classical hypothesis } h' \geq h, \\ \mathbf{f} & \text{if } d \text{ is a nonsentence, or } d \text{ is a sentence and } \mathit{Val}^{\tau}_{\mathcal{M}+h'}(d) = \mathbf{f} \\ & \text{for every strongly consistent classical hypothesis } h' \geq h, \\ \mathbf{n} & \text{otherwise.} \end{cases}$$

Both $\sigma_{1\mathcal{M}}$ and $\sigma_{2\mathcal{M}}$ are monotone, and have both a least least fixed point and a greatest intrinsic fixed point. Following [5], σ_1 and σ_2 are treated as two three-valued valuation schemes.

Let $\mathcal{L}^+$ be a truth language, and $\mathcal{M}$ be a ground model of $\mathcal{L}^+$. Kremer ([5]) has considered ten fixed-point theories, $\mathbf{T}^{\mathit{lfp},\rho}$ and $\mathbf{T}^{\mathit{gifp},\rho}$ for each $\rho \in \{\kappa, \mu, \sigma, \sigma_1, \sigma_2\}$. The fixed-point theory $\mathbf{T}^{\mathit{lfp},\rho}$ dictates that the interpretation of $\boldsymbol{T}$ in the ground model $\mathcal{M}$ is the least fixed point of $\rho_{\mathcal{M}}$. The fixed-point theory $\mathbf{T}^{\mathit{gifp},\rho}$ dictates that the interpretation of $\boldsymbol{T}$ in the ground model $\mathcal{M}$ is greatest intrinsic fixed point of $\rho_{\mathcal{M}}$.

For any hypothesis (relative to $\mathcal{M} = \langle D, I\rangle$) h and any $d \in D$, if $h(d) = \mathbf{t}$ ($h(d) = \mathbf{f}$, $h(d) = \mathbf{n}$), then say that h *declares* d *true* (*false*, *neither true nor false*).

Definition 1. Let $\mathcal{L}^+$ be a truth language, $\mathcal{M}$ be a ground model of $\mathcal{L}^+$, $\rho = \kappa$, μ, σ, σ_1 or σ_2, and $A \in \mathit{Sent}(\mathcal{L}^+)$. Say that A is *paradoxical in the ground model* $\mathcal{M}$ *according to the theory* $\boldsymbol{T}^{\mathit{lfp},\rho}$ (or *the theory* $\boldsymbol{T}^{\mathit{gifp},\rho}$) if all fixed points of $\rho_{\mathcal{M}}$ declare A neither true nor false.

3 Revision Theories of Truth

Given a truth language $\mathcal{L}^+$ and a ground model $\mathcal{M} = \langle D, I\rangle$ of $\mathcal{L}^+$, a *revision rule* for truth, $\tau_{\mathcal{M}}$, is a function from classical hypotheses to classical hypotheses such that for every classical hypothesis h, every $d \in D$,

$$\tau_{\mathcal{M}}(h)(d) = \begin{cases} \mathit{Val}^{\tau}_{\mathcal{M}+h}(d) & \text{if } d \in \mathit{Sent}(\mathcal{L}^+), \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

For any truth language $\mathcal{L}^+$ and any ground model $\mathcal{M}$ of $\mathcal{L}^+$, the following propositons hold: (1) $\mu_{\mathcal{M}}(h) \leq \kappa_{\mathcal{M}}(h) \leq \sigma_{\mathcal{M}}(h)$ for every hypothesis h; (2) $\sigma_{\mathcal{M}}(h) \leq \sigma_{1\mathcal{M}}(h)$ for every weakly consistent hypothesis h; (3) $\sigma_{1\mathcal{M}}(h) \leq \sigma_{2\mathcal{M}}(h)$ for every strongly consistent hypothesis h; (4) $\tau_{\mathcal{M}}(h) = \mu_{\mathcal{M}}(h) = \kappa_{\mathcal{M}}(h) = \sigma_{\mathcal{M}}(h)$ for every classical hypothesis h; (5) $\tau_{\mathcal{M}}(h) = \sigma_{1\mathcal{M}}(h)$ for every weakly consistent classical hypothesis h; and (6) $\tau_{\mathcal{M}}(h) = \sigma_{2\mathcal{M}}(h)$ for every strongly consistent classical hypothesis h.

For any sequence $\mathscr{S}$, $\mathit{lh}(\mathscr{S})$ is used to denote the length of $\mathscr{S}$, and for any $\beta < \mathit{lh}(\mathscr{S})$, $\mathscr{S}_\beta$ is used to denote the βth object of $\mathscr{S}$. Given a ground model $\mathcal{M} = \langle D, I\rangle$ of a truth language $\mathcal{L}^+$, and a sequence of classical hypotheses $\mathscr{S}$ whose length is some limit ordinal or On. Suppose that α is a limit ordinal such that $\alpha \leq lh(\mathscr{S})$. If there is an ordinal β such that $\beta < \alpha$, and for every $\beta \leq \delta < \alpha$, $\mathscr{S}_\delta(d) = \mathbf{t}$ ($\mathscr{S}_\delta(d) = \mathbf{f}$), then say that d is *stably true* (*stably false*) *up to* α in $\mathscr{S}$. If d is either stably true or stably false up to α, then say that d is *stable up to* α in $\mathscr{S}$; otherwise, d is unstable up to α in $\mathscr{S}$.

