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Logics for Modally Real and Modally Nonreal Events

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Abstract. An event is modally real in one world if it occurs either in the world or in one of its possible worlds; accordingly, an event is modally nonreal in one world if it does not occur in the world or in any one of its possible worlds. We call a place where all modally nonreal events of a world occur or exist as a *modally black hole*. This paper presents logical systems for modally real events and modally nonreal events, proves their soundness, and establishes their completeness.

1 Introduction

Modal realists, extreme or moderate, admit the reality of numerous worlds. For example, D. Lewis said, "Possible worlds are what they are, and not some other thing. If asked what sort of thing they are, I cannot give the kind of reply my questioner probably expects: that is, a proposal to reduce possible worlds to something else. I can only ask him to admit that he knows what sort of thing our actual world is, and then explain that possible worlds are more things of *that* sort, differing not in kind but only in what goes on at them." ([5], p. 85) Because any possible world constitutes things, admitting that possible worlds are just as real as our world means admitting that things in any possible world are just as real as things in our world. Hence, a thing or an event is regarded as modal reality if it exists or occurs either in our world or in one of the possible worlds of our world, and a thing or an event is regarded as modal nonreality if it does not exist or occur either in our world or in any of the possible worlds of our world. Thus, we have two notions: modal reality and modal nonreality.

Because a modally nonreal thing does not exist in the world or any of its possible worlds, where does it inhabit? We suppose there is such a place where all modally nonreal events of the world inhabit, and we call the place a *modally black hole*. What we focus on here is not questions related to the modally black hole, such as whether the modally black hole exists, but the logical structures of modally real events and modally nonreal events.

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2 Proof Systems for Modally Real and Modally Nonreal Events

The definitions of modal reality and modal nonreality are as follows: an event is modally real in our world if it occurs either in our world or in one of its possible worlds, and an event is modally nonreal in our world if it does not occur either in our world or in any of its possible worlds. We use p to represent an event, R for a modal reality operator, and B for a modal nonreality operator. Rp and Bp represent that "p is modally real" and "p is modally nonreal" respectively. The formal language L is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid R\varphi \mid B\varphi$$

The language in L is interpreted by the standard possible world semantics.

Definition 1 (Frames, Models, and Satisfaction**).** A Kripke frame $F = \langle W, R\rangle$ is a tuple where W is a set of possible worlds and $R \subseteq W \times W$ is an accessibility relation. A Kripke model $M = (F, \pi)$ is a tuple where F is a Kripke frame and $\pi : P \to 2^w$ is an interpretation for a set of propositional variables P. A formula φ is true in model M in the world w if

$M, w \models p$ iff $w \in \pi(p)$,
$M, w \vDash \neg\varphi$ iff it is not the case that $M, w \vDash \varphi$,
$M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \vDash \psi$,
$M, w \vDash R\varphi$ iff $M, w \vDash \varphi$, or for some w' with Rww', $M, w' \vDash \varphi$, and
$M, w \vDash B\varphi$ iff $M, w \vDash \neg\varphi$, and for any w' with Rww', $M, w' \vDash \neg\varphi$.

Semantically, the relations between the modal reality operator R or the modal nonreality operator B and the necessity operator or the possibility operator are as follows:

$Rp \equiv (p \vee \Diamond p)$ and $Bp \equiv (\neg p \wedge \Box\neg p)$

The relation between R and B is as follows:

$Rp \leftrightarrow \neg Bp$

Because the modally real operator R and the modally nonreal operator B are interdefinable (i.e., $Rp \leftrightarrow \neg Bp$), we use B as the primitive operator, and R can be defined by B.

