A Labelled Sequent Calculus for Public Announcement Logic

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Abstract. Public announcement logic (PAL) is an extension of epistemic logic (EL) with some reduction axioms. In this paper, we propose a cut-free labelled sequent calculus for PAL, which is an extension of that for EL with sequent rules adapted from the reduction axioms. This calculus admits cut and allows terminating proof search.

1   Introduction

The modern modal approach to the logic of knowledge and belief was extensively developed by Hintikka ([10]) to interpret epistemic notions utilizing possible world semantics. The standard multi-agent epistemic logic EL is usually identified with the poly-modal modal logic S5 for a group of agents. Public announcement logic (PAL), introduced by Plaza ([17]), is an extension of EL that studies logical dynamics of epistemic information after the action of public announcement. More general actions are studied in action model logic (see e.g. [4, 5, 7]).

PAL is an extension of EL with announcement operators of the form $[\varphi]$. As EL, formulas in PAL are interpreted in Kripke models in which all relations are reflexive, transitive and symmetric. In particular, formulas of the form $[\varphi]\psi$ are interpreted in the restrictions of Kripke models induced by the announcement φ. The Hilbert-style axiomatization of PAL is obtained by adding to that of EL the so called *reduction axioms* for announcement operators, which can be used to eliminate announcement operators in a PAL-formula.

Generally speaking, it is difficult to prove whether proof search using a Hilbert-style axiomatization is decidable. In view of these, many proof systems for PAL are proposed in the literature, e.g., tableau systems ([3]), labelled sequent calculi ([2, 16]).

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In this paper, we propose another labelled sequent calculus for PAL. This calculus is based on a labelled sequent calculus for EL proposed by Hakli and Negri ([9]) and rules for announcement operators designed according to the reduction axioms. This calculus admits structural rules (including cut) and allows terminating proof search. Unlike [2, 16], which are based on another semantics for PAL, our calculus is based on the original semantics and takes into account the conditions of reflexivity, transitivity and symmetry in EL.

The rest of the paper is structured as follows: Section 2 presents some basic notions and concepts. Section 3 presents our labelled sequent calculus G_{PAL} for PAL. Section 4 shows that G_{PAL} admits some structural rules, including cut. Section 5 shows that G_{PAL} allows terminating proof search. Section 6 compares G_{PAL} with related works, concludes the paper and discusses future work.

2   Preliminaries

2.1   EL and PAL

Let Prop be a denumerable set of variables and Ag a finite set of agents. Language $\mathcal{L}_{\mathrm{EL}}$ for epistemic logic is defined inductively as follows:

$$\mathcal{L} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid K_a\varphi$$

Moreover, language $\mathcal{L}_{\mathrm{PAL}}$ for public announcement logic is defined inductively as follows:

$$\mathcal{L} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid K_a\varphi \mid [\varphi]\varphi$$

where $p \in \mathrm{Prop}$ and $a \in \mathrm{Ag}$. We use $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. $\mathcal{L}_{\mathrm{PAL}}$ is an extension of $\mathcal{L}_{\mathrm{EL}}$ with announcement formulas.

An *epistemic frame* $\mathcal{F}$ is a tuple $(W, \{\sim_a\}_{a\in\mathrm{Ag}})$, where W is a set of states and $\sim_a \subseteq W \times W$ is a reflexive, transitive and symmetric relation for each $a \in \mathrm{Ag}$.

An *epistemic model* $\mathcal{M}$ is a tuple $(W, \{\sim_a\}_{a\in\mathrm{Ag}}, V)$ where $(W, \{\sim_a\}_{a\in\mathrm{Ag}})$ is an epistemic frame and V is a function from Prop to $\mathcal{P}(W)$.

Let $\mathcal{M} = (W, \{\sim_a\}_{a\in\mathrm{Ag}}, V)$ be an epistemic model and $w \in W$. The notion of φ being *true* at w in $\mathcal{M}$ (notation: $\mathcal{M}, w \Vdash \varphi$) is defined inductively as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \Vdash \neg\varphi & \text{iff} & \mathcal{M}, w \nVdash \varphi \\
\mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \to \psi & \text{iff} & \mathcal{M}, w \nVdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash K_a\varphi & \text{iff} & \text{for all } v \in W, w \sim_a v \text{ implies } \mathcal{M}, v \Vdash \varphi \\
\mathcal{M}, w \Vdash [\varphi]\psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ implies } \mathcal{M}^\varphi, w \Vdash \psi
\end{array}
$$

where $\mathcal{M}^\varphi = (W^\varphi, \{\sim_a^\varphi\}_{a\in\mathrm{Ag}}, V^\varphi)$ is the *model restricted to* φ with $W^\varphi = \{w \in W \mid \mathcal{M}, w \Vdash \varphi\}$, $\sim_a^\varphi = \sim_a \cap (W^\varphi \times W^\varphi)$ and $V^\varphi = V(p) \cap W^\varphi$.

A formula φ is *globally true* in an epistemic model (notation: $\mathcal{M} \Vdash \varphi$) if $\mathcal{M}, w \Vdash \varphi$ for all $w \in W$. A formula φ is *valid* in an epistemic frame $\mathcal{F}$ if $\mathcal{F}, V \Vdash \varphi$ for all valuations V.

Epistemic logic EL is the set of $\mathcal{L}_{\mathrm{EL}}$-formulas that are valid on the class of all epistemic frames. Public announcement logic PAL is the set of $\mathcal{L}_{\mathrm{PAL}}$-formulas that are valid on the class of all epistemic frames.

EL is axiomatized by (Tau), (K), (4), (T), (5), (MP) and (GK$_a$). PAL is axiomatized by the axiomatization for EL plus *reduction axioms* (R1)–(R6):

(Tau)	Classical propositional tautologies.	(R1)	$[\varphi]p \leftrightarrow (\varphi \to p)$
(K)	$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	(R2)	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
(4)	$K_a\varphi \to K_aK_a\varphi$	(R3)	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
(T)	$K_a\varphi \to \varphi$	(R4)	$[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$
(5)	$\neg K_a\varphi \to K_a\neg K_a\varphi$	(R5)	$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$
(MP)	From φ and $\varphi \to \psi$ infer ψ.	(R6)	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$
(GK$_a$)	From φ infer $K_a\varphi$.		

Remark 1. The standard language for PAL does not contain $\to$. To simplify our writing, we add $\to$ to our language. Because of its existence, (R4) is added to the axiomatization.

2.2 Labelled sequent calculus

A labelled sequent calculus for a logic with Kripke semantics is based on the internalization of Kripke semantics.

Let $\mathcal{F} = (W, \{\sim_a\}_{a\in\mathrm{Ag}})$ be an epistemic frame. A *relational atom* is of the form $x \sim_a y$, where $x, y \in W$ and $a \in \mathrm{Ag}$. A *labelled formula* is of the form $x : \varphi$, where $x \in W$ and φ is an $\mathcal{L}_{\mathrm{EL}}$-formula. We use σ, δ with or without subscript to denote relational atoms or labelled formulas.