Given a ground model $\mathcal{M} = \langle D, I\rangle$ of a truth language $\mathcal{L}^+$, if $\mathscr{S}$ is a sequence of classical hypotheses of length On such that for every ordinal α, (1) if $\alpha = \beta + 1$, then $\mathscr{S}_\alpha = \tau_{\mathcal{M}}(\mathscr{S}_\beta)$, and (2) if α is a limit ordinal, then for every $d \in D$, every $\mathbf{x} \in \{\mathbf{t}, \mathbf{f}\}$, if d is stably $\mathbf{x}$ up to α in $\mathscr{S}$ then $\mathscr{S}_\alpha(d) = \mathbf{x}$, then say that $\mathscr{S}$ is a *revision sequence*. According to this notion of revision sequence, any element stably true up to a limit ordinal α is declared true at stage α, and any element stably false up to α is declared false at stage α, and any element unstable up to α can be arbitrarily declared either true or false at stage α. This notion of revision sequence adopts Belnap's limit rule policy proposed in [1].

Given a revision sequence $\mathscr{S}$ and any element d in the domain, if there is an ordinal α such that for every $\alpha \leq \beta$, $\mathscr{S}_\beta(d) = \mathbf{t}$ ($\mathscr{S}_\beta(d) = \mathbf{f}$), then say that d is *stably true* (*false*) in $\mathscr{S}$. If d is stably true or stably false in $\mathscr{S}$, then say that d is *stable* in $\mathscr{S}$; otherwise, d is unstable in $\mathscr{S}$.

Given a revision sequence $\mathscr{S}$ and any element d in the domain, if there is an ordinal α such that for every $\alpha \leq \beta$, there is a natural number n such that for every $m \geq n$, $\mathscr{S}_{\beta+m}(d) = \mathbf{t}$ ($\mathscr{S}_{\beta+m}(d) = \mathbf{f}$), then say that d is *nearly stably true* (*nearly stably false*) in $\mathscr{S}$. If d is either nearly stably true or nearly stably false in $\mathscr{S}$, then say that d is *nearly stable* in $\mathscr{S}$; otherwise, d is not nearly stable in $\mathscr{S}$.

Given a ground model $\mathcal{M}$ of a truth language $\mathcal{L}^+$ and a hypothesis h (relative to $\mathcal{M}$), if the set $\{A \in \mathit{Sent}(\mathcal{L}^+) \mid h(A) = \mathbf{t}\}$ is maximally consistent, then say that h is *maximally consistent*. Given any classical hypothesis h, h is strongly consistent iff h is maximally consistent. Given a revision sequence $\mathscr{S}$, if for every ordinal α, $\mathscr{S}_\alpha$ is maximally consistent, then say that $\mathscr{S}$ is a C*-sequence*.[4]

$\mathbf{T}^*$ is the revision theory that classifies sentences according to their stability in all revision sequences. $\mathbf{T}^\#$ is the revision theory that classifies sentences according to their near stability in all revision sequences. $\mathbf{T}^C$ is the revision theory that classifies sentences according to their stability in all C-sequences.

Definition 2. Let $\mathcal{L}^+$ be a truth language, $\mathcal{M}$ be a ground model of $\mathcal{L}^+$, and $A \in \mathit{Sent}(\mathcal{L}^+)$. Say that A is *paradoxical in the ground model* $\mathcal{M}$ *according to the theory* $\boldsymbol{T}^*$ if A is unstable in any revision sequence. Say that A is *paradoxical in the ground model* $\mathcal{M}$ *according to the theory* $\boldsymbol{T}^\#$ if A is not nearly stable in any revision sequence. Say that A is *paradoxical in the ground model* $\mathcal{M}$ *according to the theory* $\boldsymbol{T}^C$ if A is unstable in any C-sequence.

[4] Note that for any revision sequence $\mathscr{S}$, every ordinal α, $\mathscr{S}_{\alpha+1}$ is maximally consistent.

4 Relationship among the Theories of Truth

In this section, the thirteen theories of truth are compared from the perspective of paradoxicality: $\mathbf{T}^{lfp,\mu}$, $\mathbf{T}^{lfp,\kappa}$, $\mathbf{T}^{lfp,\sigma}$, $\mathbf{T}^{lfp,\sigma_1}$, $\mathbf{T}^{lfp,\sigma_2}$, $\mathbf{T}^{gifp,\mu}$, $\mathbf{T}^{gifp,\kappa}$, $\mathbf{T}^{gifp,\sigma}$, $\mathbf{T}^{gifp,\sigma_1}$, $\mathbf{T}^{gifp,\sigma_2}$, $\mathbf{T}^{*}$, $\mathbf{T}^{\#}$ and $\mathbf{T}^{C}$.