Definition 2. System B_0 comprises the following axioms and transformation rules:

Ax0 all tautologies of propositional logic.
Ax1 $\vdash_{B_0} B\varphi \to \neg\varphi$
Ax2 $\vdash_{B_0} B(\varphi \wedge \psi) \to \neg B\neg\varphi \vee B\psi$
Ax3 $\vdash_{B_0} B\varphi \wedge B\psi \to B(\varphi \vee \psi)$
MP $\vdash_{B_0} \varphi, \vdash_{B_0} \varphi \to \psi \Rightarrow \vdash_{B_0} \psi$
RE $\vdash_{B_0} \varphi \leftrightarrow \psi \Rightarrow B\varphi \leftrightarrow B\psi$
RC $\vdash_{B_0} \varphi \Rightarrow \vdash_{B_0} B\neg\varphi$

Note that B_0 is the propositional calculus plus the axioms Ax1, Ax2, and Ax3 and the transformation rules RE and RC.

Theorem 1. *B_0 is sound w.r.t. arbitrary frames.*

Proof. We only demonstrate that Ax1, Ax2, Ax3, RE and RC are valid with respect to arbitrary frames.

Suppose that M is a model that is based on an arbitrary frame and w is a world in M.

For Ax1, suppose that $M, w \nvDash B\varphi \to \neg\varphi$. Consequently, $M, w \vDash \neg(B\varphi \to \neg\varphi)$. According to Ax0, $M, w \vDash B\varphi \wedge \varphi$. Hence, (a) $M, w \vDash B\varphi$ and (b) $M, w \vDash \varphi$. From (a), according to the definition of $B\varphi$ in Definition 1, $M, w \vDash \neg\varphi$, which contradicts (b).

For Ax2, suppose that $M, w \nvDash B(\varphi \wedge \psi) \to \neg B\neg\varphi \vee B\psi$. Therefore, $M, w \vDash B(\varphi\wedge \psi) \wedge B\neg\varphi \wedge \neg B\psi$. Hence, (a) $M, w \vDash B(\varphi\wedge\psi)\wedge B\neg\varphi$ and (b) $M, w \vDash \neg B\psi$. Hence, from (a), $M, w \vDash \neg(\varphi \wedge \psi)$ and $M, w \vDash \varphi$, and for any world w' with Rww', $M, w' \vDash \neg\ (\varphi \wedge \psi)$ and $M, w' \vDash \varphi$. Then, $M, w \vDash \neg\psi$, and for any world w' with Rww', $M, w' \vDash \neg\psi$. Hence, we have $M, w \vDash B\psi$, which contradicts (b).

For Ax3, suppose that $M, w \nvDash B\varphi\wedge B\psi \to B(\varphi\vee\psi)$. Then, $M, w \vDash B\varphi\wedge B\psi\wedge \neg B(\varphi \vee \psi)$. Hence, $M, w \vDash B\varphi \wedge B\psi$ and $M, w \vDash \neg B(\varphi \vee \psi)$. Consequently, from $M, w \vDash B\varphi \wedge B\psi$, $M, w \vDash \neg\varphi$ and $M, w \vDash \neg\psi$, and for any world w' with Rww', $M, w' \vDash \neg\varphi$ and $M, w' \vDash \neg\psi$. Hence, $M, w \vDash \neg(\varphi \wedge \psi)$, and for any world w' with Rww', $M, w' \vDash \neg(\varphi \wedge \psi)$. Thus, we have $M, w \vDash B(\varphi \vee \psi)$, which contradicts $M, w \vDash \neg B(\varphi \vee \psi)$.

For RE, suppose that $\vDash \varphi \leftrightarrow \psi$. Consequently, $M, w \vDash \varphi \leftrightarrow \psi$, and for any w' such that Rww', $M, w' \vDash \varphi \leftrightarrow \psi$. (a) Assume that $M, w \vDash B\varphi$. According to the definition of $B, M, w \vDash \neg\varphi$, and for any w' such that $Rww', M, w' \vDash \neg\varphi$. Hence, according to $M, w \vDash \varphi \leftrightarrow \psi$ and $M, w' \vDash \varphi \leftrightarrow \psi$, we have $M, w \vDash \neg\psi$ and $M, w' \vDash \neg\psi$. Therefore, according to the definition of B, $M, w \models B\psi$ is obtained. (b) Assume that $M, w \vDash B\psi$. The same reason as that in (a) ensures that $M, w \vDash B\varphi$. Thus, by (a) and (b), we have $M, w \vDash B\varphi \leftrightarrow B\psi$.