A *multiset* is a 'set with multiplicity', or put the other way round, a sequence modulo the ordering. A *labelled sequent* is of the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are finite multisets of relational atoms and labelled formulas. A *sequent rule* is of the form

$$(\mathcal{R}) \frac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_m \Rightarrow \Delta_m}{\Gamma \Rightarrow \Delta}$$

where $m \geq 0$. $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_m \Rightarrow \Delta_m$ are called *premises* of this rule and $\Gamma \Rightarrow \Delta$ is called the *conclusion*. If $m = 0$, we simply write $\Gamma \Rightarrow \Delta$ and call it an *initial sequent*. The formula with the connective in a rule is the *principal* formula of that rule, and its components in the premises are the *active* formulas. A *labelled sequent calculus* is a set of sequent rules. A derivation in a labelled sequent calculus G is defined as usual. The *derivation height h* of a sequent is defined as the length of

longest branch in the derivation of the sequent. We use $G \vdash \Gamma \Rightarrow \Delta$ to denote that $\Gamma \Rightarrow \Delta$ is derivable in G and $G \vdash_h \Gamma \Rightarrow \Delta$ to denote that $\Gamma \Rightarrow \Delta$ is derivable in G with a derivation of height which is at most h.

A sequent rule $(\mathcal{R})$ is *admissible* in G if $G \vdash \Gamma_i \Rightarrow \Delta_i$ for $1 \leq i \leq m$ implies $G \vdash \Gamma \Rightarrow \Delta$. It is *height-preserving admissible* if $G \vdash_h \Gamma_i \Rightarrow \Delta_i$ for $1 \leq i \leq m$ implies $G \vdash_h \Gamma \Rightarrow \Delta$.

Let $\mathcal{M} = (W, \{\sim_a\}_{a \in \mathrm{Ag}}, V)$ be an epistemic model. The *interpretation* $\tau_{\mathcal{M}}$ of relational atoms and labelled formulas are defined as follows:

$$\begin{aligned}
\tau_{\mathcal{M}}(x \sim_a y) &= & x \sim_a y; \\
\tau_{\mathcal{M}}(x : \varphi) &= & \mathcal{M}, x \Vdash \varphi.
\end{aligned}$$

A labelled sequent $\sigma_1, \ldots, \sigma_m \Rightarrow \delta_1, \ldots, \delta_n$ is *valid* if

$$\forall \mathcal{M} \forall x_1 \cdots \forall x_k [\tau_{\mathcal{M}}(\sigma_1) \wedge \cdots \wedge \tau_{\mathcal{M}}(\sigma_m) \to \tau_{\mathcal{M}}(\delta_1) \vee \cdots \vee \tau_{\mathcal{M}}(\delta_n)]$$

is true, where $\mathcal{M} = (W, \{\sim_a\}_{a \in \mathrm{Ag}}, V)$ is an epistemic model, and $x_1, \ldots, x_k$ are variables occurring in $\sigma_1, \ldots, \sigma_m \Rightarrow \delta_1, \ldots, \delta_n$ ranging over W.

A sequent rule $\mathcal{R}$ is *valid* if the validity of all the premises implies the validity of the conclusion.

Given a labelled sequent calculus G and a logic Λ, we say G is a *labelled sequent calculus for* Λ if for all φ, $G \vdash \Rightarrow \varphi$ if and only if $\varphi \in \Lambda$.

2.3 Labelled sequent calculus for EL

Definition 1.   Labelled sequent calculus G_{EL} for EL consists of the following initial sequents and rules[1]:

(1) Initial sequents:

$$x : p, \Gamma \Rightarrow \Delta, x : p \qquad\qquad x \sim_a y, \Gamma \Rightarrow \Delta, x \sim_a y$$

(2) Propositional rules:

$$(\neg\Rightarrow) \frac{\Gamma \Rightarrow \Delta, x : \varphi}{x : \neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\neg) \frac{x : \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg\varphi}$$

$$(\wedge\Rightarrow) \frac{x : \varphi_1, x : \varphi_2, \Gamma \Rightarrow \Delta}{x : \varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\wedge) \frac{\Gamma \Rightarrow \Delta, x : \varphi_1 \quad \Gamma \Rightarrow \Delta, x : \varphi_2}{\Gamma \Rightarrow \Delta, x : \varphi_1 \wedge \varphi_2}$$

$$(\to\Rightarrow) \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \to \psi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow\to) \frac{x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \to \psi}$$

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[1]It is mentioned without proof in [9] that this is a labelled sequent calculus for EL. This calculus is a multi-agent version of the labelled sequent calculus for S5 proposed in [14].

(3) Modal rules:

$$(K_a \Rightarrow) \frac{y : \varphi, x : K_a\varphi, x \sim_a y, \Gamma \Rightarrow \Delta}{x : K_a\varphi, x \sim_a y, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow K_a) \frac{x \sim_a y, \Gamma \Rightarrow \Delta, y : \varphi}{\Gamma \Rightarrow \Delta, x : K_a\varphi}$$

where y does not occur in the conclusion of $(\Rightarrow K_a)$.

(4) Relational rules:

$$(\text{Ref}_a) \frac{x \sim_a x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (\text{Trans}_a) \frac{x \sim_a z, x \sim_a y, y \sim_a z, \Gamma \Rightarrow \Delta}{x \sim_a y, y \sim_a z, \Gamma \Rightarrow \Delta}$$

$$(\text{Sym}_a) \frac{y \sim_a x, x \sim_a y, \Gamma \Rightarrow \Delta}{x \sim_a y, \Gamma \Rightarrow \Delta}$$

Proposition 1. *For any $\mathcal{L}_{\text{EL}}$-formula φ, $G_{\text{EL}} \vdash x : \varphi, \Gamma \Rightarrow \Delta, x : \varphi$.*

Proof.   It can be proved by induction on φ.   □

By proofs similar to those in [14], we have Theorems 2 and 3.

Theorem 2. *Structural rules $(w \Rightarrow)$, $(\Rightarrow w)$, $(c \Rightarrow)$, $(\Rightarrow c)$, $(c_R \Rightarrow)$ and $(\Rightarrow c_R)$ are height-preserving admissible in G_{EL}. The cut rule (Cut) is admissible in G_{EL}.*

$$(w \Rightarrow) \frac{\Gamma \Rightarrow \Delta}{x : \varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow w) \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \varphi}$$

$$(c \Rightarrow) \frac{x : \varphi, x : \varphi, \Gamma \Rightarrow \Delta}{x : \varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow c) \frac{\Gamma \Rightarrow \Delta, x : \varphi, x : \varphi}{\Gamma \Rightarrow \Delta, x : \varphi}$$

$$(c_R \Rightarrow) \frac{x \sim_a y, x \sim_a y, \Gamma \Rightarrow \Delta}{x \sim_a y, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow c_R) \frac{\Gamma \Rightarrow \Delta, x \sim_a y, x \sim_a y}{\Gamma \Rightarrow \Delta, x \sim_a y}$$

$$(Cut) \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Theorem 3. *G_{EL} allows terminating proof search.*

Example 1.   A derivation for axiom (5) in G_{EL} is as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{z : \varphi, y : K_a\varphi, y \sim_a z, x \sim_a z, y \sim_a x, x \sim_a y \Rightarrow z : \varphi}{y : K_a\varphi, y \sim_a z, x \sim_a z, y \sim_a x, x \sim_a y \Rightarrow z : \varphi} (K_a \Rightarrow)}{y : K_a\varphi, x \sim_a z, y \sim_a x, x \sim_a y \Rightarrow z : \varphi} (Tran_a)}{y : K_a\varphi, x \sim_a z, x \sim_a y \Rightarrow z : \varphi} (Sym_a)}{x \sim_a z, x \sim_a y \Rightarrow y : \neg K_a\varphi, z : \varphi} (\Rightarrow \neg)}{x \sim_a y \Rightarrow y : \neg K_a\varphi, x : K_a\varphi} (\Rightarrow K_a)}{\Rightarrow x : K_a\neg K_a\varphi, x : K_a\varphi} (\Rightarrow K_a)}{x : \neg K_a\varphi \Rightarrow x : K_a\neg K_a\varphi} (\neg \Rightarrow)}{\Rightarrow x : \neg K_a\varphi \rightarrow K_a\neg K_a\varphi} (\Rightarrow\rightarrow)$$

3 Labelled Sequent Calculus for PAL

In this section, we introduce a labelled sequent calculus for PAL. The Hilbert-style axiomatization for PAL is the extension of that for EL with reduction axioms. Can we obtain a labelled sequent calculus for PAL by adding some rules adapted from reduction axioms to the labelled sequent calculus G_{EL} for EL? The answer is yes and this is what we do.

