Studies in Logic, Vol. 13, No. 3 (2020): 32–61
PII: 1674-3202(2020)-03-0032-30

Some Results on Rewritability in Modal Logics over Tree Models*

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Abstract. We have investigated locally equivalent and m-conservative rewritabilities in modal logics over tree models. The modal languages studied in this paper are ML, MLI, MLG and MLGI.

1 Introduction

Over the past 100 years, many artificial languages in distinct expressiveness powers and complexity degrees have been introduced. They range from classical first-order and higher-order predicate languages to a large variety of modal languages.

Different types of languages may be "expressed" by other languages. For example, it is well known that each ML-formula is locally equivalently rewritable into a first-order formula over models. However, the converse does not hold:

- There are fisrt-order formulas that cannot be locally equivalently rewritable into an ML-formula.

van Benthem Characterization Theorem in [2] characterizes when exactly a first-order formula is locally equivalently rewritable into an ML-formula. Following van Benthem Characterization Theorem, (locally) equivalent rewritability has become an important and active research problem in modal logic ([4]) and computer science ([11, 12]) over the past 40 years.

E. Rosen has proved characterization theorems on (locally) equivalent rewritability over finite Kripke models in [16]. Characterization theorems over any class of Kripke models are proved by M. Otto in [14], where different versions of characterization theorems for MLI, ML plus a global modality and MLI plus a global modality are also proved. M. de Rijke proves characterization theorems for MLG in [15]. Otto proves similar theorems for μML [1] in [13]. A. Dawar and Otto proves several

Received 2019-02-26

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*This paper is supported by "the Fundamental Research Funds for the Central Universities" (No.410500139).

Give my sincere appreciation to Professor Frank Wolter and Ana Ozaki for their great contribution to this paper.

[1] The modal language μML is ML plus a monadic least fixed point operator.

characterization theorems over kinds of classes of frames in [6]. A model-theoretic characterization of MSO [2] to μML is proved by D. Janin and I. Walukiewicz in [9]. G. Fontaine proves a model-theoretic characterization of MSO to μML over tree models in [7]. Some theorems on globally equivalent rewritability of MLI to ML, MLG to ML, ML to EL [3] over models are proved by F. Wolter in [17].

Equivalent rewritability is an important notion, but it is rather strict for it cannot introduce additional non-logical symbols. So it is necessary to introduce a weaker notion admitting additional non-logical symbols, i.e., "conservative rewritability", which aims at a conservative extension rather than an equivalent one. Conservative rewritability is often studied in description logics ([1]). Some important theorems in this area are proved by [10] and [12]. [7] resolves the global case of m-conservative (i.e., model conservative) rewritability of MSO to μML over tree models. The locally m-conservative rewritability of MLG to ML over tree models can be inferred from some results in [7]. [17] characterizes the global cases of s-conservative [4] and m-conservative rewritability of MLI to ML, MLG to ML and ML to EL over models.

Local equivalent and m-conservative rewritability over tree models are studied in this paper. Modal lauguages considered include ML, MLI, MLG and MLGI. Section 3 of this paper proves that each MLI-formula is equivalently rewritable into an ML-formula at roots over tree models. Section 4 resolves whether each MLGI-formula is equivalently rewritable into an MLG-formula at roots over tree models. Section 5 characterizes the equivalent and m-conservative rewritability of "MLGI to MLI" over tree models. Section 6 resolves m-conservative rewritability of "MLGI to ML" over tree models.

2 Preliminaries

Syntax ML-formulas are formed according to the rule:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi^5$$

where p is a propositional variable. Other connectives are defined as follows: $\varphi \vee \psi ::= \neg(\neg\varphi \wedge \neg\psi)$, $\top ::= p \vee \neg p$, $\bot ::= \neg\top$, $\Box\varphi ::= \neg\Diamond\neg\varphi$.

MLI is ML plus $\Diamond^{-}$,[6] MLG is ML plus $\Diamond^{\geq n}$ and MLGI is ML plus $\Diamond^{\geq n}$ and

[2]MSO represents "monadic second-order".

[3]ML is the standard modal language; MLI is ML plus inverse modalities; MLG is ML plus graded modalities; MLGI is ML plus graded and inverse modalities; EL is a tractable modal language. For reference, see [17].

[4]S-conservative rewritability is another notion of conservative rewritability, being different from m-conservative rewritability. However, it is not studied in this paper. For reference, see [10].

[5]$\Diamond$ represents $\Diamond^{\geq 1}$.

[6]$\Diamond^{-}$ represents $\Diamond^{-\geq 1}$.

$\Diamond^{-\geq n}$. It should be noticed that

$$\Diamond^{\leq n}\varphi ::= \neg(\Diamond^{\geq n+1}\varphi)$$

and

$$\Diamond^{-\leq n}\varphi := \neg(\Diamond^{-\geq n+1}\varphi),$$

while $n \in N$, $n \geq 1$ and N is the set of natural numbers.

Model A (Kripke) *frame* F is a pair (W, R) and a (Kripke) *model* M is a triple (W, R, V), where W is a non-empty set of states, R is a binary relation on W and V is a valuation. A pointed model is a pair (M, d), where M is a model and $d \in W$.

Let R^{rt} be the reflexive transitive closure of R. If there is a unique $d^* \in W$ such that $d^* R^{rt} d$ for each $d \in W$, then the frame (W, R) is called *rooted* and d^* is its root. A rooted frame (W, R) with d^* being its root is a *tree* if each state $d \in W$ is reachable from d^* by a unique R-path $d^* R \cdots Rd$. A model (W, R, V) is a tree model if its underlying frame (W, R) is a tree.

The truth-relations for ML-formulas (MLI-formulas, MLG-formulas and MLGI-formulas) are defined in the familiar way for the atomic and boolean cases. The other cases are as follows:

- $(M, d) \models \Diamond^{\geq n}\alpha$ iff $(M, d') \models \alpha$ for at least n different points $d' \in W$ such that dRd';
- $(M, d) \models \Diamond^{-\geq n}\alpha$ iff $(M, d') \models \alpha$ for at least n different points $d' \in W$ such that $d'Rd$;

$V(\varphi)$ is defined as $\{d \in W : (M, d) \models \varphi\}$ for each formula φ.

Rewritability Let L_i ($i \in \{1, 2\}$) be a modal language. An L_1-formula φ is *locally equivalently rewritable* into an L_2-formula (or a set of L_2-formulas Δ^*) over a class of models C if there is an L_2-formula ψ (or a set of L_2-formulas Δ^*) such that

- for each model $M = (W, R, V) \in C$ and $d \in W$, $(M, d) \models \varphi$ iff $(M, d) \models \psi$ (or $(M, d) \models \Delta^*$).

An L_1-formula φ is *locally m-conservatively rewritable*[7] into an L_2-formula (or a set of L_2-formulas Δ^*) over a class of models C if there is an L_2-formula ψ (or a set of L_2-formulas Δ^*) such that

- for each model $M = (W, R, V) \in C$ and $d \in W$, if $(M, d) \models \psi$ (or $(M, d) \models \Delta^*$), then $(M, d) \models \varphi$.
- for each model $M = (W, R, V) \in C$ and $d \in W$ such that $(M, d) \models \varphi$, there is a model $M' \in C$ such that $M' = (W, R, V')$ and $M =_{sig(\varphi)} M'$ and $(M', d) \models \psi$ (or $(M', d) \models \Delta^*$).

[7] For reference, see [10].

Here $M =_{sig(\varphi)} M'$ means that $V(p) = V'(p)$ for each propositional variable $p \in sig(\varphi)$ and $sig(\varphi)$ represents the set of propositional variables occurring in φ.

According to the definition of locally m-conservative rewritability, the unique difference between M and M' is valuations of propositional variables, i.e., $M = M'$ iff $V = V'$. Therefore, if an L_1-formula is locally equivalently rewritable into an L_2-formula ψ(or a set of L_2-formulas Δ^*) over a class of models C, then it is locally m-conservatively rewritable into the L_2-formula ψ(or the set of L_2-formulas Δ^*) over the class of models C.

When C is the set of all Kripke models, "over a class of models C" is omitted. When $C = \{(M, d^*) : M$ is a tree model and d^* is its root $\}$, an L_1-formula φ is said to be equivalently (or m-conservatively) rewritable into an L_2-formula (or a set of L_2-formulas) *at roots over tree models*.

Degree The degree of an MLGI-formula is defined as follows:

- $Deg(p) = 0$,
- $Deg(\bot) = 0$,
- $Deg(\neg\varphi) = Deg(\varphi)$,
- $Deg(\varphi \wedge \psi) = \max\{Deg(\varphi), Deg(\psi)\}$,
- $Deg(\Diamond^{\geq n}\varphi) = Deg(\varphi) + 1$,
- $Deg(\Diamond^{-\geq n}\varphi) = Deg(\varphi) + 1$.

The degree of an MLI-formula (or an MLG-formula) is defined similarly.

Bisimulation Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be two Kripke models.

A non-empty relation $S \subseteq W_1 \times W_2$ is *a bisimulation*[8] between (M_1, d_1) and (M_2, d_2) if the following conditions are satisfied:

- $(d_1, d_2) \in S$;
- if $(u, v) \in S$, u and v satisfy the same propositional variables;
- if $(u, v) \in S$ and uR_1x_1, there is an x_2 such that vR_2x_2 such that $(x_1, x_2) \in S$ (*the forth condition*);
- if $(u, v) \in S$ and vR_2x_2, there is an x_1 such that uR_1x_1 such that $(x_1, x_2) \in S$ (*the back condition*).

A non-empty relation $S \subseteq W_1 \times W_2$ is *an i-bisimulation* between (M_1, d_1) and (M_2, d_2) if S satisfies all the conditions for bisimulation and the following conditions:

- if $(u, v) \in S$ and x_1R_1u, there is an x_2 such that x_2R_2v such that $(x_1, x_2) \in S$ (*the inverse forth condition*);

[8] For reference, see [5].

- if $(u, v) \in S$ and $x_2 R_2 v$, there is an x_1 such that $x_1 R_1 u$ such that $(x_1, x_2) \in S$ (*the inverse back condition*).

A non-empty relation $S \subseteq W_1 \times W_2$ is *an n-bisimulation* between (M_1, d_1) and (M_2, d_2)[9] if there is a sequence of relations $S_n \subseteq \cdots \subseteq S_0$ such that: $S = \bigcup_{0 \leq i \leq n} S_i$ and for each $0 \leq i < n$

- $(d_1, d_2) \in S_n$;
- if $(u, v) \in S_0$, u and v satisfy the same propositional variables;
- if $(u, v) \in S_{i+1}$ and $u R_1 x_1$, there is x_2 such that $v R_2 x_2$ and $(x_1, x_2) \in S_i$ (*the forth condition*);
- if $(u, v) \in S_{i+1}$ and $v R_2 x_2$, there is x_1 such that $u R_1 x_1$ and $(x_1, x_2) \in S_i$ (*the back condition*).