Definition 3. Given any two theories $\mathbf{T}$ and $\mathbf{T}'$ among the thirteen theories, say that $\mathbf{T} \leq_4 \mathbf{T}'$ iff for every truth language $\mathcal{L}^+$, every ground model $\mathcal{M}$ of $\mathcal{L}^+$, and every sentence $A \in \mathit{Sent}(\mathcal{L}^+)$, if A is paradoxical in $\mathcal{M}$ according to $\mathbf{T}$ then A is paradoxical in $\mathcal{M}$ according to $\mathbf{T}'$. Say that $\mathbf{T} \equiv_4 \mathbf{T}'$ iff $\mathbf{T} \leq_4 \mathbf{T}'$ and $\mathbf{T}' \leq_4 \mathbf{T}$. Note that $\leq_4$ is reflexive and transitive.

Theorem 4. *Figure 1 uses an arrow to represent that the relation $\leq_4$ holds between the theory at the start of the arrow and the one at the end of the arrow. When restricted to the ten fixed-point theories, the relation $\leq_4$ is the reflexive transitive closure of the relation shown in Figure 1.*

$$\begin{tikzcd}
\mathbf{T}^{gifp,\sigma_2} \arrow[d, leftrightarrow] & \mathbf{T}^{gifp,\sigma_1} \arrow[d, leftrightarrow] & \mathbf{T}^{gifp,\sigma} \arrow[d, leftrightarrow] & \mathbf{T}^{gifp,\kappa} \arrow[d, leftrightarrow] & \mathbf{T}^{gifp,\mu} \arrow[d, leftrightarrow] \\
\mathbf{T}^{lfp,\sigma_2} \arrow[r] & \mathbf{T}^{lfp,\sigma_1} \arrow[r] & \mathbf{T}^{lfp,\sigma} \arrow[r] & \mathbf{T}^{lfp,\kappa} \arrow[r] & \mathbf{T}^{lfp,\mu}
\end{tikzcd}$$

Figure 1

Proof Since $\leq_4$ is reflexive and transitive, it is enough to prove the following claims: (1) $\mathbf{T}^{lfp,\mu} \equiv_4 \mathbf{T}^{gifp,\mu}$, $\mathbf{T}^{lfp,\kappa} \equiv_4 \mathbf{T}^{gifp,\kappa}$, $\mathbf{T}^{lfp,\sigma} \equiv_4 \mathbf{T}^{gifp,\sigma}$, $\mathbf{T}^{lfp,\sigma_1} \equiv_4 \mathbf{T}^{gifp,\sigma_1}$, and $\mathbf{T}^{lfp,\sigma_2} \equiv_4 \mathbf{T}^{gifp,\sigma_2}$; (2) $\mathbf{T}^{lfp,\sigma_2} \leq_4 \mathbf{T}^{lfp,\sigma_1}$, $\mathbf{T}^{lfp,\sigma_1} \leq_4 \mathbf{T}^{lfp,\sigma}$, $\mathbf{T}^{lfp,\sigma} \leq_4 \mathbf{T}^{lfp,\kappa}$, and $\mathbf{T}^{lfp,\kappa} \leq_4 \mathbf{T}^{lfp,\mu}$; and (3) $\mathbf{T}^{lfp,\sigma_1} \not\leq_4 \mathbf{T}^{lfp,\sigma_2}$, $\mathbf{T}^{lfp,\sigma} \not\leq_4 \mathbf{T}^{lfp,\sigma_1}$, $\mathbf{T}^{lfp,\kappa} \not\leq_4 \mathbf{T}^{lfp,\sigma}$, and $\mathbf{T}^{lfp,\mu} \not\leq_4 \mathbf{T}^{lfp,\kappa}$.

(1) holds by Definition 1.

The proof of (2) is as follows. Choose ρ and ρ' from the list μ, κ, σ, σ_1, σ_2 such that ρ is to the left of ρ'. Next, we prove that $\mathbf{T}^{lfp,\rho'} \leq_4 \mathbf{T}^{lfp,\rho}$. Given any truth language $\mathcal{L}^+$ and any ground model $\mathcal{M}$ of $\mathcal{L}^+$, and any fixed point h of $\rho_{\mathcal{M}}$, note that h is strongly consistent, and h is sound relative to $\rho'_{\mathcal{M}}$ since $h = \rho_{\mathcal{M}}(h) \leq \rho'_{\mathcal{M}}(h)$. Hence a fixed point of $\rho'_{\mathcal{M}}$ can be constructed according to Kripke's inductive construction, which is the least fixed point larger than or equal to h. In other words, for any fixed point h of $\rho_{\mathcal{M}}$, there is a fixed point of $\rho'_{\mathcal{M}}$ larger than or equal to h. Given any $A \in \mathit{Sent}(\mathcal{L}^+)$, suppose that A is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{lfp,\rho'}$, i.e., all fixed points of $\rho'_{\mathcal{M}}$ declare A neither true nor false. Hence all fixed points of $\rho_{\mathcal{M}}$ must declare A neither true nor false. A is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{lfp,\rho}$. So $\mathbf{T}^{lfp,\rho'} \leq_4 \mathbf{T}^{lfp,\rho}$.