Take reduction axiom $(\mathrm{R1})[\varphi]p \leftrightarrow (\varphi \rightarrow p)$ as an example. The equivalence symbol in the axiom means that $[\varphi]p$ and $\varphi \rightarrow p$ are equivalent in PAL. Therefore, the most direct sequent rules for (R1) are

$$\frac{x : \varphi \rightarrow p, \Gamma \Rightarrow \Delta}{x : [\varphi]p, \Gamma \Rightarrow \Delta} \quad \text{and} \quad \frac{\Gamma \Rightarrow \Delta, x : \varphi \rightarrow p}{\Gamma \Rightarrow \Delta, x : [\varphi]p}$$

The rule on the left is sound because of $[\varphi]p \rightarrow (\varphi \rightarrow p)$ and the rule on the right is sound because of $(\varphi \rightarrow p) \rightarrow [\varphi]p$. These rules can be written in a more neat way if we see $\varphi \rightarrow p, \Gamma \Rightarrow \Delta$ as the conclusion of an application of $(\rightarrow\Rightarrow)$ and $\Gamma \Rightarrow \Delta, \varphi \rightarrow p$ as the conclusion of an application of $(\Rightarrow\rightarrow)$. Applying the reverse of $(\rightarrow\Rightarrow)$ and $(\Rightarrow\rightarrow)$ to their premises, we have the rules for (R1) that will be added to G_{EL}:

$$\frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : p, \Gamma \Rightarrow \Delta}{x : [\varphi]p, \Gamma \Rightarrow \Delta} \quad \text{and} \quad \frac{x : \varphi, \Gamma \Rightarrow \Delta, p}{\Gamma \Rightarrow \Delta, x : [\varphi]p}$$

In a similar way, we can have sequent rules for other reduction axioms. Therefore, we have:

Definition 2. Labelled sequent calculus G_{PAL} for PAL is G_{EL} plus the following sequent rules:

$$(R1\Rightarrow) \ \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : p, \Gamma \Rightarrow \Delta}{x : [\varphi]p, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow R1) \ \frac{x : \varphi, \Gamma \Rightarrow \Delta, x : p}{\Gamma \Rightarrow \Delta, x : [\varphi]p}$$

$$(R2\Rightarrow) \ \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : \neg[\varphi]\psi, \Gamma \Rightarrow \Delta}{x : [\varphi]\neg\psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow R2) \ \frac{x : \varphi, \Gamma \Rightarrow \Delta, x : \neg[\varphi]\psi}{\Gamma \Rightarrow \Delta, x : [\varphi]\neg\psi}$$

$$(R3\Rightarrow) \ \frac{x : [\varphi]\psi_1, x : [\varphi]\psi_2, \Gamma \Rightarrow \Delta}{x : [\varphi](\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow R3) \ \frac{\Gamma \Rightarrow \Delta, x : [\varphi]\psi_1 \quad \Gamma \Rightarrow \Delta, x : [\varphi]\psi_2}{\Gamma \Rightarrow \Delta, x : [\varphi](\psi_1 \wedge \psi_2)}$$

$(R4 \Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, x : [\varphi]\psi_1 \quad x : [\varphi]\psi_2, \Gamma \Rightarrow \Delta}{x : [\varphi](\psi_1 \rightarrow \psi_2), \Gamma \Rightarrow \Delta}$

$(\Rightarrow R4)$ $\dfrac{x : [\varphi]\psi_1, \Gamma \Rightarrow \Delta, x : [\varphi]\psi_2}{\Gamma \Rightarrow \Delta, [\varphi](\psi_1 \rightarrow \psi_2)}$

$(R5 \Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : K_a[\varphi]\psi, \Gamma \Rightarrow \Delta}{x : [\varphi]K_a\psi, \Gamma \Rightarrow \Delta}$ $\quad (\Rightarrow R5)$ $\dfrac{x : \varphi, \Gamma \Rightarrow \Delta, x : K_a[\varphi]\psi}{\Gamma \Rightarrow \Delta, x : [\varphi]K_a\psi}$

$(R6 \Rightarrow)$ $\dfrac{x : [\varphi \wedge [\varphi]\psi]\chi, \Gamma \Rightarrow \Delta}{x : [\varphi][\psi]\chi, \Gamma \Rightarrow \Delta}$ $\quad (\Rightarrow R6)$ $\dfrac{\Gamma \Rightarrow \Delta, x : [\varphi \wedge [\varphi]\psi]\chi}{\Gamma \Rightarrow \Delta, x : [\varphi][\psi]\chi}$

We call these sequent rules the *reduction rules*.

There are six pairs of reduction rules in G_{PAL}, each pair dealing with a kind of announcement formulas. Each left rule introduces a formula on the left of $\Rightarrow$, and each right rule introduces one on the right of $\Rightarrow$. Another desirable property for sequent rules is that the complexity of each premise should be less than that of the conclusion. If we define the complexity of a sequent to be the the sum of relational atoms and labelled formulas occurring in it, then the definition of formula complexity that counts the number of connectives will make (R5) and (R6) fail to satisfy the complexity increasing property. The following definition for formula complexity can solve this problem[2]:

Definition 3. Let φ be an $\mathcal{L}_{\mathrm{PAL}}$ formula, The *complexity* $c(\varphi)$ of φ is defined as follows:

$$c(p) = 1 \qquad\qquad\qquad c(\varphi \rightarrow \psi) = 1 + \max\{c(\varphi), c(\psi)\}$$
$$c(\neg\varphi) = 1 + c(\varphi) \qquad\quad c(K_a\varphi) = 1 + c(\varphi)$$
$$c(\varphi \wedge \psi) = 1 + \max\{c(\varphi), c(\psi)\} \quad c([\varphi]\psi) = (4 + c(\varphi)) \cdot c(\psi).$$

Then we have the following lemma[3]:

Lemma 1. *For all $\mathcal{L}_{\mathrm{PAL}}$-formulas φ, ψ and χ:*

(1) $c([\varphi]p) > c(\varphi \rightarrow p)$;
(2) $c([\varphi]\neg\psi) > c(\varphi \rightarrow \neg[\varphi]\psi)$;
(3) $c([\varphi](\psi \wedge \chi)) > c([\varphi]\psi \wedge [\varphi]\chi)$;
(4) $c([\varphi]K_a\psi) > c(\varphi \rightarrow K_a[\varphi]\psi)$;
(5) $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$.

Lemma 2. *For any $\mathcal{L}_{\mathrm{PAL}}$-formula φ, $G_{\mathrm{PAL}} \vdash x : \varphi, \Gamma \Rightarrow \Delta, x : \varphi$.*

Proof. We prove this by induction on the structure of φ with a subinduction on ψ of the inductive case where φ equals $[\varphi]\psi$. All inductive case not involving an-

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[2]This is Definition 7.21 in [7].

[3]This is Lemma 7.22 in [7].

nouncement are the same as in the proof of Proposition 1. When φ involves a public announcement operator, there are 6 subcases. We show two representative cases.