A non-empty relation $S \subseteq W_1 \times W_2$ is *an n-i-bisimulation* between (M_1, d_1) and (M_2, d_2) if there is a sequence of binary relations $S_n \subseteq \cdots \subseteq S_0$ such that $S = \bigcup_{0 \leq i \leq n} S_i$ and it satisfies all the conditions for n-bisimulation and the following conditions, i.e., for each $0 \leq i < n$

- if $(u, v) \in S_{i+1}$ and $x_1 R_1 u$, there is x_2 such that $x_2 R_2 v$ and $(x_1, x_2) \in S_i$ (*the inverse forth condition*);
- if $(u, v) \in S_{i+1}$ and $x_2 R_2 v$, there is x_1 such that $x_1 R_1 u$ and $(x_1, x_2) \in S_i$ (*the inverse back condition*).

van Benthem Characterization Theorem equivalently rewrites a first-order formula into an ML-formula by bisimulation. (See [2] and [3].)

Theorem 1 (van Benthem Characterization Theorem). *A first-order formula $A(x)$ is invariant under bisimulations iff it is locally equivalently rewritable into the standard translation of an ML-formula.*

A non-empty binary relation $S \subseteq W_1 \times W_2$ is *a counting bisimulation* between (M_1, d_1) and (M_2, d_2) if the following conditions are satisfied:

- $(d_1, d_2) \in S$;
- if $(u, v) \in S$, u and v satisfy the same propositional variables;
- if $(u, v) \in S$ and $X_1 \subseteq u\uparrow$ is finite[10], there is an $X_2 \subseteq v\uparrow$ such that S contains a bijection between X_1 and X_2 (*the forth condition*);
- if $(u, v) \in S$ and $X_2 \subseteq v\uparrow$ is finite, there is an $X_1 \subseteq u\uparrow$ such that S contains a bijection between X_1 and X_2 (*the back condition*).

[9] For reference, see [5].

[10] $x\uparrow = \{y \in W : xRy\}$.

A non-empty binary relation $S \subseteq W_1 \times W_2$ is *an n_m-counting bisimulation*[11] between (M_1, d_1) and (M_2, d_2) if there is a sequence of binary relations $S_n \subseteq \cdots \subseteq S_0$ such that $S = \bigcup_{0 \leq i \leq n} S_i$ and it satisfies the following conditions, i.e., for each $0 \leq i < n$

- $(d_1, d_2) \in S_n$;
- if $(u, v) \in S_0$, u and v satisfy the same propositional variables;
- if $(u, v) \in S_{i+1}$, $\mid X_1 \mid \leq m$ and $X_1 \subseteq u\uparrow$, there is an $X_2 \subseteq v\uparrow$ such that S_i contains a bijection between X_1 and X_2 (*the forth condition*);
- if $(u, v) \in S_{i+1}$, $\mid X_2 \mid \leq m$ and $X_2 \subseteq v\uparrow$, there is an $X_1 \subseteq u\uparrow$ such that S_i contains a bijection between X_1 and X_2 (*the back condition*).

A non-empty binary relation $S \subseteq W_1 \times W_2$ is an *n_m-i_k-counting bisimulation*[12]between (M_1, d_1) and (M_2, d_2) if there is a sequence of binary relations $S_n \subseteq \cdots \subseteq S_0$ such that $S = \bigcup_{0 \leq i \leq n} S_i$ and it satisfies the conditions for n_m-counting bisimulation and the following conditions, i.e., for each $0 \leq i < n$:

- if $(u, v) \in S_{i+1}$, $\mid Y_1 \mid \leq k$ and $Y_1 \subseteq u\downarrow$[13], then there is a $Y_2 \subseteq v\downarrow$ such that S_i contains a bijection between Y_1 and Y_2 (*the inverse forth condition*);
- if $(u, v) \in S_{i+1}$, $\mid Y_2 \mid \leq k$ and $Y_2 \subseteq v\downarrow$, then there is a $Y_1 \subseteq u\downarrow$ such that S_i contains a bijection between Y_1 and Y_2 (*the inverse back condition*).

Let us discuss these different but resembled bisimulations. According to their definitions, it is known that

- each counting bisimulation is also a bisimulation. However, the inverse does not hold, i.e., not each bisimulation is a counting one.
- each (n-)i-bisimulation is also an (n-)bisimulation since i only means the extra conditions for predecessors. However, it is obvious that the inverse does not hold.
- each n_m-i_k-counting bisimulation is also an n_m-counting bisimulation. The inverse does not hold.
- when $m = 1$, the n_m-counting bisimulation becomes an n-bisimulation. It means that an n-bisimulation is in fact a special case of n_m-counting bisimulations when $m = 1$.
- when $m = 1$ and $k = 1$, the n_m-i_k-counting bisimulation becomes an n-i-bisimulation. In fact, an n-i-bisimulation is a special case of n_m-i_k-counting bisimulation when $m = 1$ and $k = 1$. It should be noticed that each n_m-i_k-

[11] If "m" is changed into "finite", it is an n-counting bisimulation.

[12] If "k" is changed into "finite", it is *an n_m-i-counting bisimulation*. If "m" and "k" are both changed into "finite", it is *an n-i-counting bisimulation*.

[13] $x\downarrow = \{y \in W : yRx\}$.

counting bisimulation between two tree models is in fact an n_m-i_1-counting bisimulation since any point, except the root[14], in a tree model has only one predecessor.

Figure 1 gives an example of 2_2-i_1-counting bisimulation S. Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be two (tree) models showed in Figure 1 with $V_1(p) = V_2(p)$ for each propositional variable p. Define a sequence of binary relations $S_2 \subseteq S_1 \subseteq S_0$ as follows:

$$S_2 = \{(a_0, b_0)\}$$
$$S_1 = \{(a_0, b_0), (a_1, b_2), (a_2, b_1), (a_3, b_2)\}$$
$$S_0 = \{(a_0, b_0), (a_1, b_2), (a_2, b_1), (a_3, b_2), (a_4, b_3), (a_5, b_4), (a_6, b_4)\}.$$

Let $S = \bigcup_{0 \leq i \leq 2} S_i$. It is easy to prove that S is a 2_2-i_1-counting bisimulation between M_1 and M_2.

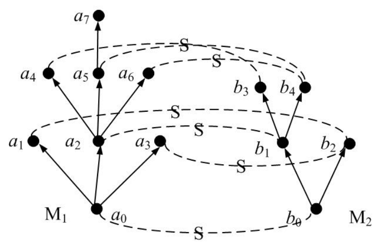

Figure 1

A *p-morphism* from (M_1, d_1) to (M_2, d_2) is a special bisimulation S[15] between (M_1, d_1) and (M_2, d_2) if S is a surjective function from W_1 to W_2.

An *i-p-morphism* from (M_1, d_1) to (M_2, d_2) is a special i-bisimulation S between (M_1, d_1) and (M_2, d_2) if S is a surjective function from W_1 to W_2.

Let Σ be a set of propositional variables. Each type of #-bisimulation is called a Σ-#-bisimulation, if no truth of propositional variables except those in Σ are considered.

Invariance An L-formula φ is *invariant under #-bisimulations over a class of models C* [16], if

$$(M_1, d_1) \models \varphi \text{ iff } (M_2, d_2) \models \varphi.^{17}$$

[14]The root of a tree model has no predecessor.

[15]For reference, see [5].

[16]# represents a type of bisimulation.

[17]If it is substituted by "if $(M_1, d_1) \models \varphi$, then $(M_2, d_2) \models \varphi$", then it means "preservation under #-bisimulations over a class of models C".

for each #-bisimulation between $(M_1, d_1), (M_2, d_2) \in C$.

An L-formula φ is *locally preserved under inverse (i-)p-morphisms over a class of models* C if there is a(n) (i-)p-morphism f from the pointed model $(M_1, d_1) \in C$ to the pointed model $(M_2, f(d_1)) \in C$ such that $(M_2, f(d_1)) \models \varphi$, then $(M_1, d_1) \models \varphi$. When C is the class of all models, "over a class of models C" is omitted.

This paper focuses on the class of all tree models. In some sections, the case that $C = \{(M, d^*) : M$ is a tree model and d^* is its root $\}$ will be considered and it says to be ***invariant (or preserved) under #-bisimulations at roots over tree models***.

Clearly invariance (or perservation) under bisimulations implies invariance (or perservation) under n-bisimulations for each $n \in N$.[18]

3 MLI to ML

The following theorem is proved in this section: "each MLI-formula is equivalently rewritable into an ML-formula at roots over tree models". However, Lemma 1 has to be proved first.

Lemma 1 From each bisimulation between tree models with their roots being mapped to each other, an i-bisimulation is constructed between the same two tree models with their roots being mapped to each other.

Proof Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be two tree models with d_1^* and d_2^* being their roots respectively. Assume that S is a bisimulation between (M_1, d_1^*) and (M_2, d_2^*) with $(d_1^*, d_2^*) \in S$. Now $S_i \subseteq S$ ($i \in N$) is defined as follows:

$S_0 = \{(d_1^*, d_2^*)\}$.
$S_{i+1} = \{(u, v) \in S : \exists x \exists y (x R_1 u \wedge y R_2 v \wedge (x, y) \in S_i)\}$.

Let

$$S^* = \bigcup_{i \in N} S_i.$$

Since each $S_i \subseteq S$, $S^* \subseteq S$. For constructing S^*, those pairs having no predecessor pairs that belong to S are deleted from S .

Assume that $(d, e) \in S^*$ and $d R_1 d'$. Then $(d, e) \in S_i$ for some $i \in N$. So $(d, e) \in S$. By the assumption that S is a bisimulation between (M_1, d_1^*) and (M_2, d_2^*) with $(d_1^*, d_2^*) \in S$, there is a point $e' \in W_2$ such that $e R_2 e'$ and $(d', e') \in S$. By the definition of S^*, $(d', e') \in S_{i+1}$ and then $(d', e') \in S^*$. That is, S^* satisfies the forth condition. The back condition can be proved similarly. Now consider the inverse forth and inverse back conditions. Since M_1 and M_2 are both tree models, each point except their roots has only one predecessor. If a pair in S^* is not the root pair (d_1^*, d_2^*),

[18] For reference, see p. 265 in [8].

the inverse forth and back conditions hold by the definition of S^*. If a pair in S^* is the root pair (d_1^*, d_2^*), the inverse forth and back conditions also hold because the roots have no predecessors at all. □

Proposition 2 follows from Lemma 1 directly.

Proposition 2 From each n-bisimulation between tree models with their roots being mapped to each other, an n-i-bisimulation is constructed between the same two tree models with their roots being mapped to each other.

Proposition 3 Each MLI-formula φ with *Deg*$(\varphi) \leq n$ is invariant under n-i-bisimulations.