For RC, suppose that $\vDash \varphi$. Then, $M, w \vDash \varphi$, and for any w' such that Rww', $M, w' \vDash \varphi$. Hence, by Definition 1, $M, w \vDash \neg\neg\varphi$, and for any w' such that Rww', $M, w' \vDash \neg\neg\varphi$. Thus, by the definition of $B\varphi$, $M, w \vDash B\neg\varphi$. □

To obtain a new and useful derived rule, suppose that $\vdash_{B_0} \varphi \to \psi$. Consequently, by RC, $\vdash_{B_0} B\neg(\varphi \to \psi)$. Because $\neg(\varphi \to \psi)$ is equivalent to $\neg\psi \wedge \varphi$, we, by RE and MP, obtain $\vdash_{B_0} B(\neg\psi \wedge \varphi)$. Applying Ax2 to $\vdash_{B_0} B(\neg\psi \wedge \varphi)$ and using MP, we obtain $\vdash_{B_0} \neg\ B\psi \vee B\varphi$. Thus, we follow the derived rule:

RC1 $\vdash_{B_0} \varphi \to \psi, \Rightarrow \vdash_{B_0} B\psi \to B\varphi$.

Theorem 2. *The following formulae are provable in system B_0 :*

1. $\vdash_{B_0} B(\varphi \vee \psi) \to B(\varphi \wedge \psi)$
2. $\vdash_{B_0} B\varphi \to \neg B \neg \varphi$
3. $\vdash_{B_0} B(\varphi \vee \psi) \leftrightarrow B\varphi \wedge B\psi$
4. $\vdash_{B_0} B\varphi \to B(\varphi \wedge \psi)$

To extend B_0, a weak relation R^w over possible worlds must be defined by the relation R and the identical relation R^0. $R^w ww'$ is defined as Rww' or $R^0 ww'$. Formally, $R^w ww' \equiv_{def.} Rww' \vee R^0 ww'$.

Definition 3 (Weak Frames)**.**

1. A frame $\langle W, R\rangle$ is weakly transitive if for any $w, w', w'' \in W$, if $R^w ww'$ and $R^w w'w''$, then $R^w ww''$.
2. A frame $\langle W, R\rangle$ is semiweakly Euclidean if for any $w, w', w'' \in W$, if $R^w ww'$ and Rww'', then $R^w w'w''$.
3. A frame $\langle W, R\rangle$ is weakly Euclidean if for any $w, w', w'' \in W$, if $R^w ww'$ and $R^W ww''$, then $R^w w'w''$.
4. A frame $\langle W, R\rangle$ is weakly dead if for any $w, w' \in W$, if $R^w ww'$, then $R^0 ww'$.

Four notes:

(a) A frame that is transitive (semiweakly Euclidean and weakly Euclidean) must be weakly transitive (semiweakly Euclidean and weakly Euclidean), and not vice versa.

(b) A semiweakly Euclidean frame must be weakly Euclidean, and not vice versa.

(c) In a weakly Euclidean frame for any $w, w' \in W$, if Rww', we, by $R^0 ww$, have $Rw'w$. It means that a weakly Euclidean frame must be symmetric, and not vice versa.

(d) A weakly symmetric frame is identical to a symmetric frame.

We do not present weakly reflexive frames in Definition 3. In fact, if we define a weakly reflexive frame in which $R^w ww$ holds for any $w \in W$, such a frame is arbitrary, and vice versa. This means that a frame is weakly reflexive if and only if it is arbitrary. We can say that B_0 is sound with respect to weakly reflexive frames. This indicates that our language L is weaker and a model based on reflexive frames is indistinguishable.

Theorem 3.