The proof of (3) is as follows. Consider a truth language $\mathcal{L}^+$ without any nonlogical symbols except for $\boldsymbol{T}$, quote names, and a nonquote names a. Let $\mathcal{M} = \langle D, I\rangle$ be the ground model of $\mathcal{L}^+$ where $I(a) = \neg\boldsymbol{T}a$. Note that for each $\rho \in \{\mu, \kappa, \sigma, \sigma_1, \sigma_2\}$, and any fixed point h of $\rho_{\mathcal{M}}$, $h(\neg\boldsymbol{T}a) = \mathbf{n}$, and $h(\boldsymbol{T}a) = \mathbf{n}$.[5]

(3.1) Consider the sentence $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$. Let $\mathscr{S}$ be the unique ordinal-length sequence of hypotheses obtained in the construction of the least fixed point of $\sigma_{2\mathcal{M}}$ according to Kripke's inductive construction, where $\mathscr{S}_0$ is the empty hypothesis. Note that $\mathscr{S}$ is an increasing sequence of hypotheses. For every strongly consistent classical hypothesis $h \geq \mathscr{S}_0$, h must declare only one of $\neg\boldsymbol{T}a$ and $\boldsymbol{T}a$ true. Hence, $\sigma_{2\mathcal{M}}(\mathscr{S}_0)$ must declare $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ true. So *lfp*$(\sigma_{2\mathcal{M}})$ declares $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ true. Therefore $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ is not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\sigma_2}$. For any fixed point h of $\sigma_{1\mathcal{M}}$, it is not the case that h declares $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ false, since there is a fixed point of $\sigma_{2\mathcal{M}}$ larger than or equal to h. And since h is strongly consistent, there is a weakly consistent classical h' such that $h \leq h'$ and $h'(\neg\boldsymbol{T}a) = \mathbf{f}$, and $h'(\boldsymbol{T}a) = \mathbf{f}$. Hence, $\tau_{\mathcal{M}}(h')$ declares $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ false. So h must declare $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ neither true nor false. Because of the arbitrariness of h, all fixed points of $\sigma_{1\mathcal{M}}$ declare $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ neither true nor false. Hence, $\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\sigma_1}$. So $\mathbf{T}^{\mathit{lfp},\sigma_1} \not\leq_4 \mathbf{T}^{\mathit{lfp},\sigma_2}$.

(3.2) Consider the sentence $\neg\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \neg\boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$. Similar to (3.1), it can be prove that *lfp*$(\sigma_{1\mathcal{M}})$ declares $\neg\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \neg\boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ true, and that $\neg\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \neg\boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$ is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\sigma}$. So $\mathbf{T}^{\mathit{lfp},\sigma} \not\leq_4 \mathbf{T}^{\mathit{lfp},\sigma_1}$.

(3.3) Consider the sentence $\neg\boldsymbol{T}a \vee \boldsymbol{T}a$. Since for any fixed point h of $\kappa_{\mathcal{M}}$, $h(\neg\boldsymbol{T}a) = \mathbf{n}$, and $h(\boldsymbol{T}a) = \mathbf{n}$, $\kappa_{\mathcal{M}}(h)(\neg\boldsymbol{T}a \vee \boldsymbol{T}a) = \mathbf{n}$. Hence $\neg\boldsymbol{T}a \vee \boldsymbol{T}a$ is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\kappa}$. Let $\mathscr{S}$ be the unique ordinal-length sequence of hypotheses obtained in the construction of the least fixed point of $\sigma_{\mathcal{M}}$ according to Kripke's inductive construction, where $\mathscr{S}_0$ is the empty hypothesis. $\mathscr{S}$ is an increasing sequence of hypotheses. For every classical hypothesis $h \geq \mathscr{S}_0$, $\tau_{\mathcal{M}}(h)$ must declare $\neg\boldsymbol{T}a \vee \boldsymbol{T}a$ true. So *lfp*$(\sigma_{\mathcal{M}})$ declares $\neg\boldsymbol{T}a \vee \boldsymbol{T}a$ true. Therefore $\neg\boldsymbol{T}a \vee \boldsymbol{T}a$ is not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\sigma}$. So $\mathbf{T}^{\mathit{lfp},\kappa} \not\leq_4 \mathbf{T}^{\mathit{lfp},\sigma}$.