Proof Let φ be an MLI-formula with *Deg*$(\varphi) \leq n$. Assume that there is an n-i-bisimulation S_0 with a sequence $S_n \subseteq \cdots \subseteq S_0$ between (M, w) and (M', w') and $(w, w') \in S_n$. We should prove that

$$(M, w) \models \varphi \text{ iff } (M', w') \models \varphi. \tag{1}$$

We prove (1) by induction on the construction of MLI-formulas. The basis and boolean cases are trivial.

Now consider the case that $\varphi = \Diamond\psi$. Assume that $(M, w) \models \varphi$. Then there is a successor v of w in M such that $(M, v) \models \psi$. By the definition of an n-i-bisimulation, there is a successor v' of w' in M' such that $(v, v') \in S_{n-1}$. So there is an $(n-1)$-i-bisimulation S_0 with $S_{n-1} \subseteq \cdots \subseteq S_0$ between (M, v) and (M', v') and $(v, v') \in S_{n-1}$. Since *Deg*$(\psi) \leq n-1$, by induction hypothesis,

$$(M, v) \models \psi \text{ iff } (M', v') \models \psi.$$

By $(M, v) \models \psi$, we have that $(M', v') \models \psi$. Thus $(M', w') \models \varphi$. The inverse is proved similarly.

Now consider the case that $\varphi = \Diamond^-\psi$. Assume that $(M, w) \models \varphi$. Then there is a predecessor v of w in M such that $(M, v) \models \psi$. By the definition of an n-i-bisimulation, there is a predecessor v' of w' in M' such that $(v, v') \in S_{n-1}$. Then there is an $(n-1)$-i-bisimulation S_0 with $S_{n-1} \subseteq \cdots \subseteq S_0$ between (M, v) and (M', v') and $(v, v') \in S_{n-1}$. Since *Deg*$(\psi) \leq n-1$, by induction hypothesis,

$$(M, v) \models \psi \text{ iff } (M', v') \models \psi.$$

By $(M, v) \models \psi$, we have that $(M', v') \models \psi$. Thus $(M', w') \models \varphi$. The inverse is proved similarly. □

We introduce characteristic ML-formulas $\chi^n_{[M,d]}$ in [8].

Definition 4 (**Characteristic ML-Formula**) Let Φ be a finite set of propositional variables and (M, d) be a pointed model with $M = (W, R)$. The characteristic ML-formula $\chi^n_{[M,d]}$ $(n \in N)$ is defined as follows:

- $\chi^0_{[M,d]}$ is purely propositional, consisting of the conjunction of all $p \in \Phi$ that are true at the point d and all $\neg p$ for those $p \in \Phi$ that are false at d;
- $$\chi^{n+1}_{[M,d]} = \chi^0_{[M,d]} \wedge \bigwedge_{dRd'} \Diamond \chi^n_{[M,d']} \wedge \Box \bigvee_{dRd'} \chi^n_{[M,d']}$$
  .

The main result of this section Theorem 5 is proved now.

Theorem 5. *Each MLI-formula is equivalently rewritable into an ML-formula at roots over tree models.*

Proof Assume that φ is an MLI-formula and *Deg*$(\varphi) \leq n$. Let C be the class of tree models (M, d^*) with d^* being its root such that $(M, d^*) \models \varphi$. Assume that $(M_1, d_1^*) \in C$ and there is an n-bisimulation S between the tree model (M_1, d_1^*) and a tree model (M', d') with d' being its root such that $(d_1^*, d') \in S$. By Proposition 2, an n-i-bisimulation S^* between (M_1, d_1^*) and (M', d') with $(d_1^*, d') \in S^*$ is constructed. Then by Proposition 3, $(M_1, d_1^*) \in C$ and $(M_1, d_1^*) \models \varphi$, we have that $(M', d') \models \varphi$. Therefore, $(M', d') \in C$. That is, C is closed under n-bisimulations at roots over tree models. By Corollary 34 of [8][19], since C is closed under n-bisimulations, C is definable by the ML-formula

$$\bigvee_{(M,d^*) \in C} \chi^n_{[M,d^*]}$$

with $\Phi = sig(\varphi)$ [20]of Definition 4. Therefore, the MLI-formula φ is equivalently rewritable into an ML-formula at roots over tree models. □

Proposition 6 follows directly from Theorem 5.

Proposition 6 Each MLI-formula is m-conservatively rewritable into an ML-formula at roots over tree models.

However, is each MLI-formula equivalently rewritable into an ML-formula **at any point** over tree models? The answer is "**No**", answered by Example 7.

[19]Corollary 34 in [8] says that a class of pointed Kripke structures being closed under n-bisimulations is definable by an ML-formula in a finite vocabulary.

[20]The ML-formula is finite as there are only finitely many such $\chi^n_{[M,d^*]}$ up to logical equivalence in the vocabulary *sig*(φ) of φ.

Example 7 Assume that $\Diamond^{-}\top$ is equivalently rewritable into an ML-formula ψ at each point over tree models, i.e.,

$$(M, d) \models \Diamond^{-}\top \text{ iff } (M, d) \models \psi$$

for each tree model (M, d). Figure 2 says that $(M_1, a) \not\models \Diamond^{-}\top$ and then $(M_1, a) \not\models \psi$. The tree model M_1 is a generated submodel of M_2 in Figure 2. Since ψ is an ML-formula, $(M_2, a) \not\models \psi$. However, $(M_2, a) \models \Diamond^{-}\top$. Thus $\Diamond^{-}\top$ is not equivalently rewritable into an ML-formula at each point over tree models.

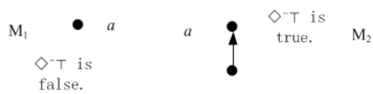

Figure 2

4 MLGI to MLG

For an MLGI-formula φ, let

$$Ind(\varphi) = \max\{n \in N : \Diamond^{\geq n} \text{ occurring in } \varphi\}$$

and

$$Ind^{-}(\varphi) = \max\{n \in N : \Diamond^{-\geq n} \text{ occurring in } \varphi\}.$$

Proposition 8 Each MLGI-formula φ with *Deg*$(\varphi) \leq n$, $Ind(\varphi) \leq m$ and $Ind^{-}(\varphi) \leq k$ is invariant under n_m-i_k-counting bisimulations.

Proof This proposition is proved by induction on the construction of MLGI-formulas φ with *Deg*$(\varphi) \leq n$, $Ind(\varphi) \leq m$ and $Ind^{-}(\varphi) \leq k$. Assume that there is an n_m-i_k-counting bisimulation S between (M, d) and (M', d'). The basis and boolean cases are trivial.

Now consider the case that $\varphi = \Diamond^{\geq l}\psi$. Assume that $(M, d) \models \varphi$. Then there are at least l different successors $d_1, \cdots, d_l$ of d in M such that $(M, d_i) \models \psi$ for each $1 \leq i \leq l$. By the definition of an n_m-i_k-counting bisimulation, $n \geq 1$ and $l \leq m$, there are at least l different successors $d'_1, \cdots, d'_l$ of d' in M' such that $(d_1, d'_1), \cdots, (d_l, d'_l) \in S$. Then there is an $(n-1)_m$-i_k-counting bisimulation $S'_i \subseteq S$ between (M, d_i) and (M', d'_i) for each $1 \leq i \leq l$. Since *Deg*$(\psi) \leq n-1$, $Ind(\psi) \leq m$ and $Ind^{-}(\psi) \leq k$, by induction hypothesis, $(M, d_i) \models \psi$ iff $(M', d'_i) \models \psi$ for each $1 \leq i \leq l$. By $(M, d_i) \models \psi$ for each $1 \leq i \leq l$,

$$(M', d'_i) \models \psi$$

for each $1 \leq i \leq l$. Thus $(M', d') \models \varphi$. The inverse can be proved similarly.

Now consider the case that $\varphi = \Diamond^{-\geq l}\psi$. Assume that $(M, d) \models \varphi$. Then there are at least l different predecessors $d_1, \cdots, d_l$ of d in M such that $(M, d_i) \models \psi$ for each $1 \leq i \leq l$. By the definition of an n_m-i_k-counting bisimulation, $n \geq 1$ and $l \leq k$, there are at least l different predecessors $d'_1, \cdots, d'_l$ of d' in M' such that $(d_1, d'_1), \cdots, (d_l, d'_l) \in S$. Then there is an $(n-1)_m$-i_k-counting bisimulation $S'_i \subseteq S$ between (M, d_i) and (M', d'_i) for each $1 \leq i \leq l$. Since $Deg(\psi) \leq n-1$, $Ind(\psi) \leq m$ and $Ind^-(\psi) \leq k$, by induction hypothesis, $(M, d_i) \models \psi$ iff $(M', d'_i) \models \psi$ for each $1 \leq i \leq l$. By $(M, d_i) \models \psi$ for each $1 \leq i \leq l$,

$$(M', d'_i) \models \psi$$

for each $1 \leq i \leq l$. Thus $(M', d') \models \varphi$. The inverse can be proved similarly. $\square$

Proposition 9 From each n_m-counting bisimulation between two tree models with their roots being mapped to each other, an n_m-i_k-counting bisimulation ($k \geq 1$) is constructed between these two tree models with their roots being mapped to each other.

Proof Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be two tree models with d_1^* and d_2^* being their roots respectively. Assume that

$$S = \bigcup_{0 \leq i \leq n} S_i$$

is an n_m-counting bisimulation between (M_1, d_1^*) and (M_2, d_2^*) with $(d_1^*, d_2^*) \in S$. Let $S'_0, \cdots, S'_n$ be as follows:

$S'_n = \{(d_1^*, d_2^*)\}$,
$S'_i = \{(u', v') : uR_1u', vR_2v', u'S_iv' \;\&\; uS'_{i+1}v\}$.