1. *The formula* $B\varphi \to B\neg B\varphi$ *is valid w.r.t. weakly transitive frames.*
2. $\varphi \to BB\varphi$ *is valid w.r.t. symmetric frames.*
3. $\neg\varphi \wedge \neg B\varphi \to BB\varphi$ *is valid w.r.t. semiweakly Euclidean frames.*
4. $\neg B\varphi \to BB\varphi$ *is valid w.r.t. weakly Euclidean frames.*

Proof. For 1. Let M be an arbitrary model that is based on a weakly transitive frame and w be a world in M. Suppose that $M, w \nvDash B\varphi \to B\neg B\varphi$. Consequently, (a) $M, w \vDash B\varphi$ and (b) $M, w \vDash \neg B\neg B\varphi$. From (b), together with (a), we have for some w^* with Rww^*, $M, w^* \vDash \neg B\varphi$. Hence, $M, w^* \vDash \neg B\varphi$ and $M, w^* \vDash \neg\varphi$. Therefore, for some w^{**} with Rw^*w^{**}, $M, w^{**} \vDash \varphi$. Because R is weakly transitive, Rww^{**} or R^0ww^{**}. However, from (a), we have $M, w \vDash \neg\varphi$, and for any w' with Rww', $M, w' \vDash \neg\varphi$. This means that we have M, $w^{**} \vDash \neg\varphi$ and $M, w \vDash \neg\varphi$. A contradiction arises.

For 2. Let M be an arbitrary model that is based on a symmetric frame and w be a world in M, and suppose that $M, w \vDash \varphi$. Then, $M, w \nvDash B\varphi$, and $M, w' \nvDash B\varphi$ for any world w' in M that "sees" w. Because M is symmetrical, the fact that $M, w' \nvDash B\varphi$ and $M, w \nvDash B\varphi$ causes $BB\varphi$ to be false in w. Hence, we have $M, w \vDash BB\varphi$.

For 3. Let M be an arbitrary model that is based on a semiweakly Euclidean frame and w be a world in M, and suppose that $M, w \vDash \neg\varphi \wedge \neg B\varphi$. Consequently, $M, w \vDash \neg\varphi$, and $M, w \vDash \neg B\varphi$ from which either $M, w \vDash \varphi$ or there must be a world w^* in M such that Rww^* and $M, w^* \vDash \varphi$. Because $M, w \vDash \neg\varphi$, we have $M, w^* \vDash \varphi$. To demonstrate that $M, w \vDash BB\varphi$, because we know $M, w \vDash \neg B\varphi$, we must demonstrate that for any world w' in M such that Rww', $M, w' \vDash \neg B\varphi$. It is the case because for any w' in M such that Rww', we have $Rw'w^*$ or $R^0w'w^*$, and w^* is a world in which φ is true.

For 4. Let M be an arbitrary model that is based on a weakly Euclidean frame and w be a world in M, and suppose that $M, w \vDash \neg B\varphi$. There exist two cases: (a) $M, w \vDash \varphi$ and (b) for some w^* with Rww^*, $M, w^* \vDash \varphi$.

Case (a). Because a Euclidean frame is symmetric, for any w' with Rww', we have that $Rw'w$, and $M, w' \vDash \neg B\varphi$. Thus, $M, w \vDash BB\varphi$.

Case (b). Because M is weakly Euclidean, for any w' such that $R^w ww'$, $R^w w'w^*$. By $M, w^* \vDash \varphi, M, w' \vDash \neg B\varphi$. Therefore, we have $M, w \vDash BB\varphi$. □

Definition 4. We have the extensions of B_0:

1. $B_1 = B_0 \oplus B\varphi \to B\neg B\varphi$.
2. $B_2 = B_0 \oplus \varphi \to BB\varphi$.
3. $B_3 = B_0 \oplus \neg\varphi \wedge \neg B\varphi \to BB\varphi$.
4. $B_4 = B_0 \oplus \neg B\varphi \to BB\varphi$.

