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Algorithmic Correspondence Theory for Sabotage Modal Logic*

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Abstract. Sabotage modal logic (SML) is a kind of dynamic logic. It extends static modal logic with a dynamic modality which is interpreted as "after deleting an arrow in the frame, the formula is true". In the present paper, we are aiming at solving an open problem stated in Aucher, van Benthem and Grossi (2018), namely giving a Sahlqvist-type correspondence theorem (Sahlqvist 1975) for sabotage modal logic. In this paper, we define sabotage Sahlqvist formulas and give an algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ to compute the first-order correspondents of sabotage Sahlqvist formulas. We give some remarks and future directions at the end of the paper.

1 Introduction

Sabotage modal logic (SML, [6]) belongs to the class of logics collectively called dynamic logics. It extends static modal logic with a dynamic modality $\blacklozenge$ such that $\blacklozenge\varphi$ is interpreted as "after deleting an arrow in the frame, φ is true". There are several existing works on sabotage modal logic. In [5], a bisimulation characterization theorem as well as a tableau system were given for sabotage modal logic, [13] proved the undecidability of the satisfiability problem and gave the complexity of the model-checking problem, and [14] gave the complexity of solving sabotage game. Several similar formalisms are also investigated, such as graph modifiers logic ([4]), swap logic ([1]) and arrow update logic ([11]), modal logic of definable link deletion ([12]), modal logic of stepwise removal ([17]). These logics are collectively called relation changing modal logics.([2, 3]) In the present paper, we are aiming at solving an open problem stated in [5], namely giving a Sahlqvist-type correspondence theorem ([16]) for sabotage modal logic. We define the Sahlqvist formulas in the sabotage modal language and give the sabotage counterpart of the algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ (Ackermann Lemma Based Algorithm), which is sound over Kripke frames and is successful on sabotage Sahlqvist formulas, to show that every Sahlqvist formula in the sabotage modal language has a first-order correspondent.

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The structure of the paper is as follows: in Section 2, we give a brief sketch on the preliminaries of sabotage modal logic, including its syntax, semantics as well as the standard translation. In Section 3–8 we define sabotage Sahlqvist formulas and the algorithm $\mathrm{ALBA}^{\mathrm{SML}}$, show its soundness over Kripke frames and its success on sabotage Sahlqvist inequalities. In Section 9 we give some further directions.

2 Preliminaries on Sabotage Modal Logic

In this section, we collect some preliminaries on sabotage modal logic. For further details, see [5].

Given a set Prop of propositional variables, the set of sabotage modal formulas is recursively defined as follows:

$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \blacklozenge\varphi$$

where $p \in \mathrm{Prop}$, $\Diamond$ is the alethic connective of ordinary modal logic and $\blacklozenge$ is the sabotage connective of sabotage modal logic. We will follow the standard rules for omission of the parentheses. $\vee, \rightarrow, \leftrightarrow, \Box, \blacksquare$ can be defined in the standard way. We call a formula *static* if it does not contain $\blacklozenge$ or $\blacksquare$. An occurence of p is said to be *positive* (resp. *negative*) in φ if p is under the scope of an even (resp. odd) number of negations in the original sabotage modal language. A formula φ is positive (resp. negative) if all occurences of propositional variables in φ are positive (resp. negative).

For the semantics of sabotage modal logic, we use Kripke frames $\mathbb{F} = (W, R)$ and Kripke models $\mathbb{M} = (\mathbb{F}, V)$ where $V : \mathrm{Prop} \rightarrow \mathcal{P}(W)$. The satisfaction relation is defined as follows:

$(W, R, V), w \nVdash \bot$	:	always;
$(W, R, V), w \Vdash \top$	:	always;
$(W, R, V), w \Vdash p$	iff	$w \in V(p)$;
$(W, R, V), w \Vdash \neg\varphi$	iff	$(W, R, V), w \nVdash \varphi$;
$(W, R, V), w \Vdash \varphi \wedge \psi$	iff	$(W, R, V), w \Vdash \varphi$ and $(W, R, V), w \Vdash \psi$;
$(W, R, V), w \Vdash \Diamond\varphi$	iff	there exists a $v \in W$, such that $(w, v) \in R$ and $(W, R, V), v \Vdash \varphi$;
$(W, R, V), w \Vdash \blacklozenge\varphi$	iff	there exists $(w_0, w_1) \in R$, such that $(W, R \setminus \{(w_0, w_1)\}, V), w \Vdash \varphi$.