Let

$$S' = \bigcup_{0 \leq i \leq n} S'_i.$$

By the proof of Lemma 1, $S' \subseteq S$ is an n_1-i_1-bisimulation between (M_1, d_1^*) and (M_2, d_2^*). We prove first that

$$S' \text{ is an } n_m\text{-}i_1\text{-counting bisimulation .} \tag{1}$$

Assume the contrary, i.e., S' is not an n_m-i_1-counting bisimulation. We can assume without loss of generality that there is a pair $u'S'_jv'$ ($1 \leq j \leq n$) and a set $D_1 \subseteq u'\uparrow$ with $\mid D_1 \mid \leq m$, but there is no $D_2 \subseteq v'\uparrow$ such that S' contains a bijection between D_1 and D_2. Since $u'S'_jv'$, $u'S_jv'$ holds according to the definition of S'. By the definition of n_m-counting bisimulation, there is a set $D_2 \subseteq v'\uparrow$ such that S contains a bijection between D_1 and D_2. According to the definition of S', S' contains a bijection between D_1 and D_2, which is contrary to our assumption. So (1) holds. Since each point in a tree has only one predecessor, by the definition of n_m-i_k-counting bisimulation, S' is any n_m-i_k-counting bisimulation for each $k \geq 1$ between (M_1, d_1^*) and (M_2, d_2^*) with their roots being mapped to each other. $\square$

In order to prove the main theorem of this section Theorem 11, Theorem 4.11 of [15] is introduced first:

Theorem 10 (Theorem 4.11 **in [15]).** *Assume that the language of MLG contains finitely many propositional variables. Let K be a class of pointed models. Then K is definable by a single MLG-formula iff K is closed under n_m-counting bisimulations for some $n, m \in N$.*

Theorem 11. *Each MLGI-formula is equivalently rewritable into an MLG-formula at roots over tree models.*

Proof Give an MLGI-formula φ with $Deg(\varphi) \leq n$, $Ind(\varphi) \leq m$ and $Ind^-(\varphi) \leq k$ for $n, m, k \geq 1$. By Theorem 10, it needs to prove that each MLGI-formula is invariant under n_m-counting bisimulations at roots over tree models. By Proposition 9, from each n_m-counting bisimulation between two tree models with their roots being mapped to each other, an n_m-i_k-counting bisimulation is constructed between these two tree models with their roots being mapped to each other. By Proposition 8, it can be easily proved that each MLGI-formula φ is invariant under n_m-counting bisimulations at roots over tree models. $\square$

The following proposition follows directly from Theorem 11.

Proposition 12 Each MLGI-formula is m-conservatively rewritable into an MLG-formula at roots over tree models.

However, not each MLGI-formula is locally equivalently rewritable into an MLG-formula at any point over tree models. Our example is still $\Diamond^- \top$ in Example 7. $\Diamond^- \top$ is also an MLGI-formula. Since each MLG-formula is invariant under counting bisimulations at any point over tree models[21], if $\Diamond^- \top$ can be locally equivalently rewritable into an MLG-formula, it should be invariant under counting bisimulations at any point over tree models. Now Figure 2 shows that it is not the truth, for $(M_2, a) \models \Diamond^- \top$, $(M_1, a) \not\models \Diamond^- \top$ and there is a counting bisimulation $S = \{(a, a)\}$ between the two tree models (M_1, a) and (M_2, a).

[21]For reference, see Proposition 3.3 in [15], which says that each MLG-formula is invariant under counting bisimulations.

Instead, the following theorem can be proved from Proposition 3.3 in [15], Theorem 10 (i.e., Theorem 4.11 in [15]) and a similar proof of Theorem 17.[22]

Theorem 13. *Let φ be an MLGI-formula with Deg(φ) $\leq n$. Then the following conditions are equivalent:*

(i) φ is locally equivalently rewritable into an MLG-formula over tree models;
(ii) φ is locally preserved (or invariant) under n-counting bisimulations over tree models;
(iii) φ is locally preserved (or invariant) under counting bisimulations over tree models.

5 MLGI to MLI

5.1 Equivalent rewritability of MLGI to MLI

Definition 14 (Height of States in Rooted Models) Let $M = (W, R, V)$ be a rooted model with the root d^*. The height $H(d^*)$ of the root d^* of M is 0; if the height $H(d)$ of d in M is n ($n \in \mathbf{N}$), then for each immediate successor[23] d' of d in M, the height $H(d')$ of d' in M that has not been assigned a height smaller than $n+1$ is $n+1$. The height $H(M)$ of a rooted model M is n if the maximum height of points in M is n. Otherwise, $H(M)$ is infinite.

Definition 15 (Submodel of M Induced by X) The submodel $M_{|X}$ of a model $M = (W, R, V)$ induced by $X \subseteq W$ is defined as $M_{|X} = (X, R_{|X}, V_{|X})$, where $R_{|X} = R \cap (X \times X)$ and $V_{|X} = V(p) \cap X$ for each propositional variable p.

Proposition 16 Let $M = (W, R, V)$, $d \in W$ and $X = \{e \in W : H(e) \leq \max\{H(d') : d' \in X_{d,n}\}\}$, where $X_{d,n} = d{\uparrow}^0 \cup \cdots \cup d{\uparrow}^n$. Then there are an n-bisimulation and an n-i-bisimulation between $(M_{|X}, d)$ and (M, d).

Proof A sequence of binary relations $S_n \subseteq \cdots \subseteq S_0$ is defined as follows ($1 \leq i \leq n$):

$$S_n = \{(d, d)\},$$
$$S_{i-1} = S_i \cup \{(e, e) \in X \times X : e \in d{\uparrow}^{n-i+1}\}.$$

It is easy to prove that $(M_{|X}, d)$ and (M, d) is both n-bisimular and n-i-bisimular. □

The following theorem holds for MLGI-formulas, also for MLG-formulas and MLI-formulas.

[22] We should add "counting" before the word "bisimulations" in Theorem 17 and a quite similar theorem to Theorem 17 can be proved by a similar way of Theorem 17.

[23] A successor y of x is an immediate successor of x if $x \neq y$, $\neg yRx$ and $xRzRy$ implies $z = x$ or $z = y$ for each $z \in W$.

Theorem 17. *Let φ be an MLGI-formula with Deg$(\varphi) \leq n$. The following two conditions are equivalent:*

(i) φ is locally preserved (or invariant) under n-bisimulations over tree models;
(ii) φ is locally preserved (or invariant) under bisimulations over tree models.

Proof We only need to prove ($\Rightarrow$). Assume that an MLGI-formula φ with $Deg(\varphi) \leq n$ is locally preserved[24] under n-bisimulations over tree models. Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be two tree models, S be a bisimulation between (M_1, d) and (M_2, e) and $(M_1, d) \models \varphi$. By Proposition 16, there is an n-bisimulation between $(M_{1|X_1}, d)$ and (M_1, d), where $X_1 = \{d'' \in W_1 : H(d'') \leq \max\{H(d') : d' \in X^1_{d,n}\}\}$ and $X^1_{d,n} = d{\uparrow}^0 \cup \cdots \cup d{\uparrow}^n$. Since φ is locally preserved under n-bisimulations over tree models, by $(M_1, d) \models \varphi$, we have that $(M_{1|X_1}, d) \models \varphi$. Similarly, there is an n-bisimulation between $(M_{2|X_2}, e)$ and (M_2, e), where $X_2 = \{e'' \in W_2 : H(e'') \leq \max\{H(e') : e' \in X^2_{e,n}\}\}$ and $X^2_{e,n} = e{\uparrow}^0 \cup \cdots \cup e{\uparrow}^n$. Define a sequence of binary relations $S_n \subseteq S_{n-1} \cdots \subseteq S_0$ as follows ($1 \leq i \leq n$):

$$S_n = \{(d, e)\},$$
$$S_{i-1} = S_i \cup \{(d'', e'') \in X_1 \times X_2 : (d', e') \in S_i, d'R_1d'', e'R_2e'' \ \&\ (d'', e'') \in S\}.$$

Let

$$S^* = \bigcup_{0 \leq j \leq n} S_j.$$

Since $(d, e) \in S$, $S^* \subseteq S$. Then it is easy to prove that S^* is an n-bisimulation between $(M_{1|X_1}, d)$ and $(M_{2|X_2}, e)$. By our assumption that φ is locally preserved under n-bisimulations over tree models, from $(M_{1|X_1}, d) \models \varphi$ we have that $(M_{2|X_2}, e) \models \varphi$. Since there is an n-bisimulation between $(M_{2|X_2}, e)$ and (M_2, e), $(M_2, e) \models \varphi$. Therefore, φ is locally preserved under bisimulations over tree models. $\square$

Not each MLGI-formula is equivalently rewritable into an MLI-formula at roots over tree models. For example, $\Diamond^{\geq 2}\top$. Assume that $\Diamond^{\geq 2}\top$ is equivalently rewritable into an MLI-formula at roots over tree models. Since each MLI-formula is invariant under i-bisimulations at roots over tree models, $\Diamond^{\geq 2}\top$ should be invariant under i-bisimulations at roots over tree models. However, it is not the truth. We show it as follows.

Let $M_1 = (W_1, R_1, V_1)$ and $M_2 = (W_2, R_2, V_2)$ be the two tree models in Figure 3 respectively. Here $V_1(p) = V_2(p) = \emptyset$ for each propositional variable p. It is obvious that $(M_1, a_0) \models \Diamond^{\geq 2}\top$, $(M_2, b_0) \not\models \Diamond^{\geq 2}\top$, but there is an i-bisimulation $S = \{(a_0, b_0), (a_1, b_1), (a_2, b_1)\}$ between the two tree models (M_1, a_0) and (M_2, b_0).

The following theorem is proved, instead.

[24] The "invariant"-case can be proved similarly.

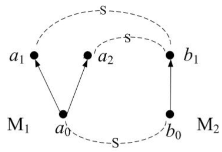

Figure 3

Theorem 18. *Let φ be an MLGI-formula with Deg$(\varphi) \leq n$. Then the following conditions are equivalent:*

(i) φ is equivalently rewritable into an MLI-formula at roots over tree models;
(ii) φ is preserved (or invariant) under bisimulations at roots over tree models;
(iii) φ is preserved (or invariant) under n-bisimulations at roots over tree models;
(iv) φ is preserved (or invariant) under n-i-bisimulations at roots over tree models;
(v) φ is preserved (or invariant) under i-bisimulations at roots over tree models.

Proof $2 \Leftrightarrow 3$ can be proved by a very similar proof of Theorem 17.

$3 \Leftrightarrow 4$ is prove as follows: $3 \Rightarrow 4$ follows directly from the fact that each n-i-bisimulation is also an n-bisimulation by the definitions of n-bisimulation and n-i-bisimulation. $4 \Rightarrow 3$ follows from Proposition 2.

$2 \Leftrightarrow 5$ is proved as follows: $2 \Rightarrow 5$ follows directly from the fact that each i-bisimulation is also a bisimulation by the definitions of bisimulation and i-bisimulation. $5 \Rightarrow 2$ follows from Lemma 1.

Now we prove that $1 \Leftrightarrow 5$. ($1 \Rightarrow 5$) Assume that an MLGI-formula φ is equivalently rewritable into an MLI-formula ψ at roots over tree models. Since each MLI-formula is preserved (or invariant) under i-bisimulations at roots over tree models, φ is preserved (or invariant) under i-bisimulations at roots over tree models. ($5 \Rightarrow 1$) Assume that φ is an MLGI-formula with *Deg*$(\varphi) \leq n$ and is preserved (or invariant) under i-bisimulations at roots over tree models. There are only finitely many non-equivalent MLI-formulas β with *Deg*$(\beta) \leq m$ and *sig*$(\beta) \subseteq$ *sig*(φ) for each $m \in N$. For each tree model $M = (W, R, V)$ and $d \in W$, let the MLI-formula $\alpha^m_{(M,d)}$ be the conjunction of all these finitely many non-equivalent MLI-formulas β with *Deg*$(\beta) \leq m$, *sig*$(\beta) \subseteq$ *sig*(φ) and $(M, d) \models \beta$. Now let

$$\alpha = \bigvee_{(M,d)\models\varphi} \alpha^n_{(M,d)},$$

where M is a tree model with d being its root such that $(M, d) \models \varphi$. Being a disjunction of finitely many non-equivalent MLI-formulas, α is a proper MLI-formula.