By Theorem 2 and 3, we obtain the following:

Theorem 4 (Soundness)**.**

1. *B_1 is sound w.r.t. weakly transitive frames.*
2. *B_2 is sound w.r.t. symmetric frames.*
3. *B_3 is sound w.r.t. semiweakly Euclidean frames.*
4. *B_4 is sound w.r.t. weakly Euclidean frames.*

Note that because $\varphi \to BB\varphi$ and $\neg\varphi \wedge \neg B\varphi \to BB\varphi$ are provable in B_4, B_4 is stronger than both B_2 and B_3.

There exists a trivial axiom $\varphi \to B\neg\varphi$ to the systems. If $\varphi \to B\neg\varphi$ is added to B_0 or other systems, the resulting system will collapse into the propositional calculus. This can be demonstrated as follows. Because of Ax1 in Definition 1, we have $\varphi \leftrightarrow B\neg\varphi$, by which the operator B in all formulae will be replaced. If a weakly dead end frame is defined as for any $w \in W$, if Rww', then R^0ww'; the trivial axiom $\varphi \to B\neg\varphi$ is valid in weakly dead end frames.[1]

By $R\varphi \leftrightarrow \varphi$ and $\varphi \to B\neg\varphi, R\varphi \to \varphi$. Philosophically, if our world is the only world, $R\varphi \to \varphi$ means that the modally real thing must be actual, and not vice versa. However, if φ is actual, its negation is modally nonreal.

3 Completeness

The modal logics we present in the previous section are nonstandard. Following logicians who have dealt with other nonstandard modal logics such as logics of contingency and noncontingency ([2, 3, 4, 9]) and the ones of essence and accident ([6, 7]), we establish ad hoc canonical models for logics of modal reality and modal nonreality.

For the purpose of presenting a general result, we use S to stand for B_0 or one of its extensions.

A set Γ of well-formed formulae is maximally consistent with respect to a system S, if and only if for every formula α, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$, and there is no finite collection $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma$ such that $\vdash_S \neg(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n)$. We simply call Γ a maximally S-consistent set.

To establish the completeness of the systems, we construct the successor of a maximally S-consistent set.

Definition 5. Let Γ be a maximally S-consistent set. The successor of $\Gamma, D(\Gamma)$, is defined as $D(\Gamma) = \{\alpha \mid \text{for every } \alpha, B\neg\alpha \in \Gamma\}$.

Lemma 1. *For a maximally S-consistent set Γ of formulae, the successor $D(\Gamma)$ is closed under conjunction.*

Proof. Let Γ be a maximally S-consistent set and $D(\Gamma)$ be the successor of Γ.

Suppose that $\alpha \in D(\Gamma)$ and $\beta \in D(\Gamma)$. According to the construction of $D(\Gamma), B\neg\alpha \in \Gamma$ and $B\neg\beta \in \Gamma$. By Ax3, $B\neg\alpha \wedge B\neg\beta \to B(\neg\alpha \vee \neg\beta)$, and by the maximality of $\Gamma, B(\neg\alpha \vee \neg\beta) \in \Gamma$. By RE, $B(\neg\alpha \vee \neg\beta) \leftrightarrow B\neg(\alpha \wedge \beta)$. Hence, $B\neg(\alpha \wedge \beta) \in \Gamma$. Thus, $(\alpha \wedge \beta) \in D(\Gamma)$, which is required. □

[1] Let M be an arbitrary model based on a weakly dead end frame and w be a world in M. Assume that $M, w \vDash \varphi$. Because w is only a possibly accessible world of w and $M, w \vDash \varphi$, we obtain $M, w \vDash B\neg\varphi$.

Lemma 2. *Let Γ be a maximally S-consistent set of formulae. Assume that for some $\beta, \neg B\beta \in \Gamma$. Then, $\{\beta\} \cup D(\Gamma)$ is S-consistent.*

Proof. Suppose that $\neg B\beta \in \Gamma$ but $\{\beta\} \cup D(\Gamma)$ is not S-consistent. Consequently, there must be $\psi_1, \psi_2, \ldots, \psi_n \in D(\Gamma)$ such that

$\vdash_S \neg(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n \wedge \beta)$. (a)

By Ax0,

$\vdash_S \beta \rightarrow \neg(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n)$. (b)