When $\varphi = [\phi]K_a\psi$, the derivation is as follows:

$$\cfrac{\cfrac{x:\phi,\Gamma \Rightarrow \Delta, x:K_a[\phi]\psi, x:\phi \qquad x:K_a[\phi]\psi, x:\phi, \Gamma \Rightarrow \Delta, x:K_a[\phi]\psi}{\cfrac{x:\phi, x:[\phi]K_a\psi, \Gamma \Rightarrow \Delta, x:K_a[\phi]\psi}{x:[\phi]K_a\psi, \Gamma \Rightarrow \Delta, x:[\phi]K_a\psi}\ (\Rightarrow R5)}\ (R5 \Rightarrow)}$$

When $\varphi = [\phi][\chi]\psi$, the derivation is as follows:

$$\cfrac{\cfrac{x:[\phi \wedge [\chi]\psi], \Gamma \Rightarrow \Delta, x:[\phi \wedge [\phi]\chi]\psi}{x:[\phi][\chi]\psi, \Gamma \Rightarrow \Delta, x:[\phi \wedge [\phi]\chi]\psi}\ (R6 \Rightarrow)}{x:[\phi][\chi]\psi, \Gamma \Rightarrow \Delta, x:[\phi][\chi]\psi}\ (\Rightarrow R6)}$$

Other cases can be proved analogously.   □

Example 2. Now we show that $(R5)[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$ is derivable in G_{PAL}. A derivation for $[\varphi]K_a\psi \rightarrow (\varphi \rightarrow K_a[\varphi]\psi)$ in G_{PAL} is as follows:

$$\cfrac{y:[\varphi]\psi, x:K_a[\varphi]\psi, x \sim_a y, x:\varphi \Rightarrow y:[\varphi]\psi}{x:K_a[\varphi]\psi, x \sim_a y, x:\varphi \Rightarrow y:[\varphi]\psi}\ (K_a \Rightarrow)$$

and then

$$\cfrac{\cfrac{\cfrac{\cfrac{x \sim_a y, x:\varphi \Rightarrow y:[\varphi]\psi, x:\varphi \qquad x:K_a[\varphi]\psi, x \sim_a y, x:\varphi \Rightarrow y:[\varphi]\psi}{x \sim_a y, x:[\varphi]K_a\psi, x:\varphi \Rightarrow y:[\varphi]\psi}\ (R5 \Rightarrow)}{x:[\varphi]K_a\psi, x:\varphi \Rightarrow x:K_a[\varphi]\psi}\ (\Rightarrow K_a)}{x:[\varphi]K_a\psi \Rightarrow x:\varphi \rightarrow K_a[\varphi]\psi}\ (\Rightarrow\rightarrow)}{\Rightarrow x:[\varphi]K_a\psi \rightarrow (\varphi \rightarrow K_a[\varphi]\psi)}\ (\Rightarrow\rightarrow)}$$

A derivation for $(\varphi \rightarrow K_a[\varphi]\psi) \rightarrow [\varphi]K_a\psi$ in G_{PAL} is as follows:

$$\cfrac{\cfrac{\cfrac{x:\varphi \Rightarrow x:K_a[\varphi]\psi, x:\varphi \qquad x:K_a[\varphi]\psi, x:\varphi \Rightarrow x:K_a[\varphi]\psi}{x:\varphi, x:\varphi \rightarrow K_a[\varphi]\psi \Rightarrow x:K_a[\varphi]\psi}\ (\rightarrow\Rightarrow)}{x:\varphi \rightarrow K_a[\varphi]\psi \Rightarrow x:[\varphi]K_a\psi}\ (\Rightarrow R5)}{\Rightarrow x:(\varphi \rightarrow K_a[\varphi]\psi) \rightarrow [\varphi]K_a\psi}\ (\Rightarrow\rightarrow)}$$

Example 3. $[p \wedge \neg K_a p]\neg K_a p$ is not derivable in G_{PAL}.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{D}_0}{x \sim_a y, x:K_a[p \wedge \neg K_a p]p, x:p \Rightarrow y:p}}{x:K_a[p \wedge \neg K_a p]p, x:p \Rightarrow x:K_a p}\ (\Rightarrow K_a) \qquad x:p \Rightarrow x:p \wedge \neg K_a p, x:K_a p}{x:[p \wedge \neg K_a p]K_a p, x:p \Rightarrow x:K_a p}}{\cfrac{x:[p \wedge \neg K_a p]K_a p, x:p, x:\neg K_a p \Rightarrow}{x:p \wedge \neg K_a p \Rightarrow x:\neg[p \wedge \neg K_a p]K_a p}\ (\wedge \Rightarrow, \Rightarrow \neg)}\ (\neg \Rightarrow)}{\Rightarrow x:[p \wedge \neg K_a p]\neg K_a p}$$

where $\mathcal{D}_0$ is:

$$\dfrac{x \sim_a y, x{:}p \Rightarrow y{:}p, y{:}p \qquad \dfrac{\dfrac{y{:}K_a p, x \sim_a y, x{:}p \Rightarrow y{:}p}{x \sim_a y, x{:}p \Rightarrow y{:}p, y{:}\neg K_a p}}{x \sim_a y, x{:}p \Rightarrow y{:}p, y{:}p \wedge \neg K_a p} \qquad y{:}p, x \sim_a y, x{:}p \Rightarrow y{:}p}{\dfrac{\dfrac{y{:}[p \wedge \neg K_a p]p, x \sim_a y, x{:}p \Rightarrow y{:}p}{}}{x \sim_a y, x{:}K_a[p \wedge \neg K_a p]p, x{:}p \Rightarrow y{:}p}}$$

4 Admissibility of Some Structural Rules

In light of the reduction axioms, we can define a translation from $\mathcal{L}_{\mathrm{PAL}}$-formulas to $\mathcal{L}_{\mathrm{EL}}$-formulas.[4]

Definition 4. The translation $t : \mathcal{L}_{\mathrm{PAL}} \to \mathcal{L}_{\mathrm{EL}}$ is defined as follows:

$$
\begin{aligned}
t(p) &= p \\
t(\neg\varphi) &= \neg t(\varphi) \\
t(\varphi \wedge \psi) &= t(\varphi) \wedge t(\psi) \\
t(\varphi \to \psi) &= t(\varphi) \to t(\psi) \\
t(K_a\varphi) &= K_a t(\varphi)
\end{aligned}
\qquad
\begin{aligned}
t([\varphi]p) &= t(\varphi \to p) \\
t([\varphi]\neg\psi) &= t(\varphi \to \neg[\varphi]\psi) \\
t([\varphi](\psi \wedge \chi)) &= t([\varphi]\psi \wedge [\varphi]\chi) \\
t([\varphi](\psi \to \chi)) &= t([\varphi]\psi \to [\varphi]\chi) \\
t([\varphi]K_a\psi) &= t(\varphi \to K_a[\varphi]\psi) \\
t([\varphi][\psi]\chi) &= t([\varphi \wedge [\varphi]\psi]\chi)
\end{aligned}
$$

Now we extend the translation t to relational atoms and labelled $\mathcal{L}_{\mathrm{PAL}}$-formulas: for any relational atom $x \sim_a y$, let $t(x \sim_a y) = x \sim_a y$; for any labelled $\mathcal{L}_{\mathrm{PAL}}$-formula $x : \varphi$, $t(x : \varphi) = x : t(\varphi)$. Moreover, for any set Γ of relational atoms and labelled formulas: $t(\Gamma) = \{t(\sigma) \mid \sigma \in \Gamma\}$.