Now we prove that φ is equivalently rewritable into the MLI-formula α at roots over tree models. Let M^* be a tree model and d^* be its root. Assume that $(M^*, d^*) \models$

φ. By the definition of α and $\alpha^n_{(M,d)}$, it is clear that $(M^*, d^*) \models \alpha^n_{(M^*,d^*)}$ and then $(M^*, d^*) \models \alpha$.

Now assume that $M^* = (W^*, R^*, V^*)$ is a tree model, d^* is the root of M^* and $(M^*, d^*) \models \alpha$. Then there is a tree model $M' = (W', R', V')$ with d' being its root and $(M', d') \models \varphi$ such that $(M^*, d^*) \models \alpha^n_{(M',d')}$. Now we prove the following claim:

- There is an n-i-bisimulation S between (M', d') and (M^*, d^*) with $(d', d^*) \in S$.

Since $(M^*, d^*) \models \alpha^n_{(M',d')}$, it is easy to prove that $(M^*, d^*) \models \delta$ iff $(M', d') \models \delta$ for each MLI-formula δ with *Deg*$(\delta) \leq n$ and *sig*$(\delta) \subseteq$ *sig*(φ).

Assume that $d^* R^* v$. By $(M^*, v) \models \alpha^{n-1}_{(M^*,v)}$ and then $(M^*, d^*) \models \Diamond\alpha^{n-1}_{(M^*,v)}$. From $(M^*, d^*) \models \alpha^n_{(M',d')}$, we have that $(M^*, d^*) \models \delta$ iff $(M', d') \models \delta$ for each MLI-formula δ with *Deg*$(\delta) \leq n$ and *sig*$(\delta) \subseteq$ *sig*(φ). So $(M', d') \models \Diamond\alpha^{n-1}_{(M^*,v)}$. Thus there is a point $v' \in W'$ such that $d' R' v'$ and $(M', v') \models \alpha^{n-1}_{(M^*,v)}$. Then for each MLI-formula δ with *Deg*$(\delta) \leq n - 1$ and *sig*$(\delta) \subseteq$ *sig*(φ), $(M^*, v) \models \delta$ iff $(M', v') \models \delta$. By a similar argument, we can also prove that if $d' R' v'$, there is a point $v \in W^*$ such that $d^* R^* v$ and for each MLI-formula δ with *Deg*$(\delta) \leq n - 1$ and *sig*$(\delta) \subseteq$ *sig*(φ), $(M^*, v) \models \delta$ iff $(M', v') \models \delta$.

Now let S_{n-1} be the union of $S_n = \{(d', d^*)\}$ and the set of all the above selected pairs (v', v) such that $d' R' v'$, $d^* R^* v$ and $(M^*, v) \models \delta$ iff $(M', v') \models \delta$ for each MLI-formula δ with *Deg*$(\delta) \leq n - 1$ and *sig*$(\delta) \subseteq$ *sig*(φ). Similarly, a sequence of binary relations $S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_0$ is defined as follows:

for each $1 \leq i \leq n$, S_{i-1} is the union of S_i and the set of all the selected pairs (v', v) satisfying that $w' R' v'$, $w R^* v$ for some $(w', w) \in S_i$ and $(M^*, v) \models \delta$ iff $(M', v') \models \delta$ for each MLI-formula δ with *Deg*$(\delta) \leq i - 1$ and *sig*$(\delta) \subseteq$ *sig*(φ). It is easy to prove that

$$S_0 = \bigcup_{0 \leq i \leq n} S_i$$

is an n-i-bisimulation between (M', d') and (M^*, d^*) with $(d', d^*) \in S_0$.

Since φ is preserved (or invariant) under i-bisimulations at roots over tree models, by $2 \Leftrightarrow 5$, $2 \Leftrightarrow 3$ and $3 \Leftrightarrow 4$, φ is preserved (or invariant) under n-i-bisimulations at roots over tree models. Then by $(M', d') \models \varphi$, we have that $(M^*, d^*) \models \varphi$. $\square$

If being preserved (or invariant) at each point of a tree model is considered, we have the following theorem:

Theorem 19. *Let φ be an MLGI-formula with Deg$(\varphi) \leq n$. Then the following conditions are equivalent:*

(i) φ is locally equivalently rewritable into an MLI-formula over tree models;

(ii) φ *is locally preserved (or invariant) under* n*-*i*-bisimulations over tree models;*
(iii) φ *is locally preserved (or invariant) under* i*-bisimulations over tree models.*

Proof 2 $\Leftrightarrow$ 3 can be proved by a similar argument to the proof of Theorem 17. 3 $\Leftrightarrow$ 1 follows from a similar argument to the proof of 1 $\Leftrightarrow$ 5 of Theorem 18. $\square$

5.2 m-Conservative rewritability of MLGI to MLI

Lemma 2 follows from the fact that each i-p-morphism is an i-bisimulation by their definitions and the fact that each MLI-formula is preserved (or invariant) under i-bisimulations.

Lemma 2 Let Δ be a set of propositional variables, f be a Δ-i-p-morphism from M_1 to M_2. Then $(M_1, d) \models \varphi$ iff $(M_2, f(d)) \models \varphi$ for each MLI-formula φ with *sig*$(\varphi) \subseteq \Delta$.

We prove Theorem 20 by Lemma 2.

Theorem 20. *Let* φ *be an MLGI-formula,* Δ^* *be a set of MLI-formulas and* Δ *be a set of propositional variables such that sig*$(\varphi) \subseteq \Delta$ *and sig*$(\alpha) \subseteq \Delta$ *for each MLI-formula* $\alpha \in \Delta^*$*. If* φ *is locally m-conservatively rewritable into* Δ^**, then it is locally preserved under inverse* Δ*-*i*-*p*-morphisms.*

Proof Let φ be an MLGI-formula, Δ^* be a set of MLI-formulas and Δ be a set of propositional variables such that *sig*$(\varphi) \subseteq \Delta$ and *sig*$(\alpha) \subseteq \Delta$ for each MLI-formula $\alpha \in \Delta^*$. Assume that φ is locally m-conservatively rewritable into Δ^* and there is a Δ-i-p-morphism f from a model $M_1 = (W_1, R_1, V_1)$ to a model $M_2 = (W_2, R_2, V_2)$ with $d_1 \in W_1$, $d_2 \in W_2$, $f(d_1) = d_2$ and $(M_2, d_2) \models \varphi$. We need to prove that $(M_1, d_1) \models \varphi$. According to our assumption that $(M_2, d_2) \models \varphi$ and the definition of locally m-conservative rewritability, there is a pointed model (M_2', d_2) with $M_2' = (W_2, R_2, V_2')$ such that $(M_2', d_2) \models \Delta^*$ and $M_2 =_{sig(\varphi)} M_2'$. By $M_2 =_{sig(\varphi)} M_2'$, we have that

$$V_2'(p) = V_2(p)$$

for each propositional variable $p \in$ *sig*(φ). Let $M_1' = (W_1, R_1, V_1')$, while

$$V_1'(p) = f^{-1}(V_2'(p)) = \{e \in W_1 : f(e) \in V_2'(p)\}$$

for each propositional variable $p \in \Delta$. It is obvious that f is also a Δ-i-p-morphism from (M_1', d_1) to (M_2', d_2) with $f(d_1) = d_2$. From $(M_2', d_2) \models \Delta^*$ and *sig*$(\alpha) \subseteq \Delta$ for each MLI-formula $\alpha \in \Delta^*$, by Lemma 2 we have that

$$(M_1', d_1) \models \Delta^*.$$

By the definition of locally m-conservative rewritability, $(M, d) \models \Delta^*$ implies (M, d) $\models \varphi$ for each pointed model (M, d), then by $(M_1', d_1) \models \Delta^*$ we have that

$$(M_1', d_1) \models \varphi.$$

Since $V_1'(p) = f^{-1}(V_2'(p)) = f^{-1}(V_2(p)) = V_1(p)$ for each propositional variable $p \in sig(\varphi)$, we get that

$$(M_1, d_1) \models \varphi.$$

□

Give an MLGI-formula φ with $Deg(\varphi) \leq \ell$. Let $\Sigma^*(\varphi)$ be the set of all subformulas of φ. Take new propositional variables $p^\psi, p_1^\psi, \ldots, p_n^\psi$ for each subformla $\psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$, and let Σ be the union of $sig(\varphi)$ and the set of all the new propositional variables $p^\psi, p_1^\psi, \ldots, p_n^\psi$. For each $\chi \in \Sigma^*(\varphi)$, let $\chi^\sharp$ be the MLI-formula obtained from χ by replacing all the topmost subformulas $\psi = \Diamond^{\geq n}\psi'$ and $\Diamond^{-\geq n}\psi'$ of χ $(n \geq 2)$ with p^ψ and $\perp$ respectively. $\Sigma_{\varphi^\dagger}$ is defined as the set of the MLI-formula $\varphi^\sharp$ and the following infinite many formulas for each $\psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$:

$$\bigwedge_{0 \leq i \leq \ell} \Box^i(p^\psi \rightarrow (\bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^\sharp \wedge p_i^\psi \wedge \bigwedge_{1 \leq j \neq i \leq n} \neg p_j^\psi))))$$

and

$$\bigwedge_{0 \leq i \leq \ell} \Box^i((\bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^\sharp \wedge \psi_i \wedge \bigwedge_{1 \leq j \neq i \leq n} \neg \psi_j))) \rightarrow p^\psi),$$

while each ψ_i $(1 \leq i \leq n)$ is an MLI-formula with $sig(\psi_i) \subseteq \Sigma$ and $\Box^i$ represents a sequence of i operators $\Box$ $(i \in N)$.

Now we can prove the main result Theorem 21 of this subsection.