By RC1, from (b),

$\vdash_S B\neg(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n) \rightarrow B\beta$. (c)

According to $\psi_1, \psi_2, \cdots, \psi_n \in D(\Gamma)$ and Lemma 1,

$\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n \in D(\Gamma)$ (d)

By the construction of $D(\Gamma)$,

$B\neg(\psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_n) \in \Gamma$. (e)

$B\beta \in \Gamma$ (f)

(f) means $\neg B\beta \notin \Gamma$, which contradicts the supposition of $\neg B\beta \in \Gamma$. □

To prove the completeness of B_0 and its extensions, we must construct the special canonical model for them.

Definition 6 (Canonical Model)**.** The canonical model $M^C = \langle W^C, R^C, \pi^C \rangle$ for the logic S is defined as follows:

1. W^C is the set of all maximally S-consistent sets of formulae;
2. $R^C \subseteq W^C X W^C$ is defined by $R^C\Gamma\Gamma_1$ or $R^0\Gamma\Gamma_1$ iff $D(\Gamma) \subseteq \Gamma_1$; and
3. $p \in \pi^C(\Gamma)$ iff $p \in \Gamma$.

The canonical model for S is *ad hoc*. However, according to Definition 6, we have $D(\Gamma) \subseteq \Gamma_1$ by $R^C\Gamma\Gamma_1$; we cannot specify $R^C\Gamma\Gamma_1$ by $D(\Gamma) \subseteq \Gamma_1$, which is different from normal modal logics.

Lemma 3 (Truth Lemma)**.** *Let $M^C = \left(W^C, R^C, \pi^C\right)$ be the canonical model for S. For all formulae φ and all maximally S-consistent sets Γ s, $M^C, \Gamma \vDash_S \varphi \Leftrightarrow \varphi \in \Gamma$.*

Proof. The theorem can be proven by induction on the structure of wff. Here, we prove the case of $B\varphi$ only:

$$M^C, \Gamma \vDash_S B\varphi \Leftrightarrow B\varphi \in \Gamma.$$

We assume that the theorem holds for φ and for all Γ: $M^C, \Gamma \vDash_S \varphi \Leftrightarrow \varphi \in \Gamma$. Suppose that $B\varphi \in \Gamma$. By the definition of $D(\Gamma)$, $\neg\varphi \in D(\Gamma)$. According to Definition 6, we have that if $R^C\Gamma\Gamma_1$ or $R^0\Gamma\Gamma_1$, $\neg\varphi \in \Gamma_1$. This means that if $R^C\Gamma\Gamma_1$, $\neg\varphi \in \Gamma_1$ and $\neg\varphi \in \Gamma$. Hence, according to the assumption in the beginning, we have

(a) if $R^C\Gamma\Gamma_1$, $M^C, \Gamma_1 \vDash \neg\varphi$ and (b) $M^C, \Gamma \vDash_S \neg\varphi$. According to the definition of the truth value of $B\varphi$, (a) and (b) yield $M^C, \Gamma \models B\varphi$.

Suppose that $B\varphi \notin \Gamma$. Subsequently, because Γ is maximal and S-consistent, $\neg B\varphi \in \Gamma$. According to Lemma 2, $\{\varphi\} \cup D(\Gamma)$ is S-consistent. Thus, there is some Γ_1 such that (a) $D(\Gamma) \subseteq \Gamma_1$ and (b) $\varphi \in \Gamma_1$. According to the definition of R^C, (a) yields $R^C\Gamma\Gamma_1$ or $R^0\Gamma\Gamma_1$, but according to the assumption, (b) yields $M^C, \Gamma_1 \vDash \varphi$. Hence, by the definition of the truth value of $B\varphi$, $M^C, \Gamma \nvDash_S B\varphi$. □

Theorem 5 (Completeness). *Given the system S, for any formula φ, we have*

$$\vdash_S \varphi \Leftrightarrow \vDash_S \varphi.$$

Proof. Soundness (i.e., $\vdash_S \varphi \Rightarrow \vDash_S \varphi$) is illustrated in Theorem 1 and Theorem 4.