(3.4) Consider the sentence $\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)$. Since for any fixed point h of $\mu_{\mathcal{M}}$, $h(\neg\boldsymbol{T}a) = \mathbf{n}$, and $h(\neg\exists x(x = x)) = \mathbf{f}$, $\mu_{\mathcal{M}}(h)(\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)) = \mathbf{n}$. $h(\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)) = \mu_{\mathcal{M}}(h)(\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)) = \mathbf{n}$. Because of the arbitrariness of h, all fixed points of $\mu_{\mathcal{M}}$ declare $\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)$ neither true nor false. Hence $\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)$ is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\mu}$. Since for any fixed point h' of $\kappa_{\mathcal{M}}$, $h'(\neg\boldsymbol{T}a) = \mathbf{n}$, and $h'(\neg\exists x(x = x)) = \mathbf{f}$, $\kappa_{\mathcal{M}}(h')(\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)) = \mathbf{f}$. Hence $h'(\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)) = \mathbf{f}$. Hence $\neg\boldsymbol{T}a \wedge \neg\exists x(x = x)$ is not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\mathit{lfp},\kappa}$. So $\mathbf{T}^{\mathit{lfp},\mu} \not\leq_4 \mathbf{T}^{\mathit{lfp},\kappa}$. □

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[5] $\neg\boldsymbol{T}a$ can be regarded as a formalization of the Liar sentence.

Lemma 1. $\boldsymbol{T}^C \not\leq_4 \boldsymbol{T}^*$.

Proof Consider a truth language $\mathcal{L}^+$ without any nonlogical symbols except for $\boldsymbol{T}$, quote names, a nonquote names a, and and a 1-place predicate symbol G. Let A be the sentence $\neg\boldsymbol{T}\ulcorner\neg\boldsymbol{T}a\urcorner \vee \neg\boldsymbol{T}\ulcorner\boldsymbol{T}a\urcorner$. Define sentences $\boldsymbol{T}^n(A)(n \geq 0)$ as follows: $\boldsymbol{T}^0(A) = A$, and $\boldsymbol{T}^{n+1}(A) = \boldsymbol{T}\ulcorner\boldsymbol{T}^n(A)\urcorner$.

Let $\mathcal{M} = \langle D, I\rangle$ be the ground model of $\mathcal{L}^+$ where $I(a) = \neg\boldsymbol{T}a$ and $I(G) = \{A, \boldsymbol{T}^1(A), \boldsymbol{T}^2(A), \boldsymbol{T}^3(A), \ldots\}$.

Consider the sentence $\forall x(Gx \to \boldsymbol{T}x) \wedge A$. Let $\mathscr{S}$ be a revision sequence such that, $\mathscr{S}_0$ declares $\neg\boldsymbol{T}a$, $\boldsymbol{T}a$ and all elements in $I(G)$ true, and declares $\forall x(Gx \to \boldsymbol{T}x) \wedge A$ false, and for any limit ordinal α, $\mathscr{S}_\alpha$ declares both $\neg\boldsymbol{T}a$ and $\boldsymbol{T}a$ true.[6] The behavior of sentences in the set $I(G) \cup \{\neg\boldsymbol{T}a, \boldsymbol{T}a, \forall x(Gx \to \boldsymbol{T}x) \wedge A\}$ in $\mathscr{S}$ is shown in Table 1. Clearly, $\forall x(Gx \to \boldsymbol{T}x) \wedge A$ is stably false in $\mathscr{S}$. Hence $\forall x(Gx \to \boldsymbol{T}x) \wedge A \wedge \neg\boldsymbol{T}a$ is stably false in $\mathscr{S}$. So $\forall x(Gx \to \boldsymbol{T}x) \wedge A \wedge \neg\boldsymbol{T}a$ is not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^*$.

	$\mathscr{S}_0$	$\mathscr{S}_1$	$\mathscr{S}_2$	$\mathscr{S}_3$	$\mathscr{S}_4$	$\mathscr{S}_5$	$\ldots$	$\mathscr{S}_\omega$	$\mathscr{S}_{\omega+1}$	$\mathscr{S}_{\omega+2}$	$\mathscr{S}_{\omega+3}$	$\mathscr{S}_{\omega+4}$	$\ldots$	$\mathscr{S}_{\omega^2}$	$\ldots$
$\boldsymbol{T}a$	**t**	**t**	**f**	**t**	**f**	**t**	$\ldots$	**t**	**t**	**f**	**t**	**f**	$\ldots$	**t**	$\ldots$
$\neg\boldsymbol{T}a$	**t**	**f**	**t**	**f**	**t**	**f**	$\ldots$	**t**	**f**	**t**	**f**	**t**	$\ldots$	**t**	$\ldots$
A	**t**	**f**	**t**	**t**	**t**	**t**	$\ldots$	**t**	**f**	**t**	**t**	**t**	$\ldots$	**t**	$\ldots$
$\boldsymbol{T}^1(A)$	**t**	**t**	**f**	**t**	**t**	**t**	$\ldots$	**t**	**t**	**f**	**t**	**t**	$\ldots$	**t**	$\ldots$
$\boldsymbol{T}^2(A)$	**t**	**t**	**t**	**f**	**t**	**t**	$\ldots$	**t**	**t**	**t**	**f**	**t**	$\ldots$	**t**	$\ldots$
$\boldsymbol{T}^3(A)$	**t**	**t**	**t**	**t**	**f**	**t**	$\ldots$	**t**	**t**	**t**	**t**	**f**	$\ldots$	**t**	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$
$\forall x(Gx \to \boldsymbol{T}x) \wedge A$	**f**	**f**	**f**	**f**	**f**	**f**	$\ldots$	**f**	**f**	**f**	**f**	**f**	$\ldots$	**f**	$\ldots$