Theorem 21. *Let φ be an MLGI-formula, Δ^* be a set of MLI-formulas and Δ be a set of propositional variables such that sig$(\varphi) \subseteq \Delta$ and sig$(\alpha) \subseteq \Delta$ for each MLI-formula $\alpha \in \Delta^*$. Then the MLGI-formula φ is locally m-conservatively rewritable into Δ^* over tree models iff φ is locally preserved under inverse Δ-i-p-morphisms over tree models.*

Proof ($\Rightarrow$) It follows directly from Theorem 20. ($\Leftarrow$) Let φ be an MLGI-formula, $\Sigma^*(\varphi)$ be the set of all subformulas of φ, Σ be $sig(\varphi)$ together with all the fresh propositional variables $p^\psi, p_1^\psi, \ldots, p_n^\psi$ and $\Sigma_{\varphi^\dagger}$ be the set of MLI-formulas being defined above. Assume that the MLGI-formula φ with $Deg(\varphi) \leq \ell$ is locally preserved under inverse Δ-i-p-morphisms over tree models with $sig(\varphi) \subseteq \Delta$ and $sig(\alpha) \subseteq \Delta$ for each MLI-formula $\alpha \in \Sigma_{\varphi^\dagger}$. We prove that φ can be locally m-conservatively rewritable into the set $\Sigma_{\varphi^\dagger}$ of MLI-formulas over tree models. We need to prove that

Claim 1 for each tree model $M = (W, R, V)$ and $d \in W$ such that $(M, d) \models \varphi$, there is a tree model $M' = (W, R, V')$ such that $M =_{sig(\varphi)} M'$ and $(M', d) \models \Sigma_{\varphi^\dagger}$;

Claim 2 for each tree model $M = (W, R, V)$ and $d \in W$, if $(M, d) \models \Sigma_{\varphi^\dagger}$, then $(M, d) \models \varphi$.

To prove Claim 1, we should notice that each point in a tree model has only one predecessor, and then each MLGI-formula $\Diamond^{-\geq n}\psi$ $(n \geq 2)$ is equivalent to $\perp$ at each point of a tree model. Assume that $M = (W, R, V)$ is a tree model, $d \in W$ and $(M, d) \models \varphi$. Let

$$V'(p) = \begin{cases} V(p), & p \in sig(\varphi) \\ V(\psi), & p = p^\psi \text{ and } \psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi) \ (n \geq 2) \\ W, & p = p_i^\psi \text{ and } \psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi) \ (1 \leq i \leq n \text{ and } n \geq 2) \end{cases}$$

Then a new model $M' = (W, R, V')$ is constructed from M. It is obvious that $M =_{sig(\varphi)} M'$ and $(M', d) \models \Sigma_{\varphi^\dagger}$.

Let's consider Claim 2. Give a tree model $M = (W, R, V)$ with $d \in W$ and d^* being its root. Assume that $(M, d) \models \Sigma_{\varphi^\dagger}$ and $(M, d) \not\models \varphi$. Let $S_0 = \{d' \in W : \exists k \in N(d' \in d{\downarrow_k})\}$.[25] Assume that $S_0 \subseteq S_1 \cdots \subseteq S_n$ have already been defined. Fix a point $e \in S_n$.

Step (i) For each $\Diamond^{\geq m}\psi' \in \Sigma^*(\varphi)$ such that $(M, e) \models \Diamond^{\geq m}\psi'$, select m points $e_1, \cdots, e_m \in W$ such that eRe_i and $(M, e_i) \models \psi'$ for each $1 \leq i \leq m$. For each $\Diamond^-\psi' \in \Sigma^*(\varphi)$[26] such that $(M, e) \models \Diamond^-\psi'$, select the only predecessor e' of e[27] such that $(M, e') \models \psi'$.

Step (ii) For each $\psi = \Diamond^{\geq m}\psi' \in \Sigma^*(\varphi)$ $(m \geq 2)$ such that $(M, e) \models p^\psi$, select m points $e_1, \cdots, e_m \in W$ such that eRe_i and

$$(M, e_i) \models \psi'^\sharp \wedge p_i^\psi \wedge \bigwedge_{j \neq i} \neg p_j^\psi$$

for each $1 \leq i \leq m$.

Step (iii) For each subformula $\Diamond\gamma$ of $\varphi^\sharp$ such that $(M, e) \models \Diamond\gamma$, select a point $e' \in W$ such that eRe' and $(M, e') \models \gamma$. For each subformula $\Diamond^-\gamma$ of $\varphi^\sharp$ such that $(M, e) \models \Diamond^-\gamma$, select the only predecessor $e' \in W$ of e such that $(M, e') \models \gamma$.

Step (iv) For each subformula $\Diamond\gamma$ of $\psi'^\sharp$ with $\Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$ such that $(M, e) \models \Diamond\gamma$, select a point $e' \in W$ such that eRe' and $(M, e') \models \gamma$. For each subformula $\Diamond^-\gamma$ of $\psi'^\sharp$ with $\Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$ such that $(M, e) \models \Diamond^-\gamma$, select the only predecessor $e' \in W$ of e such that $(M, e') \models \gamma$.

[25] $d{\downarrow_k} = \{d' \in W : \exists d_1 \cdots d_{k-1} \in W(d'Rd_{k-1} \cdots d_2Rd_1Rd)\}$ for $k \in N$. When $k = 0$, $d{\downarrow_0} = \{d\}$. We should notice that the root of M belongs to S_0, i.e., $d^* \in S_0$.

[26] If $\psi = \Diamond^{-\geq m'}\psi'$ $(m' \geq 2)$, then $(M, e) \not\models \psi$.

[27] The predecessor e' of e is unique because M is a tree model.

Repeat the above selection process for each point $e \in S_n$. Let S_{n+1} contains all these points e_i or e' selected by the above selection process (i)–(iv). Next, for each two $d_1, d_2 \in S_n$ such that d_1 is Σ-i-bisimilar to d_2 in M[28] , if $d_1 R d_1'$ (or $d_1' R d_1$) and $d_1' \in S_{n+1}$, then each successor (or the only predecessor) d_2' of d_2 being Σ-i-bisimilar to d_1' in M should be added into S_{n+1}. Let S_{n+1} be the smallest set of points satisfying all of the above conditions. Then the sequence of sets of points $S_0 \subseteq S_1 \cdots \subseteq S_n \cdots$ is defined completely.

The selection process (i)–(iv) may choose two successors of one point which are equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M but not Σ-i-bisimilar to each other in M. Assume that such a case occurs, i.e., there are two successors $d_1, d_2 \in S_{i+1}$ of $d' \in S_i$ $(i \in N)$ such that d_1, d_2 are equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M but d_1 is not Σ-i-bisimilar to d_2 in M. Let

$$B_{d_2}^{d_1} = \{e' \in W : \exists m \in N(m \geq 2\ell + 1 \,\&\, e' \in e{\uparrow}^m \& \, e \text{ is } \Sigma\text{-}i\text{-bisimilar to } d_1 \text{ or } d_2 \text{ in } M)\}.$$

We delete the points of the sets $B_{d_2}^{d_1}$ from each S_i $(i \in N)$ for each two points $d_1, d_2 \in W$. Let S_i' $(i \in N)$ be the remaining set of points after the above deletion process. Then a new sequence $S_0' \subseteq S_1' \cdots \subseteq S_n' \cdots$ is constructed from the sequence $S_0 \subseteq S_1 \cdots \subseteq S_n \cdots$.

Now a new model $M' = (W', R', V')$ can be defined as follows:

$$W' = \bigcup_{0 \leq i \in N} S_i',$$
$$R' = R \cap (W' \times W'),$$
$$V'(p) = V(p) \cap W' \text{ for each propositional variable } p.$$

According to the assumption that M is a tree model with d^* being its root, M' is also a tree model with d^* being its root.[29] Then by $(M, d) \not\models \varphi$, we have that $(M', d) \not\models \varphi$. We need to prove that $(M', d) \models \Sigma_{\varphi^\dagger}$. Since we have Step (ii), the only cases in $\Sigma_{\varphi^\dagger}$ needed to be considered are the formulas

$$\bigwedge_{0 \leq i \leq \ell} \Box^i \Big(\Big(\bigwedge_{1 \leq i \leq n} \big(\Diamond (\psi'^\sharp \wedge \psi_i \wedge \bigwedge_{1 \leq j \neq i \leq n} \neg \psi_j) \big) \Big) \to p^\psi \Big),$$

while each ψ_i $(1 \leq i \leq n$ and $n \geq 2)$ is an MLI-formula with $sig(\psi_i) \subseteq \Sigma$.

Assume the contrary, i.e., there are a point $d' \in d{\uparrow}^m \subseteq W'$ $(0 \leq m \leq \ell)$ and MLI-formulas $\psi_1, \cdots, \psi_n$ $(n \geq 2)$ with $sig(\psi_i) \subseteq \Sigma$ $(1 \leq i \leq n)$ such that $(M', d') \not\models p^\psi$ for some $\psi = \Diamond^{\geq n} \psi' \in \Sigma^*(\varphi)$ and

$$(M', d') \models \bigwedge_{1 \leq i \leq n} \big(\Diamond (\psi'^\sharp \wedge \psi_i \wedge \bigwedge_{1 \leq j \neq i \leq n} \neg \psi_j) \big).$$

[28] Each point is Σ-i-bisimilar to itself in M. Therefore, if $d_1 = d_2$, then d_1 is definitely Σ-i-bisimilar to d_2 in M.

[29] We should notice that $d^* \in S_0$ and d^* won't be deleted from each S_i $(i \in N)$ since it is the root of M. So $d^* \in W'$.

It means that d' has n different R'-successors that are not equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M', and then not Σ-i-bisimilar to each other in M'. By the construction of M', if $d_1 \in W'$ is Σ-i-bisimilar to $d_2 \in W'$ in M, then d_1 is Σ-i-bisimilar to d_2 in M'. So d' has n different R-successors that are not Σ-i-bisimilar to each other in M. We prove that the n different R-successors of d' are also not equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M.

Assume the contrary, i.e., there are two successors $d'_1 \in W'$ and $d'_2 \in W'$ of $d' \in W'$ satisfying that d'_1 and d'_2 are not equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M' and not Σ-i-bisimilar to each other in M, but they are equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M. Since Σ is finite[30] and d'_1 and d'_2 are equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M, d'_1 is Σ-2ℓ-i-bisimilar to d'_2 in M.[31] According to the construction of the sequence $S'_0 \subseteq S'_1 \cdots \subseteq S'_n \cdots$, d'_1 is Σ-i-bisimilar to d'_2 in M'. Therefore, d'_1 is equivalent to d'_2 over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M', which is contrary to our assumption that d'_1 and d'_2 are not equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M'. So $d' \in d{\uparrow}^m(0 \leq m \leq \ell)$ has n different R-successors that are not equivalent over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ to each other in M.

Since these n different R-successors of d'[32] satisfy $\psi'^{\sharp}$ in M', according to the construction of M', each of them also satisfies $\psi'^{\sharp}$ in M. Then there are MLI-formulas $\psi'_1, \cdots, \psi'_n$ with $sig(\psi'_i) \subseteq \Sigma$ $(1 \leq i \leq n)$ such that

$$(M, d') \models \bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^{\sharp} \wedge \psi'_i \wedge \bigwedge_{1 \leq j \neq i \leq n} \neg\psi'_j)). \tag{0*}$$

Last, from $(M, d) \models \Sigma_{\varphi^{\dagger}}$, $d' \in d{\uparrow}^m(0 \leq m \leq \ell) \subseteq W' \subseteq W$ and (0^*), we have that $(M, d') \models p^{\psi}$. It means that $(M', d') \models p^{\psi}$ by the construction of M', which is contrary to our assumption that $(M', d') \not\models p^{\psi}$. Therefore, $(M', d) \models \Sigma_{\varphi^{\dagger}}$ is proved.