For completeness, we suppose that $\nvdash_S \varphi$. Then, $\neg\varphi$ is maximally S-consistent. Therefore, by the Lindenbaum theorem, there is some maximally S-consistent set Γ in W^C such that $\neg\varphi \in \Gamma$. By Lemma 3, $M^C, \Gamma \vDash_S \neg\varphi$. Thus, $\nvDash_S \varphi$. □

Different systems are complete with respect to different frames. Thus, we have the following:

Theorem 6.

1. *B_0 is complete w.r.t. arbitrary frames.*
2. *B_1 is complete w.r.t. weakly transitive frames.*
3. *B_2 is complete w.r.t. symmetric frames.*
4. *B_3 is complete w.r.t. semiweakly Euclidean frames.*
5. *B_4 is complete w.r.t. weakly Euclidean frames.*

Note that the definitions of weakly transitive frames, semiweakly Euclidean frames, and weakly Euclidean frames are described in Definition 3.

Proof. For 1. The axioms and transformation rules of B_0 require no information about its canonical model, which means that B_0 is complete with respect to arbitrary frames.

For 2. We must demonstrate that the canonical model of B_1 is weakly transitive. Assume that $R^w\Gamma\Gamma'$ and $R^w\Gamma'\Gamma''$; thus, $D(\Gamma) \subseteq \Gamma'$ and $D(\Gamma') \subseteq \Gamma''$. Let α be an arbitrary element in $D(\Gamma)$. Hence, $B\neg\alpha \in \Gamma$. According to axiom $B\neg\alpha \to B\neg B\neg\alpha, B\neg\ B\neg\alpha \in \Gamma$. Therefore, $B\neg\alpha \in D(\Gamma)$, and because $D(\Gamma) \subseteq \Gamma'$, it follows that $B\neg\alpha \in \Gamma'$. Again, we have $\alpha \in D(\Gamma')$, and because $D(\Gamma') \subseteq \Gamma''$, it follows that $\alpha \in \Gamma''$. Because α is arbitrary, we have $D(\Gamma) \subseteq \Gamma''$, and this means $R\Gamma\Gamma''$ or $R^0\Gamma\Gamma''$.

For 3. We must demonstrate that the canonical of B_2 is symmetric. Assume that $R\Gamma\Gamma'$. Hence, $D(\Gamma) \subseteq \Gamma'$. Let α be an arbitrary element in $D(\Gamma')$. Hence, $B\neg\alpha \in \Gamma'$. Because $D(\Gamma) \subseteq \Gamma', \neg B\neg\alpha \notin D(\Gamma)$. According to the definition of

$D(\Gamma)$, it implies that $BB\neg\alpha \notin \Gamma$. Then, by axiom $\neg\alpha \to BB\neg\alpha$, we have $\neg\alpha \notin \Gamma$, which means $\alpha \in \Gamma$. Because α is arbitrary, we have $D(\Gamma') \subseteq \Gamma$, and this means $R\Gamma'\Gamma$ or $R^0\Gamma\Gamma''$. Thus, $R\Gamma'\Gamma$.

For 4. Assume that $R^w\Gamma\Gamma'$ and $R\Gamma\Gamma''$. Then, $D(\Gamma) \subseteq \Gamma'$ and $D(\Gamma) \subseteq \Gamma''$. Let α be an arbitrary formula that is not in Γ'' but $B\neg\alpha \in \Gamma'$. By Theorem 2 (4), $B\neg\alpha \to B(\neg\alpha \wedge \neg\beta)$, and RE, we obtain $B\neg\alpha \to B\neg(\alpha \vee \beta)$. By $B\neg\alpha \in \Gamma'$, $B\neg(\alpha\vee\beta) \in \Gamma'$. Because Γ is not identical to Γ'', there must exist some β, $\beta \notin \Gamma''$ but $\beta \in \Gamma$. Hence, $\alpha \vee \beta \notin \Gamma''$. Because $R\Gamma\Gamma''$, $B\neg(\alpha\vee\beta) \notin \Gamma$ by which $\neg B\neg(\alpha\vee\beta) \in \Gamma$. Because of $\beta \in \Gamma$, $(\alpha\vee\beta) \in \Gamma$. Therefore, $(\alpha\vee\beta)\wedge\neg B\neg(\alpha\vee\beta) \in \Gamma$. According to $\neg\varphi \wedge \neg B\varphi \to BB\varphi$, $BB\neg(\alpha \vee \beta) \in \Gamma$. $\neg B\neg(\alpha \vee \beta) \in \Gamma'$. However, we have $B\neg(\alpha \vee \beta) \in \Gamma'$. A contradiction arises.