Table 1

Next, we prove that $\forall x(Gx \to \boldsymbol{T}x) \wedge A \wedge \neg\boldsymbol{T}a$ is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^C$. For any C-sequence $\mathscr{S}'$, any ordinal α, since $\mathscr{S}_\alpha$ is maximally consistent, $\mathscr{S}_{\alpha+1}$ declares A true. By the definition of revision sequence, A is stably true in $\mathscr{S}'$. Clearly, for every natural number n, $\boldsymbol{T}^n(A)$ is stably true in $\mathscr{S}'$. Hence $\forall x(Gx \to \boldsymbol{T}x) \wedge A$ is stably true in $\mathscr{S}'$, and $\forall x(Gx \to \boldsymbol{T}x) \wedge A \wedge \neg\boldsymbol{T}a$ is unstable in $\mathscr{S}'$. Because of the arbitrariness of $\mathscr{S}'$, $\forall x(Gx \to \boldsymbol{T}x) \wedge A \wedge \neg\boldsymbol{T}a$ is unstable in all C-sequences. So $\forall x(Gx \to \boldsymbol{T}x) \wedge A \wedge \neg\boldsymbol{T}a$ is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^C$. Therefore, $\mathbf{T}^C \not\leq_4 \mathbf{T}^*$. $\square$

Lemma 2. $\boldsymbol{T}^{lfp,\sigma_2} \not\leq_4 \boldsymbol{T}^C$.

Proof Let a truth language $\mathcal{L}^+$ have no nonlogical symbols except for $\boldsymbol{T}$, quote names, countably many nonquote names $a_n(n \in \mathbb{N})$, and a 1-place predicate symbol G. Define sentences ϕ, $\psi_n(n \in \mathbb{N})$, $A_n(n \in \mathbb{N})$ and B as follows:

[6] Note that both $\neg\boldsymbol{T}a$ and $\boldsymbol{T}a$ are unstable up to α.

Let $\phi = \forall x \forall y(Gx \wedge Gy \rightarrow (\boldsymbol{T}x \leftrightarrow \boldsymbol{T}y))$. For any $n \in \mathbb{N}$, Let $\psi_n = \bigwedge_{j,k \leq n}(\boldsymbol{T}a_j \leftrightarrow \boldsymbol{T}a_k)$. For any $n \in \mathbb{N}$, if n is even, then $A_n = \phi \vee (\neg\phi \wedge \psi_n \wedge \neg \boldsymbol{T}a_n) \vee (\neg\phi \wedge \neg\psi_n \wedge \boldsymbol{T}a_n)$; and if n is odd, then $A_n = (\neg\phi \wedge \psi_n \wedge \neg\boldsymbol{T}a_n) \vee (\neg\phi \wedge \neg\psi_n \wedge \boldsymbol{T}a_n)$. Let $B = \forall x(Gx \rightarrow \boldsymbol{T}x)$.

Let $\mathcal{M} = \langle D, I \rangle$ be the ground model of $\mathcal{L}^+$ where $I(a_n) = A_n$ for any $n \in \mathbb{N}$, and $I(G) = \{A_n \mid n \in \mathbb{N}\}$.

By compactness of classical first-order logic, the set $\{\neg A_n \mid n \in \mathbb{N}\} \cup \{\neg B\}$ is consistent. Hence there is a maximally consistent classical hypothesis h such that h declares all elements in $I(G)$ and B false. Let $\mathscr{S}$ be a C-sequence where $\mathscr{S}_0 = h$ and for any limit ordinal α, $\mathscr{S}_\alpha$ declares all elements in $I(G)$ false.[7] The behavior of sentences in the set $I(G) \cup \{B\}$ in $\mathscr{S}$ is shown in Table 2. Clearly, B is stably false in $\mathscr{S}$. So B is not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^C$.

	$\mathscr{S}_0$	$\mathscr{S}_1$	$\mathscr{S}_2$	$\mathscr{S}_3$	$\mathscr{S}_4$	$\mathscr{S}_5$	$\dots$	$\mathscr{S}_\omega$	$\mathscr{S}_{\omega+1}$	$\mathscr{S}_{\omega+2}$	$\dots$	$\mathscr{S}_{\omega^2}$	$\mathscr{S}_{\omega^2+1}$	$\mathscr{S}_{\omega^2+2}$	$\dots$
A_0	**f**	**t**	**f**	**t**	**f**	**t**	$\dots$	**f**	**t**	**f**	$\dots$	**f**	**t**	**f**	$\dots$
A_1	**f**	**f**	**f**	**t**	**f**	**t**	$\dots$	**f**	**f**	**f**	$\dots$	**f**	**f**	**f**	$\dots$
A_2	**f**	**t**	**t**	**t**	**f**	**t**	$\dots$	**f**	**t**	**t**	$\dots$	**f**	**t**	**t**	$\dots$
A_3	**f**	**f**	**f**	**f**	**f**	**t**	$\dots$	**f**	**f**	**f**	$\dots$	**f**	**f**	**f**	$\dots$
A_4	**f**	**t**	**t**	**t**	**t**	**t**	$\dots$	**f**	**t**	**t**	$\dots$	**f**	**t**	**t**	$\dots$
A_5	**f**	**f**	**f**	**f**	**f**	**f**	$\dots$	**f**	**f**	**f**	$\dots$	**f**	**f**	**f**	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$
B	**f**	**f**	**f**	**f**	**f**	**f**	$\dots$	**f**	**f**	**f**	$\dots$	**f**	**f**	**f**	$\dots$