Intuitively, $\blacklozenge\varphi$ is true at w iff there is an edge (w_0, w_1) of R such that after deleting this edge from R, the formula φ is still true at w. It is easy to see that the semantic clause for $\blacksquare$ is defined as follows:

$(W, R, V), w \Vdash \blacksquare\varphi$ iff for all edges $(w_0, w_1) \in R$, $(W, R \setminus \{(w_0, w_1)\}, V), w \Vdash \varphi$.

The standard translation of sabotage modal language into first-order logic is given as follows (notice that we need to record the edges already deleted from R so that we know what edges could still be deleted):

Definition 1 ([5, Def. 1]). Let E be a set of pairs (y, z) of variables standing for edges and let x be a designated variable. The translation ST_x^E is recursively defined as follows:

- $ST_x^E(p) := Px$;
- $ST_x^E(\bot) := x \neq x$;
- $ST_x^E(\neg\varphi) := \neg ST_x^E(\varphi)$;
- $ST_x^E(\varphi \wedge \psi) := ST_x^E(\varphi) \wedge ST_x^E(\psi)$;
- $ST_x^E(\Diamond\varphi) := \exists y(Rxy \wedge (\bigwedge_{(v,w)\in E} \neg(x = v \wedge y = w)) \wedge ST_y^E(\varphi))$;
- $ST_x^E(\blacklozenge\varphi) := \exists y \exists z(Ryz \wedge (\bigwedge_{(v,w)\in E} \neg(y = v \wedge z = w)) \wedge ST_x^{E\cup\{(y,z)\}}(\varphi))$.

It is proved in [5, Thm. 1] that this translation is correct:

Theorem 1. *For any pointed model* $(\mathbb{M}, w)$ *and sabotage modal formula* φ*,*

$$(\mathbb{M}, w) \Vdash \varphi \quad \textit{iff} \quad \mathbb{M} \vDash ST_x^{\varnothing}(\varphi)[w].$$

3 Algorithmic Correspondence for Sabotage Modal Logic: A Sketch

We will develop the correspondence algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ for sabotage modal logic, in the style of [8, 9]. The basic idea is to use an algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ to transform the modal formula $\varphi(\vec{p})$ into an equivalent set of pure quasi-(universally quantified) inequalities which does not contain occurrences of propositional variables, and therefore can be translated into the first-order correspondence language via the standard translation of the expanded language of sabotage modal logic (which will be defined on Section 4).

The key ingredients of the algorithmic correspondence proof can be listed as follows:

- An expanded sabotage modal language as the syntax of the algorithm, as well as their interpretations in the relational semantics;
- An algorithm ALBA which transforms a given sabotage modal formula $\varphi(\vec{p})$ into equivalent pure quasi-(universally quantified) inequalities $\mathrm{Pure}(\varphi(\vec{p}))$;
- A soundness proof of the algorithm;
- A syntactically identified class of formulas on which the algorithm is successful;
- A first-order correspondence language and first-order translation which transform pure quasi-(universally quantified) inequalities into their equivalent first-order correspondents.

In the remainder of the paper, we will define an expanded sabotage modal language which the algorithm will manipulate (Section 4.1), define the first-order correspondence language of the expanded sabotage modal language and the standard translation (Section 4.2). We report on the definition of Sahlqvist inequalities (Section 5), define a modified version of the algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ (Section 6), and show its soundness (Section 7) and success on Sahlqvist inequalities (Section 8).