Lemma 3. *For any $\mathcal{L}_{\mathrm{PAL}}$-sequent $\Gamma \Rightarrow \Delta$, the following hold:*

(1) *if $G_{\mathrm{PAL}} \vdash x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$, then $G_{\mathrm{PAL}} \vdash x : \varphi, t(\Gamma) \Rightarrow t(\Delta)$;*
(2) *if $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : t(\varphi)$, then $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : \varphi$.*

Proof. We prove these claims simultaneously by induction on the height of derivation h of $G_{\mathrm{PAL}} \vdash x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$ (or $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : t(\varphi)$).

If $h = 1$, then $x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$ (or $t(\Gamma) \Rightarrow t(\Delta), x : t(\varphi)$) is an initial sequent. If $x : t(\varphi)$ is principal, then $t(\varphi) = p$ for some proposition letter p. It follows that $\varphi = p$. Therefore, $x : \varphi, t(\Gamma) \Rightarrow t(\Delta)$ (or $t(\Gamma) \Rightarrow t(\Delta), x : \varphi$) is also an initial sequent. If $x : t(\varphi)$ is not principal, it is immediate that $x : \varphi, t(\Gamma) \Rightarrow t(\Delta)$ (or $t(\Gamma) \Rightarrow t(\Delta), x : \varphi$) is an initial sequent.

If $h > 1$, the induction hypothesis is formulated as:
($1'$) for all $i < h$, if $G_{\mathrm{PAL}} \vdash_i x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$,

---

[4]This is Definition 7.20 in [7].

then $G_{\mathrm{PAL}} \vdash_i x : \varphi, t(\Gamma) \Rightarrow t(\Delta)$;

$(2')$ for all $i < h$, if $G_{\mathrm{PAL}} \vdash_i t(\Gamma) \Rightarrow t(\Delta), x : t(\varphi)$,

then $G_{\mathrm{PAL}} \vdash_i t(\Gamma) \Rightarrow t(\Delta), x : \varphi$.

In what follows we only give the proof for claim $(1')$. The proof for the other is similar.

Assume that $G_{\mathrm{PAL}} \vdash_h x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$. Then there exists a derivation $\mathcal{D}$ for $x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$ in G_{PAL}. Let the last rule applied in $\mathcal{D}$ be $\mathcal{R}$. If $x : t(\varphi)$ is not principal in the application of $\mathcal{R}$, the desired result can be obtained by applying the induction hypothesis to the premise(s) of $\mathcal{R}$ and then applying $\mathcal{R}$.

If $x : t(\varphi)$ is principal in the application of $\mathcal{R}$, we prove by an sub-induction on the complexity $c(\varphi)$ of φ. Since $x : t(\varphi)$ is principal and $h > 1$, φ is not a proposition letter. We have ten sub-cases. We divide them into two groups depending on whether φ starts with an announcement operator or not.

If φ does not start with an announcement operator, the desired result can be obtained by applying the induction hypothesis to the premise(s) of $\mathcal{R}$ and then applying $\mathcal{R}$. There are four sub-cases: φ is of the form $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \rightarrow \psi_2$ or $K_a\psi$. We illustrate this by the cases $\neg\psi$ and $K_a\psi$.

(1)   If $\varphi = \neg\psi$, then $\mathcal{R}$ is $(\neg\Rightarrow)$. Note that $t(\varphi) = t(\neg\psi) = \neg t(\psi)$. Let the derivation $\mathcal{D}$ end with

$$\frac{t(\Gamma) \Rightarrow t(\Delta), x : t(\psi)}{x : \neg t(\psi), t(\Gamma) \Rightarrow t(\Delta)} \, (\neg\Rightarrow)$$

By the induction hypothesis, we have $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : \psi$. Then by $(\neg\Rightarrow)$ we have $G_{\mathrm{PAL}} \vdash x : \neg\psi, t(\Gamma) \Rightarrow t(\Delta)$.

(2)   If $\varphi = K_a\psi$, then $\mathcal{R}$ is $(K_a \Rightarrow)$. Note that $t(\varphi) = t(K_a\psi) = K_a t(\psi)$. Let the derivation $\mathcal{D}$ end with

$$\frac{y : t(\psi), x : K_a t(\psi), x \sim_a y, t(\Gamma) \Rightarrow t(\Delta)}{x : K_a t(\psi), x \sim_a y, t(\Gamma) \Rightarrow t(\Delta)} \, (K_a \Rightarrow)$$

First apply the main induction hypothesis to $x : K_a t(\psi)$ and we have $G_{\mathrm{PAL}} \vdash y : t(\psi), x : K_a\psi, x \sim_a y, t(\Gamma) \Rightarrow t(\Delta)$. Then apply the sub-induction hypothesis to $y : t(\psi)$ and we have $G_{\mathrm{PAL}} \vdash y : \psi, x : K_a\psi, x \sim_a y, t(\Gamma) \Rightarrow t(\Delta)$. Finally by $(K_a \Rightarrow)$ we have $G_{\mathrm{PAL}} \vdash x : K_a\psi, x \sim_a y, t(\Gamma) \Rightarrow t(\Delta)$.

If φ starts with an announcement operator, then there are six sub-cases: φ is $[\phi]p$, $[\phi]\neg\psi$, $[\phi](\psi_1 \wedge \psi_2)$, $[\phi](\psi_1 \rightarrow \psi_2)$, $[\phi]K_a\psi$ or $[\phi][\psi]\chi$.

If φ is $[\phi]p$, $[\phi]\neg\psi$, $[\phi](\psi_1 \rightarrow \psi_2)$ or $[\phi]K_a\psi$, then $t(\varphi)$ is $t(\phi) \rightarrow p$, $t(\phi) \rightarrow t(\neg[\phi]\psi)$, $t([\phi]\psi_1) \rightarrow t([\phi]\psi_2)$, or $t(\phi) \rightarrow t(K_a[\phi]\psi)$, respectively. Since $x : t(\varphi)$ is principal, $\mathcal{R}$ must be $(\rightarrow\Rightarrow)$. We substitute the application of $(\rightarrow\Rightarrow)$ with an application of $(R1 \Rightarrow)$, $(R2 \Rightarrow)$, $(R4 \Rightarrow)$ and $(R5 \Rightarrow)$, respectively. We illustrate the proof by the case where φ is $[\phi]\neg\psi$ and the case where φ is $[\phi]K_a\psi$.

(1)   If φ is $[\phi]\neg\psi$, then the derivation $\mathcal{D}$ ends with

$$\frac{t(\Gamma) \Rightarrow t(\Delta), x : t(\phi) \qquad x : t(\neg[\phi]\psi), t(\Gamma) \Rightarrow t(\Delta)}{x : t(\phi) \rightarrow t(\neg[\phi]\psi), t(\Gamma) \Rightarrow t(\Delta)} \ (\rightarrow\Rightarrow)$$

Apply the induction hypothesis to the premises and we have $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : \phi$ and $G_{\mathrm{PAL}} \vdash x : \neg[\phi]\psi, t(\Gamma) \Rightarrow t(\Delta)$. Then by $(R2 \Rightarrow)$ we have $G_{\mathrm{PAL}} \vdash x : [\phi]\neg\psi, t(\Gamma) \Rightarrow t(\Delta)$.