Table 2

Next, we prove that B is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{lfp,\sigma_2}$. Given any C-sequence $\mathscr{S}'$, by the definiton of $A_n (n \in \mathbb{N})$, it is not the case that, for any $n \in \mathbb{N}$, A_n is stably true in $\mathscr{S}'$. For similar reason, it is not the case that, for any $n \in \mathbb{N}$, A_n is stably false in $\mathscr{S}'$. In fact, for any $n \in \mathbb{N}$, A_n is unstable up to any limit ordinal in $\mathscr{S}'$. Next, we prove that for any fixed point h of $\sigma_{2\mathcal{M}}$, h delcares B neither true nor false. Note that for any revision sequence $\mathscr{S}$, and any limit ordinal α, B is stably false up to α in $\mathscr{S}$. Let h be a fixed point h of $\sigma_{2\mathcal{M}}$. h is strongly consistent. Let h' be a maximally consistent classical hypothesis such that $h \leq h'$ and let $\mathscr{S}'$ be a C-sequence with $\mathscr{S}'_0 = h'$. It is easy to prove by transfinite induction that for every α, $h \leq \mathscr{S}'_\alpha$, since $\sigma_{2\mathcal{M}}$ is monotone and agrees with $\tau_{\mathcal{M}}$ on classical hypotheses. Let S_1 be the set $\{A \mid A \in \mathit{Sent}(\mathcal{L}^+)$ and A is stably true up to ω in $\mathscr{S}'\} \cup \{\neg A \mid A \in \mathit{Sent}(\mathcal{L}^+)$ and A is stably false up to ω in $\mathscr{S}'\} \cup \{\neg A_n \mid n \in \mathbb{N}\}$. Let S_2 be the set $\{A \mid A \in \mathit{Sent}(\mathcal{L}^+)$ and A is stably true up to ω in $\mathscr{S}'\} \cup \{\neg A \mid A \in \mathit{Sent}(\mathcal{L}^+)$ and A is stably false up to ω in $\mathscr{S}'\} \cup \{A_n \mid n \in \mathbb{N}\}$. By compactness of classical first-

[7] Note that all elements in $I(G)$ are unstable up to α and the set $\{A \mid A \in \mathit{Sent}(\mathcal{L}^+)$ and A is stably true up to α in $\mathscr{S}\} \cup \{\neg A \mid A \in \mathit{Sent}(\mathcal{L}^+)$ and A is stably false up to α in $\mathscr{S}\} \cup \{\neg A_n \mid n \in \mathbb{N}\}$ is consistent by compactness of classical first-order logic. Hence such a C-sequence exists.

order logic, both S_1 and S_2 are consistent. There is a maximally consistent classical hypothesis h_1 such that h_1 declares all elements in S_1 true. And there is a maximally consistent classical hypothesis h_2 such that h_2 declares all elements in S_2 true. It is clear that $h \leq h_1$ and $h \leq h_2$. However, $\tau_{\mathcal{M}}(h_1)(B) = \mathbf{f}$, and $\tau_{\mathcal{M}}(h_2)(B) = \mathbf{t}$. Hence $\sigma 2_{\mathcal{M}}(h)(B) = \mathbf{n}$. Then $h(B) = \mathbf{n}$. So B is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{lfp,\sigma_2}$.

Therefore, $\mathbf{T}^{lfp,\sigma_2} \nleq_4 \mathbf{T}^C$. □

Theorem 5. *Figure 2 uses an arrow to represent that the relation $\leq_4$ holds between the theory at the start of the arrow and the one at the end of the arrow. When restricted to the three revision theories, the relation $\leq_4$ is the reflexive transitive closure of the relation shown in Figure 2.*

$$\mathbf{T}^{\#} \longrightarrow \mathbf{T}^{*} \longrightarrow \mathbf{T}^{C}$$

Figure 2

Proof Since $\leq_4$ is reflexive and transitive, it is enough to prove the following claims: (1) $\mathbf{T}^{\#} \leq_4 \mathbf{T}^{*}$, $\mathbf{T}^{*} \leq_4 \mathbf{T}^{C}$; and (2) $\mathbf{T}^{*} \nleq_4 \mathbf{T}^{\#}$, $\mathbf{T}^{C} \nleq_4 \mathbf{T}^{*}$.