4 The Expanded Language in the Algorithm

4.1 The expanded sabotage modal language $\mathcal{L}^+_{\blacksquare}$

In the present subsection, we give the definition of the expanded sabotage modal language $\mathcal{L}^+_{\blacksquare}$ and its standard translations, which will be used in the execution of the algorithm:

$$\varphi ::= p \mid \bot \mid \top \mid \mathbf{i} \mid \neg\varphi \mid (\varphi\wedge\varphi) \mid (\varphi\vee\varphi) \mid (\varphi\to\varphi) \mid \Box\varphi \mid \Diamond\varphi \mid \blacksquare\varphi \mid \blacklozenge\varphi \mid$$

$$\Box^S\varphi \mid \Diamond^S\varphi \mid (\Box^S)^{-1}\varphi \mid (\Diamond^S)^{-1}\varphi \mid \mathrm{A}\varphi \mid \mathrm{E}\varphi \mid \forall\mathbf{i}\varphi \mid \exists\mathbf{i}\varphi$$

$$S ::= \varnothing \mid S\cup\{(\mathbf{i}_{k0},\mathbf{i}_{k1})\}$$

where $\mathbf{i}$ is called nominal as in hybrid logic ([7, Ch. 14]), $\mathbf{i}_{k0},\mathbf{i}_{k1}$ are fresh nominals not in S. We use the notation $\varphi(\vec{p})$ to indicate that the propositional variables occurring in φ are all in $\vec{p}$. We call a formula *pure* if it does not contain propositional variables.

When interpreting the formulas in the expanded language, we assume that we start from a given pointed model $((W,R_0,V),w)$, and use S to record the edges deleted from R_0. $\Box^S$, $\Diamond^S$ correspond to the relation $R_0\setminus\{(V(\mathbf{i}_{k0}),V(\mathbf{i}_{k1}))\mid(\mathbf{i}_{k0},\mathbf{i}_{k1})\in S\}$ (denoted as $R_0\setminus S$), $(\Box^S)^{-1}$, $(\Diamond^S)^{-1}$ correspond to the relation $R_0^{-1}\setminus\{(V(\mathbf{i}_{k1}),V(\mathbf{i}_{k0}))\mid(\mathbf{i}_{k0},\mathbf{i}_{k1})\in S\}$ (denoted as $(R_0\setminus S)^{-1}$), which intuitively means first delete the edges in S and then take the inverse relation. Unlabelled $\Box$ and $\Diamond$ correspond to the relation R after certain deletions of edges. Therefore, we can say that $\Box^S$, $\Diamond^S$, $(\Box^S)^{-1}$, $(\Diamond^S)^{-1}$ are "absolute connectives" whose interpretations just depend on R_0 and S, while $\Box$ and $\Diamond$ are "contextual connectives" whose interpretations depend on the concrete R after certain steps of deletions. For $\blacksquare$ and $\blacklozenge$, their interpretations depend on the context. A and E are global box and diamond modalities respectively, $(W,R,V),w\Vdash\forall\mathbf{i}\varphi$ indicates that for all valuation variant $V^{\mathbf{i}}_v$ such that $V^{\mathbf{i}}_v$ is the same as V except that $V^{\mathbf{i}}_v(\mathbf{i})=\{v\}$, $(W,R,V^{\mathbf{i}}_v),w\Vdash\varphi$, and $(W,R,V),w\Vdash\exists\mathbf{i}\varphi$ is the corresponding existential statement.

For the semantics of the expanded sabotage modal language, the valuation is defined as $V:\mathrm{Prop}\cup\mathrm{Nom}\to\mathcal{P}(W)$ similar to hybrid logic, and for the modal and dynamic connectives, the additional semantic clauses can be given as follows (notice that here R is the "actual" accessibility relation in the model $\mathbb{M}=(W,R,V)$ after some (maybe none) edges have been deleted, while R_0 is the "starting accessibility" relation when no edge has been deleted yet, and $R_0 \backslash S$ is the notation for $R_0 \backslash \{(V(\mathbf{i}_{k0}), V(\mathbf{i}_{k1})) \mid (\mathbf{i}_{k0}, \mathbf{i}_{k1}) \in S\}$):