(2)   If φ is $[\phi]K_a\psi$, then the derivation $\mathcal{D}$ ends with

$$\frac{t(\Gamma) \Rightarrow t(\Delta), x : t(\phi) \qquad x : t(K_a[\phi]\psi), t(\Gamma) \Rightarrow t(\Delta)}{x : t(\phi) \rightarrow t(K_a[\phi]\psi), t(\Gamma) \Rightarrow t(\Delta)} \ (\rightarrow\Rightarrow)$$

Apply the induction hypothesis to the premises and we have $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : \phi$ and $G_{\mathrm{PAL}} \vdash x : K_a[\phi]\psi, t(\Gamma) \Rightarrow t(\Delta)$. Then by $(R5 \Rightarrow)$ we have $G_{\mathrm{PAL}} \vdash x : [\phi]K_a\psi, t(\Gamma) \Rightarrow t(\Delta)$.

If φ is $[\phi](\psi_1 \wedge \psi_2)$, then $t(\varphi)$ is $t([\phi]\psi_1) \wedge t([\phi]\psi_2)$. Since $x : t(\varphi)$ is principal, $\mathcal{R}$ must be $(\wedge \Rightarrow)$. We substitute the application of $(\wedge \Rightarrow)$ with an application of $(R3 \Rightarrow)$. Let the derivation $\mathcal{D}$ end with

$$\frac{x : t([\phi]\psi_1), x : t([\phi]\psi_2), t(\Gamma) \Rightarrow t(\Delta)}{x : t([\phi]\psi_1) \wedge t([\phi]\psi_2), t(\Gamma) \Rightarrow t(\Delta)} \ (\wedge \Rightarrow)$$

Apply the sub-induction hypothesis to the premise twice and we have $G_{\mathrm{PAL}} \vdash x : [\phi]\psi_1, x : [\phi]\psi_2, t(\Gamma) \Rightarrow t(\Delta)$. Then by $(R3 \Rightarrow)$ we have $G_{\mathrm{PAL}} \vdash x : [\phi](\psi_1 \wedge \psi_2), t(\Gamma) \Rightarrow t(\Delta)$.

If φ is $[\phi][\psi]\chi$, by assumption, $G_{\mathrm{PAL}} \vdash x : t([\phi][\psi]\chi), t(\Gamma) \Rightarrow t(\Delta)$. Since $t([\phi][\psi]\chi) = t([\phi \wedge [\phi]\psi]\chi)$, $G_{\mathrm{PAL}} \vdash x : t([\phi \wedge [\phi]\psi]\chi), t(\Gamma) \Rightarrow t(\Delta)$. Since $c([\phi \wedge [\phi]\psi]\chi) < c([\phi][\psi]\chi)$, by the sub-induction hypothesis, we have $G_{\mathrm{PAL}} \vdash x : [\phi \wedge [\phi]\psi]\chi, t(\Gamma) \Rightarrow t(\Delta)$. Then by $(R6 \Rightarrow)$, $G_{\mathrm{PAL}} \vdash x : [\phi][\psi]\chi, t(\Gamma) \Rightarrow t(\Delta)$.

This completes the proof. $\square$

The following theorem is a bridge between G_{PAL} and G_{EL}, enabling us to prove properties of G_{PAL} through G_{EL}.

Theorem 4. *For any $\mathcal{L}_{\mathrm{PAL}}$-sequent $\Gamma \Rightarrow \Delta$,*

(1) if $G_{\mathrm{EL}} \vdash t(\Gamma) \Rightarrow t(\Delta)$, then $G_{\mathrm{PAL}} \vdash \Gamma \Rightarrow \Delta$;
(2) if $G_{\mathrm{PAL}} \vdash_h \Gamma \Rightarrow \Delta$, then $G_{\mathrm{EL}} \vdash_h t(\Gamma) \Rightarrow t(\Delta)$.

Proof.   (1) Assume that $G_{\mathrm{EL}} \vdash t(\Gamma) \Rightarrow t(\Delta)$. Since G_{PAL} is an extension of G_{EL}, $G_{\mathrm{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta)$. Since $t(\Gamma) \Rightarrow t(\Delta)$ is finite, applying Lemma 3 a finite number of times, we have $G_{\mathrm{PAL}} \vdash \Gamma \Rightarrow \Delta$.

(2) The proof is by induction on h.

If $h = 1$, then $\Gamma \Rightarrow \Delta$ is an initial sequent in G_{PAL}. By definition, it is also an initial sequent in G_{EL}.

If $h > 1$, we consider the last rule $\mathcal{R}$ applied in the derivation. If $\mathcal{R}$ is not a reduction rule, the claim can be proved by first applying the induction hypothesis to the premise(s) and then applying $\mathcal{R}$.

If $\mathcal{R}$ is a reduction rule for (R1), (R2), (R4) or (R5), we apply the induction hypothesis to the premise(s), and then apply $(\rightarrow\Rightarrow)$ or $(\Rightarrow\rightarrow)$. We illustrate this by a few cases.

If $\mathcal{R}$ is $(R1 \Rightarrow)$, let the derivation end with:

$$\frac{\Gamma' \Rightarrow \Delta, x : \varphi \qquad x : p, \Gamma' \Rightarrow \Delta}{x : [\varphi]p, \Gamma' \Rightarrow \Delta} \ (R1 \Rightarrow)$$

By the induction hypothesis, we have $G_{\text{EL}} \vdash_{h-1} t(\Gamma') \Rightarrow t(\Delta), x : t(\varphi)$ and $G_{\text{EL}} \vdash_{h-1} x : t(p), t(\Gamma') \Rightarrow t(\Delta)$. We proceed as follows:

$$\frac{t(\Gamma') \Rightarrow t(\Delta), x : t(\varphi) \qquad x : t(p), t(\Gamma') \Rightarrow t(\Delta)}{x : t(\varphi) \rightarrow t(p), t(\Gamma') \Rightarrow t(\Delta)} \ (\rightarrow\Rightarrow)$$

Since $t(x : [\varphi]p) = x : t(\varphi) \rightarrow t(p)$, we have

$$G_{\text{EL}} \vdash_h t(x : [\varphi]p), t(\Gamma') \Rightarrow t(\Delta).$$

If $\mathcal{R}$ is $(\Rightarrow R2)$, let the derivation end with:

$$\frac{x : \varphi, \Gamma \Rightarrow \Delta', x : \neg[\varphi]\psi}{\Gamma \Rightarrow \Delta', x : [\varphi]\neg\psi} \ (\Rightarrow R2)$$

By the induction hypothesis, we have $G_{\text{EL}} \vdash_{h-1} x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta'), x : t(\neg[\varphi]\psi)$. We proceed as follows:

$$\frac{x : t(\varphi), t(\Gamma) \Rightarrow t(\Delta'), x : t(\neg[\varphi]\psi)}{t(\Gamma) \Rightarrow t(\Delta'), x : t(\varphi) \rightarrow t(\neg[\varphi]\psi)} \ (\Rightarrow\rightarrow)$$

Since $t(x : [\varphi]\neg\psi) = x : t(\varphi) \rightarrow t(\neg[\varphi]\psi)$, we have

$$G_{\text{EL}} \vdash_h t(\Gamma) \Rightarrow t(\Delta'), t(x : [\varphi]\neg p).$$

If $\mathcal{R}$ is $(R5 \Rightarrow)$, let the derivation end with:

$$\frac{\Gamma' \Rightarrow \Delta, x : \varphi \qquad x : K_a[\varphi]\psi, \Gamma' \Rightarrow \Delta}{x : [\varphi]K_a\psi, \Gamma' \Rightarrow \Delta} \ (R5 \Rightarrow)$$

By the induction hypothesis, we have $G_{\text{EL}} \vdash_{h-1} t(\Gamma') \Rightarrow t(\Delta), x : t(\varphi)$ and $G_{\text{EL}} \vdash_{h-1} x : t(K_a[\varphi]\psi), t(\Gamma') \Rightarrow t(\Delta)$. We proceed as follows:

$$\frac{t(\Gamma') \Rightarrow t(\Delta), x : t(\varphi) \qquad x : t(K_a[\varphi]\psi), t(\Gamma') \Rightarrow t(\Delta)}{x : t(\varphi) \rightarrow t(K_a[\varphi]\psi), t(\Gamma') \Rightarrow t(\Delta)} \ (\rightarrow\Rightarrow)$$

Since $t(x : [\varphi]K_a\psi) = x : t(\varphi) \to t(K_a[\varphi]\psi)$, we have

$$G_{\mathrm{EL}} \vdash_h t(x : [\varphi]K_a\psi), t(\Gamma') \Rightarrow t(\Delta).$$

If $\mathcal{R}$ is a reduction rule for (R3), we apply the induction hypothesis to the premise(s), and then apply an inference rule for $\wedge$. We illustrate this by the case where $\mathcal{R}$ is $(R3 \Rightarrow)$.