For 5. Assume that $R^w\Gamma\Gamma'$ and $R^w\Gamma\Gamma''$. Then, $D(\Gamma) \subseteq \Gamma'$ and $D(\Gamma) \subseteq \Gamma''$. Let α be an arbitrary formula that is not in Γ'' but $B\neg\alpha \in \Gamma'$. Then, by $R\Gamma\Gamma'', B\neg\alpha \notin \Gamma$. According to axiom $\neg B\neg\alpha \to BB\neg\alpha$, we have $BB\neg\alpha \in \Gamma$. Therefore, $B\neg\neg B\neg\alpha \in \Gamma$, which means that $\neg B\neg\alpha \in D(\Gamma)$. Because $D(\Gamma) \subseteq \Gamma'$, we have $\neg B\neg\alpha \in \Gamma'$, which contradicts $B\neg\alpha \in \Gamma'$. Because α is arbitrary, we have $D(\Gamma') \subseteq \Gamma''$, and this means that $R^w\Gamma'\Gamma''$. □

4 Conclusions and Remarks

This paper presents formal systems for modally real events and modally nonreal events with respect to different structures or frames of possible worlds, demonstrates their soundness, and establishes their completeness. Four additional remarks are presented as follows:

First, formally, compared with normal modal logics, the expressivity of the logics presented in this paper is relatively weak. The logics cannot distinguish reflexive models. Nevertheless, the logics provide us with all formulae that we want.

Second, we regard an event to be modally real or modally nonreal with respect to a world in a model. Some events that are modally real (or nonreal) with respect to a world in a model could be modally nonreal (or real) with respect to other world(s) in the model. We, in the Introduction section, designate a place where *all* modally nonreal events of a world occur as a modally black hole. The *modally black hole* of a world, here, is not a possible world but a place where all modal nonreal events of the world are "stored". It can be seen that given a model, each world has *only* a modally black hole, and the number of modally black holes in a model is equivalent to the number of the worlds of the model.

Third, although our world consists of infinite events, there exist also infinite events that do not occur in our world. Hence, if possible worlds of our world are not infinite, there exist infinite events that do not exist in our world or in its possible worlds; alternatively, even if possible worlds of our world are infinite, any contradiction does not exist in any worlds. Consequently, our world has its modally black hole.

Fourth, as we know, the term "black hole" in physics refers to a special space-time region that has sufficiently compact mass and allows nothing to escape from the inside. We cannot observe any event in a physically black hole. How about an event in the modally black hole of our world? If the conceivable is possible and exists in possible worlds ([1, 8]), any event in the modally black hole of our world is inconceivable, even unimagined. For instance, we can describe a contradiction, but we cannot conceive a contradictory event. Nevertheless, events in the modally black hole have their own logical structure, as revealed by the aforementioned logics.

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模态实在与模态非实在事件的逻辑

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摘　要

一个事件在一个世界中是模态实在的，如果它发生在这个世界或该世界的一个可能世界中；如果一个事件没有发生在一个世界或其任何一个可能世界中，则它在这个世界中是模态非实在的。我们称一个世界中所有模态非实在事件发生或存在的地方为模态黑洞。本文提出了模态实在事件和模态非实在事件的逻辑系统，并证明了系统的可靠性和完全性。

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