(1) holds by Definition 2.

For (2), consider Example 5.7 in [5, p. 387] . From [5], we know that the sentences in Y are all nearly stably true in every revision sequence, but no sentence in Y is stably true in any revision sequence. Hence all sentences in Y are paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{*}$, and all sentences in Y are not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{\#}$. Therefore, $\mathbf{T}^{*} \nleq_4 \mathbf{T}^{\#}$. $\mathbf{T}^{C} \nleq_4 \mathbf{T}^{*}$ is Lemma 1. □

Theorem 6. *Figure 3 uses an arrow to represent that the relation $\leq_4$ holds between the theory at the start of the arrow and the one at the end of the arrow. The relation $\leq_4$ is the reflexive transitive closure of the relation shown in Figure 3.*

Proof Given that $\leq_4$ is reflexive and transitive, Theorem 4 and Theorem 5, it is enough to prove the following claims: (1) $\mathbf{T}^{C} \leq_4 \mathbf{T}^{lfp,\sigma_2}$; and (2) $\mathbf{T}^{lfp,\sigma_2} \nleq_4 \mathbf{T}^{C}$.

The proof of (1) is as follows. Suppose $\mathbf{T}^{C} \nleq_4 \mathbf{T}^{lfp,\sigma_2}$, then there is a truth language $\mathcal{L}^+$ and a ground model $\mathcal{M}$ of $\mathcal{L}^+$ and a sentence $A \in \mathit{Sent}(\mathcal{L}^+)$ such that A is paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{C}$, and A is not paradoxical in $\mathcal{M}$ according to the theory $\mathbf{T}^{lfp,\sigma_2}$. Then there is a fixed point h of $\sigma 2_{\mathcal{M}}$ such that $h(A) = \mathbf{t}$ or $\mathbf{f}$. Let h' be a maximally consistent classical hypothesis such that $h \leq h'$. Let $\mathscr{S}'$ be a C-sequence with $\mathscr{S}'_0 = h'$. It can be proved by transfinite induction that for every α, $h \leq \mathscr{S}'_\alpha$, given that $\sigma 2_{\mathcal{M}}$ is monotone and agrees with $\tau_{\mathcal{M}}$ on classical hypotheses. Hence, A is stably true or stably false in $\mathscr{S}'$. Contradiction. Therefore, $\mathbf{T}^{C} \leq_4 \mathbf{T}^{lfp,\sigma_2}$.

(2) is Lemma 2. □

$$\begin{tikzcd}
& & \mathbf{T}^{\mathit{gifp},\sigma_2} \arrow[d, leftrightarrow] & \mathbf{T}^{\mathit{gifp},\sigma_1} \arrow[d, leftrightarrow] & \mathbf{T}^{\mathit{gifp},\sigma} \arrow[d, leftrightarrow] & \mathbf{T}^{\mathit{gifp},\kappa} \arrow[d, leftrightarrow] & \mathbf{T}^{\mathit{gifp},\mu} \arrow[d, leftrightarrow] \\
& \mathbf{T}^{C} \arrow[r] & \mathbf{T}^{\mathit{lfp},\sigma_2} \arrow[r] & \mathbf{T}^{\mathit{lfp},\sigma_1} \arrow[r] & \mathbf{T}^{\mathit{lfp},\sigma} \arrow[r] & \mathbf{T}^{\mathit{lfp},\kappa} \arrow[r] & \mathbf{T}^{\mathit{lfp},\mu} \\
\mathbf{T}^{\#} \arrow[r] & \mathbf{T}^{*} \arrow[u] & & & & &
\end{tikzcd}$$

Figure 3

5 Conclusion

In this paper, we present a complete picture of comparing the thirteen theories of truth considered in Kremer's comparative work ([5]), according to the relation $\leq_4$. In our comparison, we show that among the thirteen theories, the fixed-point theories $\mathbf{T}^{\mathit{lfp},\mu}$ and $\mathbf{T}^{\mathit{gifp},\mu}$ are the largest ones, and revision theory $\mathbf{T}^{\#}$ is the least one, according to $\leq_4$.

This paper extends Kremer's comparative work by comparing these theories the perspective of paradoxicality. In the future, we will add more theories of truth to this comparative work, for example, five other revision theories considered in [8], including Gupta's one in [2], Herzberger's one in [4] and Yaqūb's one in [10].

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从悖论性的角度比较关于真的不动点理论和修正理论

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摘　要

出于“在各种各样的选择中了解情况”的目的，克莱默（2009）定义了三个不同的关系来比较文献中的十个不动点理论和三个修正理论。本文通过从另一个重要的的角度对这些理论进行比较——悖论性的角度，来延续克莱默的比较工作。悖论性这个观念对真理论很重要，它甚至影响了哲学家对具体真理论的选择。本文根据这个比较的角度定义了一个二元关系，并依据这个新关系建立了克莱默（2009）所考虑的十三个真理论之间的关系。

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