If $\mathcal{R}$ is $(R3 \Rightarrow)$, let the derivation end with:

$$\frac{x : [\varphi]\psi_1, x : [\varphi]\psi_2, \Gamma' \Rightarrow \Delta}{x : [\varphi](\psi_1 \wedge \psi_2), \Gamma' \Rightarrow \Delta} \ (R3 \Rightarrow)$$

By induction hypothesis, we have $G_{\mathrm{EL}} \vdash_{h-1} x : t([\varphi]\psi_1), x : t([\varphi]\psi_2), t(\Gamma') \Rightarrow t(\Delta)$. We proceed as follows:

$$\frac{x : t([\varphi]\psi_1), x : t([\varphi]\psi_2), t(\Gamma') \Rightarrow t(\Delta)}{x : t([\varphi]\psi_1) \wedge t([\varphi]\psi_2), t(\Gamma') \Rightarrow t(\Delta)} \ (\wedge \Rightarrow)$$

Since $t(x : [\varphi](\psi_1 \wedge \psi_2)) = x : t([\varphi]\psi_1) \wedge t([\varphi]\psi_2)$, we have

$$G_{\mathrm{EL}} \vdash_h t(x : [\varphi](\psi_1 \wedge \psi_2)), t(\Gamma') \Rightarrow t(\Delta).$$

If $\mathcal{R}$ is a reduction rule for (R6), then we simply apply the induction hypothesis to the premise.

This completes the proof. $\qquad\qquad\square$

Remark 2. Since PAL is more succinct than EL (see [8, 11]), one might expect that a derivation of a sequent $\Gamma \Rightarrow \Delta$ in G_{PAL} is strictly shorter (with respect to derivation height) than a derivation of $t(\Gamma) \Rightarrow t(\Delta)$ in G_{EL}. This is not the case for G_{PAL} and G_{EL} as shown in Theorem 4. This is because a derivation for $\Gamma \Rightarrow \Delta$ in G_{PAL} is obtained by executing the translation function t (encoded by the reduction rules) in a derivation for $t(\Gamma) \Rightarrow t(\Delta)$ in G_{EL}, which increases the length of the derivation.

Corollary 1. *The following structural rules are admissible in G_{PAL}:*

$$(w \Rightarrow) \frac{\Gamma \Rightarrow \Delta}{x : \varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow w) \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \varphi}$$

$$(c \Rightarrow) \frac{x : \varphi, x : \varphi, \Gamma \Rightarrow \Delta}{x : \varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow c) \frac{\Gamma \Rightarrow \Delta, x : \varphi, x : \varphi}{\Gamma \Rightarrow \Delta, x : \varphi}$$

$$(c_R \Rightarrow) \frac{x \sim_a y, x \sim_a y, \Gamma \Rightarrow \Delta}{x \sim_a y, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow c_R) \frac{\Gamma \Rightarrow \Delta, x \sim_a y, x \sim_a y}{\Gamma \Rightarrow \Delta, x \sim_a y}$$

$$(Cut) \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof.   It follows directly from Theorem 2 and Theorem 4. Take (Cut) as an example. Assume that $G_{\mathrm{PAL}} \vdash \Gamma \Rightarrow \Delta, x : \varphi$ and $x : \varphi, \Gamma' \Rightarrow \Delta'$. By Theorem 4, $G_{\mathrm{EL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x : t(\varphi)$ and $G_{\mathrm{EL}} \vdash x : t(\varphi), t(\Gamma') \Rightarrow t(\Delta')$. Then by admissibility of cut in G_{EL} (Theorem 2), $G_{\mathrm{EL}} \vdash t(\Gamma), t(\Gamma') \Rightarrow t(\Delta), t(\Delta')$. By Theorem 4 again, $G_{\mathrm{PAL}} \vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.   □

Theorem 5 (Soundness and Completeness). *For any $\mathcal{L}_{\mathrm{PAL}}$-formula φ, $\varphi \in \mathrm{PAL}$ iff $G_{\mathrm{PAL}} \vdash \Rightarrow \varphi$.*

Proof.   For the left-to-right direction, it suffices to show that each axiom in the Hilbert-style axiomatization for PAL is derivable in G_{PAL} and that (MP) and (GK$_a$) are admissible in G_{PAL}. Since G_{PAL} is an extension of G_{EL}, derivability of axioms in EL. Admissibility of (MP) follows from admissibility of cut in G_{PAL} (Corollary 1). Admissibility of (GK$_a$) in G_{PAL} follows from admissibility of (GK$_a$) in G_{EL} and Theorem 4. For the reduction axioms, since each reduction axiom has a corresponding pair of sequent rules in G_{PAL}, their derivations can be obtained by direct root-first search. One derivation of (R5) is given in Example 2.

For the right-to-left direction, it suffices to show that each sequent rule in G_{PAL} is valid. This is routine. We omit the proof here.   □

5   Decidability

In this section we show that G_{PAL} allows terminating proof search. This result can be proved indirectly by Theorems 3 and 4. In this section we present a direct proof.

Readers familiar with the proof for terminating proof search in G_{EL} may notice that the proof for terminating proof search in G_{PAL} is essentially the same as that in G_{EL} because G_{PAL} is an extension of G_{EL} with some reduction rules. For this reason, we sketch the proof in this section and refer to [14] for a detailed description.

To show that G_{PAL} allows terminating proof search, first we extend the notion of *subformulas* for $\mathcal{L}_{\mathrm{PAL}}$-formulas, because active formulas are not subformulas of the principal formula in the usual sense in the reduction rules of G_{PAL}.

Definition 5.   Let φ be an $\mathcal{L}_{\mathrm{PAL}}$-formula. Formula ψ is called a *semi-subformula* of φ if one of the following conditions hold:

(1) ψ is a subformula of φ;
(2) $\varphi = [\phi] * \chi$ and $\psi = *[\phi]\chi$, where $* \in \{\neg, K_a\}$;
(3) $\varphi = [\phi](\chi_1 \star \chi_2)$ and $\psi = [\phi]\chi_1$ or $\psi = [\phi]\chi_2$, where $\star \in \{\wedge, \rightarrow\}$;
(4) $\varphi = [\phi][\chi]\xi$ and $\psi = [\phi \wedge [\phi]\chi]\xi$.

The set of semi-subformulas of φ is denoted by $SSub(\varphi)$. We say that ψ is a *proper semi-subformula* of φ if ψ is a semi-subformula of φ and $\psi \neq \varphi$.

By Definition 3, the complexity of a proper semi-subformula of φ is strictly smaller than that of φ.