Since "being Σ-i-bisimilar to" is an equivalence relation, let $[e] = \{e' \in W' : (M', e)$ is Σ-i-bisimilar to $(M', e')\}$ for $e \in W'$. A new model $M'' = (W'', R'', V'')$ can be defined from M' as follows:

$W'' = \{[e] : e \in W'\}$,
$[d_1]R''[d_2]$ iff there are $e_1 \in [d_1]$ and $e_2 \in [d_2]$ such that $e_1 R' e_2$.
$V''(p) = \{[e] \in W'' : e \in V'(p)\}$ for each propositional variable $p \in \Sigma$.

According to the construction of M' and M'', M'' is of finite outdegrees, i.e., each point in M'' has only finitely many successors.

[30] If Σ is finite, there are only finitely many non-equivalent MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ and $Deg(\alpha) \leq 2\ell$.

[31] The proof of this part is similar to Proposition 2.31 of [5].

[32] These n points are also R'-successors of d'.

Now we show that $f : e \mapsto [e]$ for $e \in W'$ and $[e] \in W''$ is a Σ-i-p-morphism from M' to M''. The valuation and the forth conditions are obviously satisfied by the definition of M''. We prove the back condition as follows:

Assume that $[e_1]R''[e_2]$ for $[e_1], [e_2] \in W''$. Then there are $e'_1 \in [e_1]$ and $e'_2 \in [e_2]$ such that $e'_1 R' e'_2$ according to the definition of R''. By $e'_1 \in [e_1]$, (M', e'_1) is Σ-i-bisimilar to (M', e_1). Then from $e'_1 R' e'_2$ we have that there is an $e^*_1 \in W'$ such that $e_1 R' e^*_1$ and (M', e^*_1) is Σ-i-bisimilar to (M', e'_2). So $e^*_1 \in [e'_2] = [e_2]$. That is, $f(e^*_1) = [e^*_1] = [e_2]$. Thus the back condition holds.
The inverse forth condition follows from the definition of R''. Now we prove the inverse back condition as follows:

Assume that $[e_1]R''[e_2]$. Then there are $e'_1 \in [e_1]$ and $e'_2 \in [e_2]$ such that $e'_1 R' e'_2$ according to the definition of R''. By $e'_2 \in [e_2]$, (M', e'_2) is Σ-i-bisimilar to (M', e_2). Then from $e'_1 R' e'_2$, the unique predecessor e^*_2 of e_2 in M' satisfies that (M', e^*_2) is Σ-i-bisimilar to (M', e'_1). So $e^*_2 \in [e'_1] = [e_1]$. That is, $f(e^*_2) = [e^*_2] = [e_1]$. Thus the inverse back condition holds.

Therefore, $f : e \mapsto [e]$ for $e \in W'$ and $[e] \in W''$ is a Σ-i-p-morphism from M' to M''.

We prove that M'' is a tree model. Since M' is a tree model with d^* being its root, there is an R''-path from $[d^*]$ to $[e]$ for each $[e] \in W''$. If $[d^*]$[33] has a predecessor in M'', d^* has a predecessor in M' according to the definition of R'', which is contrary to our assumption that d^* is the root of the tree model M'. Therefore, $[d^*]$ is the root of M''. Now we prove that there is a unique path from $[d^*]$ to $[e]$ for each $[e]$ in M''. Assume the contrary, i.e., there is a $[e]$ in M'' such that $[e]$ has two different predecessors $[d_1]$ and $[d_2]$ in M''. Since f is a Σ-i-p-morphism from M' to M'', there are two points $d'_1 \in [d_1]$ and $d'_2 \in [d_2]$ such that $d'_1 R' e$ and $d'_2 R' e$. Since $[d_1] \neq [d_2]$, d'_1 is also different from d'_2. It means that the point e has two different predecessors in the tree model M', which is contrary to the definition of a tree model. So there is only one unique path from $[d^*]$ to $[e]$ for each $[e]$ in M''. Therefore, M'' is a tree model with $[d^*]$ being its root.

Next we prove the following claims:

Claim (1) $(M'', [d]) \models \Sigma_{\varphi^\dagger}$;
Claim (2) Let $[u]$ and $[v]$ be successors of a point $[w] \in W''$ in M''. For each MLI-formula α with *sig*$(\alpha) \subseteq \Sigma$, if $(M'', [u]) \models \alpha$ iff $(M'', [v]) \models \alpha$, then $[u] = [v]$;
Claim (1) $(M'', [d]) \models \varphi$.

Claim (1) follows directly from Lemma 2 and $(M', d) \models \Sigma_{\varphi^\dagger}$.[34] Claim (2) is proved as follows:

[33] $[d^*] = \{d^*\}$.

[34] We have proved that there is a Σ-i-p-morphism from M' to M''

Let $[u]$ and $[v]$ be successors of a point $[w] \in W''$ in M''. Assume that

$$(M'', [u]) \models \alpha \text{ iff } (M'', [v]) \models \alpha \tag{1*}$$

for each MLI-formula α with *sig*$(\alpha) \subseteq \Sigma$. Since M'' is of finite outdegrees, $[u]$ is Σ-i-bisimilar to $[v]$ in M''.[35] Since $[u], [v]$ are successors of the point $[w]$ in M'' and there is a Σ-i-p-morphism from M' to M'', there are points $u_1 \in [u_1] = [u]$ and $v_1 \in [v_1] = [v]$ such that $wR'u_1$ and $wR'v_1$. According to Lemma 2 and the fact that $f : e \mapsto [e]$ for $e \in W'$ and $[e] \in W''$ is a Σ-i-p-morphism from M' to M'',

$$(M', u_1) \models \alpha \text{ iff } (M'', [u]) \models \alpha$$

and

$$(M', v_1) \models \alpha \text{ iff } (M'', [v]) \models \alpha$$

for each MLI-formula α with *sig*$(\alpha) \subseteq \Sigma$. Then by (1^*), we have that

$$(M', u_1) \models \alpha \text{ iff } (M', v_1) \models \alpha$$

for each MLI-formula α with *sig*$(\alpha) \subseteq \Sigma$.

Now we prove that u_1 is Σ-i-bisimilar to v_1 in M'. Assume the contrary, i.e., u_1 is not Σ-i-bisimilar to v_1 in M'. Since u_1 and v_1 has the same unique R'-predecessor w in the tree model M', we can assume without loss of generality that there is a point $u_1' \in W'$ such that $u_1R'u_1'$ and no successor of v_1 is equivalent to u_1' over MLI-formulas α with *sig*$(\alpha) \subseteq \Sigma$ in M' . From $u_1R'u_1'$ and the fact that there is a Σ-i-p-morphism from M' to M'', $[u_1]R''[u_1']$ and then $[u]R''[u_1']$ by $[u] = [u_1]$. Since $[u]$ is Σ-i-bisimilar to $[v]$ in M'', there is a point $[v'] \in W''$ such that $[v]R''[v']$ and $[u_1']$ is Σ-i-bisimilar to $[v']$ in M''. Then

$$(M'', [u_1']) \models \alpha \text{ iff } (M'', [v']) \models \alpha \tag{2*}$$

for each MLI-formula α with *sig*$(\alpha) \subseteq \Sigma$. By Lemma 2 and the fact that there is a Σ-i-p-morphism from M' to M'',

$$(M', u_1') \models \alpha \text{ iff } (M'', [u_1']) \models \alpha$$

and

$$(M', v') \models \alpha \text{ iff } (M'', [v']) \models \alpha$$

for each MLI-formula α with *sig*$(\alpha) \subseteq \Sigma$. Thus, by (2^*), we have that

$$(M', u_1') \models \alpha \text{ iff } (M', v') \models \alpha \tag{3*}$$

[35] Since $[u]$ and $[v]$ has the same unique predecessor $[w]$ in M'', the proof of this part is similar to the proof of Theorem 2.24 (i.e., Hennessy-Milner Theorem) in [5].

for each MLI-formula α with $sig(\alpha) \subseteq \Sigma$. From $[v]R''[v']$ and $[v_1] = [v]$, $[v_1]R''[v']$ holds. By the fact that there is a Σ-i-p-morphism from M' to M'', there is a point $v'_1 \in W'$ such that $v_1 R' v'_1$ and $v'_1 \in [v'_1] = [v']$. Then from $v'_1 \in [v']$ we get that

$$(M', v'_1) \models \alpha \text{ iff } (M', v') \models \alpha \tag{4*}$$

for each MLI-formula α with $sig(\alpha) \subseteq \Sigma$. Therefore, by (3^*) and (4^*),

$$(M', u'_1) \models \alpha \text{ iff } (M', v'_1) \models \alpha \tag{5*}$$

for each MLI-formula α with $sig(\alpha) \subseteq \Sigma$. However, (5^*) is contrary to our assumption that no successors of v_1 is equivalent to u'_1 over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ in M'. Thus u_1 is Σ-i-bisimilar to v_1 in M'.

Since u_1 is Σ-i-bisimilar to v_1 in M', then $[u_1] = [v_1]$. By $[u_1] = [u]$ and $[v_1] = [v]$, we finally get that $[u] = [v]$. That is, Claim (2) is proved.

We prove Claim (3) by showing that

$$(M'', [d']) \models p^{\psi} \text{ iff } (M'', [d']) \models \psi \tag{6*}$$

for each $\psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$ and for each $[d'] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi)}$.

Since M'' is a tree model, we should notice that

$$(M'', [e]) \models \bot \text{ iff } (M'', [e]) \models \gamma$$

for each $[e] \in W''$ and for each $\gamma = \Diamond^{-\geq n}\gamma' \in \Sigma^*(\varphi)$ $(n \geq 2)$. So we can assume without loss of generality that there are no such subformulas $\gamma = \Diamond^{-\geq n}\gamma'$ $(n \geq 2)$ occurring in each $\psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$.[36] We prove (6^*) by induction on the numbers of subformulas $\Diamond^{\geq t}\beta$ $(t \geq 2)$ occurring in ψ' for $\psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$. Let $\psi = \Diamond^{\geq n}\psi' \in \Sigma^*(\varphi)$ $(n \geq 2)$ and k be the number of subformulas $\Diamond^{\geq t}\beta$ $(t \geq 2)$ occurring in ψ'.