Definition 6. A derivation in G_{PAL} satisfies the *weak subformula property* if all expressions in the derivation are either semi-subformulas of formulas φ in labelled formulas $x : \varphi$ in the endsequent of the derivation, or relational atoms of the form $x \sim_a y$.

It is easy to verify that each derivation in G_{PAL} satisfies the weak subformula property.

The potential sources of non-terminating proof search in G_{PAL} are as follows:

(1) $(Trans_a)$, (Sym_a) (read root-first) can be applied infinitely on the same principal formulas.
(2) (Ref_a) (read root-first) can be applied infinitely to introduce new relational atoms.
(3) $(K_a \Rightarrow)$ (read root-first) can be applied infinitely on the same principal formulas.
(4) By $(Trans_a)$ and its iteration with $(\Rightarrow K_a)$ that brings in new accessible worlds, we can build chains of accessible worlds on which $(K_a \Rightarrow)$ can be applied infinitely.

To show that the space of root-first proof search is finite, it is useful to look at *minimal derivations*, that is, derivations where shortenings are not possible.

For (1), since no rules, read root first, can change a relational atom, any application of $(Trans_a)$ or (Sym_a) on the same principal formulas will make the derivation fail to be minimal. Therefore, this case should be excluded when searching for minimal derivation.

For (2), we need the following lemma:

Lemma 4. *All variables in relational atoms of the form $x \sim_a x$ removed by (Ref_a) in a minimal derivation of a sequent $\Gamma \Rightarrow \Delta$ in G_{PAL} are variables in Γ or Δ.*

Proof. It can be proved by tracing the relational atom $x \sim_a x$ up in the derivation. For a detailed proof, see Lemma 6.2 of [14]. □

For (3), we need to show the following lemma:

Lemma 5. *Rule $(K_a \Rightarrow)$ permutes down with respect to $(\Rightarrow K_a)$ in case the principal relational atom of $(K_a \Rightarrow)$ is not active in $(\Rightarrow K_a)$. Moreover, rule $(K_a \Rightarrow)$ permutes down with respect to all other rules.*

Then we can prove the following proposition:

Proposition 6. $(K_a \Rightarrow)$ *and* $(\Rightarrow K_a)$ *cannot be applied more than once on the same pair of principal formulas on any branch in any derivation in* G_{PAL}.

For (4) we need first some definitions. *Positive part* and *negative part* of a sequent $\Gamma \Rightarrow \Delta$ are defined as the positive part and negative part in the formula $\bigwedge \Gamma \to \bigvee \Delta$ respectively. An occurrence of a subformula in a formula is a positive part if it is in the scope of an even number of negation signs; it is a negative part if it is in the scope of an odd number of negation signs (to apply this definition, every occurrence of implication $\varphi \to \psi$ in subformula should be rewritten as $\neg\varphi \vee \psi$). Then we are ready to prove the following proposition:

Proposition 7. *In a minimal derivation of a sequent in* G_{PAL}, *for each formula* $x :$ $K_a\varphi$ *in its positive part there are at most* $n(K_a)$ *applications of* $(\Rightarrow K_a)$ *iterated on a chain of accessible worlds* $x \sim_a x_1, x_1 \sim_a x_2, \ldots$ *with principal formula* $x_i : K_a\varphi$, *where* $n(K_a)$ *denotes the number of* K_a *in the negative part of* $\Gamma \Rightarrow \Delta$.

6   Conclusion, Comparison and Future Research

In this paper we proposed a labelled sequent calculus for PAL. It is based on a proof system for EL, namely poly-modal S5, extended with sequent rules to deal with announcement operators that directly mirror the PAL reduction rules. We also determined that the obtained system allows terminating proof search, and compared our system with the calculus for EL on matters such a height preservation of derivations.

Various proposals for sequent calculi for PAL have been made in [2, 3, 13, 16]. The labelled sequent calculi in [2, 16] for PAL are based on a reformulation of the semantics for PAL. The notion of a model restriction to a (single) formula is generalized in these works to that of a model restriction to a list of formulas. Denote by α or β finite lists $(\varphi_1, \ldots, \varphi_n)$ of formulas, and by ϵ the empty list. For any list $\alpha = (\varphi_1, \ldots, \varphi_n)$ of formulas, define $\mathcal{M}^\alpha$ recursively as follows: $\mathcal{M}^\alpha := \mathcal{M}$ (if $\alpha = \epsilon$), and $\mathcal{M}^\alpha := (\mathcal{M}^\beta)^{\varphi_n} = (W^{\beta,\varphi_n}, (\sim_a^{\beta,\varphi_n})_{a\in\mathrm{Ag}}, V^{\beta,\varphi_n})$ (if $\alpha = \beta, \varphi_n$).

Then an equivalent semantics for PAL is defined as follows:

$$
\begin{array}{lll}
\mathcal{M}^{\alpha,\varphi}, w \Vdash p & \text{iff} & \mathcal{M}^\alpha, w \Vdash \varphi \text{ and } \mathcal{M}^\alpha, w \Vdash p \\
\mathcal{M}^\alpha, w \Vdash \neg\varphi & \text{iff} & \mathcal{M}^\alpha, w \nVdash \varphi \\
\mathcal{M}^\alpha, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}^\alpha, w \Vdash \varphi \text{ and } \mathcal{M}^\alpha, w \Vdash \psi \\
\mathcal{M}^\alpha, w \Vdash \varphi \to \psi & \text{iff} & \mathcal{M}^\alpha, w \nVdash \varphi \text{ or } \mathcal{M}^\alpha, w \Vdash \psi \\
\mathcal{M}^\alpha, w \Vdash K_a\varphi & \text{iff} & \text{for all } v \in W, w \sim_a^\alpha v \text{ implies } \mathcal{M}^\alpha, v \Vdash \varphi \\
\mathcal{M}^\alpha, w \Vdash [\varphi]\psi & \text{iff} & \mathcal{M}^\alpha, w \Vdash \varphi \text{ implies } \mathcal{M}^{\alpha,\varphi}, w \Vdash \psi
\end{array}
$$

With this semantics, Balbiani ([2]) and Nomura et al. ([16])[5] developed different labelled sequent calculi for PAL, simultaneously repairing some perceived deficiencies in the previously published in Maffezioli and Negri ([13]). The calculus in [2]

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[5]Nomura et al. ([15]) extended the method developed in [16] to action modal logic.

admits cut and allows terminating proof search, and the calculus in [16] also admits cut. Note that these calculi, unlike ours, are for PAL based on the normal modal logic K, whereas we took PAL as an extension of S5 (like [9, 13]). We therefore also include the usual inference rules for S5 (i.e., (Ref_a), $(Trans_a)$ and (Sym_a)) into our calculus. Further, our calculus uses sequent rules based on PAL reduction axioms to deal with the announcement operators.

The method we proposed can be directly applied to action model logic ([7]) since this logic is also an extension of EL with some reduction axioms, of which PAL is a special case.

There are many extensions of other logics with public announcement that also involve reduction axioms for such announcements, such as intuitionistic PAL ([12]), bilattice PAL ([18]), Łukasiewicz PAL ([6]), and variations/extensions of action model logic like bilattice logic of epistemic actions and knowledge ([1]). We can try to develop calculi for these logics with the steps similar to those for PAL.

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公开宣告逻辑的一个加标矢列演算

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摘　要

公开宣告逻辑（PAL）是在认知逻辑（EL）中添加归约公理得到的拓展。本文为 PAL 提出一个无切割定理的加标矢列演算系统，该系统是在 EL 的加标矢列演算系统上添加归约公理对应的矢列规则得到的拓展，且切割可允准、支持停机搜索。

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