Assume that $k = 0$. Then $\psi' = \psi'^{\sharp}$. Let $[d'] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi)}$. Assume that $(M'', [d']) \models p^{\psi}$. By Claim (1) that $(M'', [d]) \models \Sigma_{\varphi^{\dagger}}$ and $[d'] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi)}$, we have that

$$(M'', [d']) \models \bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^{\sharp} \wedge p_i^{\psi} \wedge \bigwedge_{j \neq i} \neg p_j^{\psi})).$$

So $(M'', [d']) \models \Diamond^{\geq n}\psi'^{\sharp}$, i.e.,

$$(M'', [d']) \models \Diamond^{\geq n}\psi'.$$

[36] If such a subformula $\gamma = \Diamond^{-\geq n}\gamma'$ $(n \geq 2)$ occurs in ψ, we can substitute γ with $\bot$ immediately.

Assume that $(M'', [d']) \models \psi$. Then $[d']$ has n different successors $[d'_1], \cdots, [d'_n]$ such that $(M'', [d'_i]) \models \psi'$ $(n \geq 2)$ for each $1 \leq i \leq n$. None of $[d'_1], \cdots, [d'_n]$ is equivalent to another over MLI-formulas α with $sig(\alpha) \subseteq \Sigma$ according to Claim (2). Therefore, there are n different MLI-formulas $\psi_1, \cdots, \psi_n$ with $sig(\psi_i) \subseteq \Sigma$ such that

$$(M'', [d'_i]) \models \psi_j \text{ iff } j = i$$

for $1 \leq i, j \leq n$. By Claim (1) that $(M'', [d]) \models \Sigma_{\varphi^\dagger}$, we have that

$$(M'', [d']) \models (\bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^\sharp \wedge \psi_i \wedge \bigwedge_{j \neq i} \neg\psi_j))) \to p^\psi.$$

From $\psi' = \psi'^\sharp$, we get that $(M'', [d']) \models p^\psi$. That is, (6^*) holds for $k = 0$.

Now assume that (6^*) holds for $k \leq m \in N$. Let's consider the case that $k = m+1$. Let $[d'] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi)}$. Let $\psi''_1 = \Diamond^{\geq n_1}\delta_1, \cdots, \psi''_q = \Diamond^{\geq n_q}\delta_q$ $(q \in N)$ be the topmost subformulas having the form $\Diamond^{\geq n}\delta$ $(n \geq 2)$ occurring in ψ'. Let k_i be the number of subformulas $\Diamond^{\geq n}\delta$ $(n \geq 2)$ occurring in δ_i for each $1 \leq i \leq q$. By induction hypothesis that (6^*) holds for $k \leq m$ and the fact that each $k_i \leq k \leq m$ for $1 \leq i \leq q$, we have that

$$(M'', [d'']) \models p^{\psi''_i} \text{ iff } (M'', [d'']) \models \psi''_i$$

for each $[d''] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi''_i)}$ and each $1 \leq i \leq q$. Thus

$$(M'', [d'']) \models \psi'^\sharp \text{ iff } (M'', [d'']) \models \psi' \tag{7*}$$

for each $[d''] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - \max\{Deg(\psi''_i):1 \leq i \leq q\}}$.

Assume that $(M'', [d']) \models p^\psi$. Then from Claim (1) that $(M'', [d]) \models \Sigma_{\varphi^\dagger}$, we have that

$$(M'', [d']) \models \bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^\sharp \wedge p^\psi_i \wedge \bigwedge_{j \neq i} \neg p^\psi_j)). \tag{8*}$$

From (8^*), we get that there are n different successors $[d'_1], \cdots, [d'_n]$ of $[d']$ such that

$$(M'', [d'_i]) \models \psi'^\sharp$$

for each $1 \leq i \leq n$. Since $[d'] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi)}$, $[d'_i] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - (Deg(\psi) - 1)}$ for each $1 \leq i \leq n$. Since $Deg(\psi') = Deg(\psi) - 1$ and $\max\{Deg(\psi''_i) : 1 \leq i \leq q\} \leq Deg(\psi')$,

$$[d'_i] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - \max\{Deg(\psi''_i):1 \leq i \leq q\}}$$

for each $1 \leq i \leq n$. So by (7^*),

$$(M'', [d'_i]) \models \psi'$$

for each $1 \leq i \leq n$. It means that $(M'', [d']) \models \Diamond^{\geq n}\psi'$.

Now assume that $(M'', [d']) \models \psi$. Then $[d']$ has n different successors $[d'_1], \cdots, [d'_n]$ such that $(M'', [d'_i]) \models \psi'$ ($n \geq 2$) for each $1 \leq i \leq n$. Being similar to the case that $k = 0$, there are n different MLI-formulas $\psi_1, \cdots, \psi_n$ with *sig*$(\psi_i) \subseteq \Sigma$ such that

$$(M'', [d'_i]) \models \psi_j \text{ iff } j = i \tag{9*}$$

for $1 \leq i, j \leq n$ and $n \geq 2$. By Claim (1) that $(M'', [d]) \models \Sigma_{\varphi^\dagger}$, we have that

$$(M'', [d']) \models (\bigwedge_{1 \leq i \leq n} (\Diamond(\psi'^\sharp \wedge \psi_i \wedge \bigwedge_{j \neq i} \neg\psi_j))) \rightarrow p^\psi \tag{10*}$$

Since $[d'] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - Deg(\psi)}$, $[d'_i] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - (Deg(\psi) - 1)}$ for each $1 \leq i \leq n$. Since $Deg(\psi') = Deg(\psi) - 1$ and $\max\{Deg(\psi''_i) : 1 \leq i \leq q\} \leq Deg(\psi')$,

$$[d'_i] \in [d]{\uparrow}^0 \cup [d]{\uparrow}^1 \cup \cdots \cup [d]{\uparrow}^{\ell - \max\{Deg(\psi''_i) : 1 \leq i \leq q\}}$$

for each $1 \leq i \leq n$. Then by (7*),

$$(M'', [d'_i]) \models \psi'^\sharp \tag{11*}$$

for each $1 \leq i \leq n$. From (9*), (10*) and (11*), we get that $(M'', [d']) \models p^\psi$. Therefore, (6*) is proved.

Last, from (6*), Claim (1) and $\varphi^\sharp \in \Sigma_{\varphi^\dagger}$, Claim (3) that $(M'', [d]) \models \varphi$ is proved.

Since φ is locally preserved under inverse Δ-i-p-morphisms over tree models such that *sig*$(\varphi) \subseteq \Delta$ and *sig*$(\alpha) \subseteq \Delta$ for each MLI-formula $\alpha \in \Sigma_{\varphi^\dagger}$, from Claim (3) and the fact that there is a Σ-i-p-morphism from M' to M'', we have that $(M', d) \models \varphi$, which is contrary to what we have prove that $(M', d) \not\models \varphi$. Therefore, for each tree model $M = (W, R, V)$ with $d \in W$, if $(M, d) \models \Sigma_{\varphi^\dagger}$, then $(M, d) \models \varphi$, i.e., Claim 2 is proved. □

6 MLGI to ML

Now we consider the problem of locally m-conservative rewritability of MLGI to ML over tree models.

Theorem 22, the main result of this subsection, can be proved by Theorem 21.

Theorem 22. *Let φ be an MLGI-formula, Δ^* be a set of ML-formulas and Δ be a set of propositional variables such that sig$(\varphi) \subseteq \Delta$ and sig$(\alpha) \subseteq \Delta$ for each ML-formula $\alpha \in \Delta^*$. Then the MLGI-formula φ is locally m-conservatively rewritable into Δ^* over tree models iff φ is locally preserved under inverse Δ-p-morphisms over tree models.*

Proof ($\Rightarrow$) This part can be proved by a similar one to the proof of Theorem 20. ($\Leftarrow$) Let φ be an MLGI-formula, Δ^* be a set of ML-formulas and Δ be a set of propositional variables such that $sig(\varphi) \subseteq \Delta$ and $sig(\alpha) \subseteq \Delta$ for each ML-formula $\alpha \in \Delta^*$. Assume that an MLGI-formula φ is locally preserved under inverse Δ-p-morphisms over tree models. Since each i-p-morphism is also a p-morphism, φ is locally preserved under inverse i-p-morphisms over tree models. By Theorem 21 and the fact that each ML-formula is also an MLI-formula, φ is locally m-conservatively rewritable into Δ^* over tree models. □

Lemma 3 says, a p-morphism between two tree models f itself is also an i-p-morphism.

Lemma 3 Let M_1 and M_2 be tree models. Then each p-morphism from M_1 to M_2 is also an i-p-morphism from M_1 to M_2.

Proof Assume that M_1 and M_2 are tree models with d_1^*, d_2^* being their roots respectively. Let f be a p-morphism from M_1 to M_2. We prove that f is also an i-p-morphism from M_1 and M_2. Assume the contrary, i.e., f is not an i-p-morphism from M_1 to M_2. It means that $f(x)R_2f(y)$ with $x, y \in W_1$ but there is no point $z \in W_1$ such that zR_1y and $f(z) = f(x)$. Since M_1 and M_2 are tree models and $f(d_1^*) = d_2^*$ by the definition of p-morphisms, $d_1^* \neq y$. Then there is an R_1-path $d_1^*R_1x_1R_1x_2 \cdots R_1x_nR_1y$ from d_1^* to y in M_1. Thus, according to the definition of p-morphisms, there is an R_2-path $d_2^*R_2f(x_1)R_2f(x_2)R_2 \cdots R_2f(x_n)R_2f(y)$ from d_2^* to $f(y)$ in M_2. Since there is no point $z \in W_1$ such that zR_1y and $f(z) = f(x)$, we have that $f(x_n) \neq f(x)$. Then the point $f(y)$ in M_2 has two different predecessors $f(x)$ and $f(x_n)$. It is contrary to our assumption that M_2 is a tree model. Therefore, f itself is also an i-p-morphism from M_1 and M_2. □

From Theorem 22 and Lemma 3, the following theorem is got immediately, whose proof is omitted for its clearness.

Theorem 23. *Let φ be an MLGI-formula, Δ^* be a set of ML-formulas and Δ be a set of propositional variables such that sig$(\varphi) \subseteq \Delta$ and sig$(\alpha) \subseteq \Delta$ for each ML-formula $\alpha \in \Delta^*$. The following conditions are equivalent for the MLGI-formula φ:*

(i) φ is locally m-conservatively rewritable into Δ^* over tree models;
(ii) φ is locally preserved under inverse Δ-p-morphisms over tree models;
(iii) φ is locally preserved under inverse Δ-i-p-morphisms over tree models.

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与模态逻辑树模型上的重述相关的若干结果

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摘　要

本文考察模态逻辑树模型上的局部等价以及 m-保守的重述。所谓重述指的是将一种语言下的公式翻译到另一种语言中去。这种翻译可以是等价的（局部等价性），也可以是不等价的（m-保守的）。本文所研究的模态语言包括 ML、MLI、MLG 和 MLGI，所涉及的模型是模态逻辑的树模型。

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