$\mathbb{M}, w \Vdash \Box\varphi$	iff	for all v s.t. $(w, v) \in R$, $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash \Diamond\varphi$	iff	there exists a v s.t. $(w, v) \in R$ and $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacksquare\varphi$	iff	for all edges $(w_0, w_1) \in R$, $(W, R \setminus \{(w_0, w_1)\}, V), w \Vdash \varphi$
$\mathbb{M}, w \Vdash \blacklozenge\varphi$	iff	there exists $(w_0, w_1) \in R$ s.t. $(W, R \setminus \{(w_0, w_1)\}, V), w \Vdash \varphi$
$\mathbb{M}, w \Vdash \Box^S\varphi$	iff	for all v s.t. $(w, v) \in (R_0 \setminus S)$, $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash \Diamond^S\varphi$	iff	there exists a v s.t. $(w, v) \in (R_0 \setminus S)$ and $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash (\Box^S)^{-1}\varphi$	iff	for all v s.t. $(v, w) \in (R_0 \setminus S)$, $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash (\Diamond^S)^{-1}\varphi$	iff	there exists a v s.t. $(v, w) \in (R_0 \setminus S)$ and $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash \mathrm{A}\varphi$	iff	for all $v \in W$, $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash \mathrm{E}\varphi$	iff	there exists a $v \in W$ s.t. $(W, R, V), v \Vdash \varphi$
$\mathbb{M}, w \Vdash \forall\mathbf{i}\varphi$	iff	for all $v \in W$, $(W, R, V^{\mathbf{i}}_v), w \Vdash \varphi$
$\mathbb{M}, w \Vdash \exists\mathbf{i}\varphi$	iff	there exists a $v \in W$ s.t. $(W, R, V^{\mathbf{i}}_v), w \Vdash \varphi$.

Here we do not require that each pair of nominals in S denote different edges in R_0. For the convenience of the algorithm, we introduce the following definitions:

Definition 2.

- An *inequality* is of the form $\varphi \leq^S_{S'} \psi$, where φ and ψ are formulas, S and S' record the context of φ and ψ respectively, i.e. which edges have already been deleted. Its interpretation is given as follows:

$$(W, R_0, V) \Vdash \varphi \leq^S_{S'} \psi$$

iff for all $w \in W$, if $(W, R_0 \backslash S, V), w \Vdash \varphi$, then $(W, R_0 \backslash S', V), w \Vdash \psi$.

We use $\varphi \leq \psi$ to denote $\varphi \leq^{\varnothing}_{\varnothing} \psi$.

- A *quasi-inequality* is of the form $\varphi_1 \leq^{S_1}_{S'_1} \psi_1 \ \& \ \cdots \ \& \ \varphi_n \leq^{S_n}_{S'_n} \psi_n \Rightarrow \varphi \leq^S_{S'} \psi$. Its interpretation is given as follows:

$$(W, R_0, V) \Vdash \varphi_1 \leq^{S_1}_{S'_1} \psi_1 \ \& \ \cdots \ \& \ \varphi_n \leq^{S_n}_{S'_n} \psi_n \Rightarrow \varphi \leq^S_{S'} \psi$$

iff $(W, R_0, V) \Vdash \varphi \leq^S_{S'} \psi$ holds,
whenever $(W, R_0, V) \Vdash \varphi_i \leq^{S_i}_{S'_i} \psi_i$ for all $1 \leq i \leq n$.

- A *Mega-inequality* is defined inductively as follows:

$$\mathrm{Mega} ::= \mathrm{Ineq} \mid \mathrm{Mega} \,\&\, \mathrm{Mega} \mid \forall \mathbf{i} \forall \mathbf{j} (\mathbf{i} \leq^S_S \Diamond^S \mathbf{j} \Rightarrow \mathrm{Mega})$$

  where Ineq is an inequality, & is the meta-conjunction and $\Rightarrow$ is the meta-implication. Its interpretation is given as follows:

  - $(W, R_0, V) \Vdash$ Ineq iff the inequality holds as defined in the definition above;
  - $(W, R_0, V) \Vdash \mathrm{Mega}_1 \,\&\, \mathrm{Mega}_2$ iff $(W, R_0, V) \Vdash \mathrm{Mega}_1$ and $(W, R_0, V) \Vdash \mathrm{Mega}_2$;
  - $(W, R_0, V) \Vdash \forall \mathbf{i} \forall \mathbf{j} (\mathbf{i} \leq^S_S \Diamond^S \mathbf{j} \Rightarrow \mathrm{Mega})$ iff for all w, v, if $(w, v) \in (R_0 \setminus S)$ then $(W, R_0, V^{\mathbf{i},\mathbf{j}}_{w,v}) \Vdash$ Mega, where $V^{\mathbf{i},\mathbf{j}}_{w,v}$ is the same as V except that $V^{\mathbf{i},\mathbf{j}}_{w,v}(\mathbf{i}) = \{w\}$, $V^{\mathbf{i},\mathbf{j}}_{w,v}(\mathbf{j}) = \{v\}$.

- A *universally quantified inequality* is defined as $\forall \mathbf{i}_1 \cdots \forall \mathbf{i}_n (\varphi \leq^S_{S'} \psi)$, and its interpretation is given as follows:

$$(W, R_0, V) \Vdash \forall \mathbf{i}_1 \cdots \forall \mathbf{i}_n (\varphi \leq^S_{S'} \psi)$$

  iff for all $w_1, \ldots, w_n \in W$, $(W, R_0, V^{\mathbf{i}_1,\ldots,\mathbf{i}_n}_{w_1,\ldots,w_n}) \Vdash \varphi \leq^S_{S'} \psi$,
  where $V^{\mathbf{i}_1,\ldots,\mathbf{i}_n}_{w_1,\ldots,w_n}$ is the same as V except that $V^{\mathbf{i}_1,\ldots,\mathbf{i}_n}_{w_1,\ldots,w_n}(\mathbf{i}_i) = \{w_i\}$, $i = 1, \ldots, n$.

- A *quasi-universally quantified inequality* is defined as $\mathrm{UQIneq}_1 \& \cdots \& \mathrm{UQIneq}_n \Rightarrow \mathrm{UQIneq}$ where UQIneq, UQIneq_i are universally quantified inequalities. Its interpretation is given as follows:

$$(W, R_0, V) \Vdash \mathrm{UQIneq}_1 \& \cdots \& \mathrm{UQIneq}_n \Rightarrow \mathrm{UQIneq}$$

  iff $(W, R_0, V) \Vdash$ UQIneq holds,
  whenever $(W, R_0, V) \Vdash \mathrm{UQIneq}_i$ for all $1 \leq i \leq n$.

It is easy to see that $(W, R_0, V) \Vdash \varphi \leq^{\varnothing}_{\varnothing} \psi$ iff $(W, R_0, V) \Vdash \varphi \to \psi$. We will find it easy to work with inequalities $\varphi \leq \psi$ in place of implicative formulas $\varphi \to \psi$ in Section 5.

For inequalities, we have the following properties, which will be useful in the soundness proofs:

Proposition 2.

- $(W, R_0, V) \Vdash \mathbf{i} \leq^S_S \Diamond^S \mathbf{j}$ *iff* $(V(\mathbf{i}), V(\mathbf{j})) \in (R_0 \setminus S)$*;*
- $(W, R_0, V) \Vdash \mathbf{i} \leq^S_{S'} \alpha$ *iff* $(W, (R_0 \setminus S'), V), V(\mathbf{i}) \Vdash \alpha$*, where* α *is a formula in the expanded sabotage modal language;*
- $(W, R_0, V) \Vdash A(\mathbf{i} \to \Diamond^S \mathbf{j})$ *iff* $(V(\mathbf{i}), V(\mathbf{j})) \in (R_0 \setminus S)$.

4.2 The first-order correspondence language and the standard translation

In the first-order correspondence language, we have a binary relation symbol R corresponding to the binary relation, a set of constant symbols i corresponding to each nominal $\mathbf{i}$, a set of unary predicate symbols P corresponding to each propositional variable p.

The standard translation of the expanded sabotage modal language is defined as follows:

Definition 3. Let E be a finite set of pairs (y, z) of variables standing for edges and let x be a designated variable. The translation ST_x^E is recursively defined as follows:

- $ST_x^E(\bot) := x \neq x$;
- $ST_x^E(\top) := x = x$;
- $ST_x^E(\mathbf{i}) := x = i$;
- $ST_x^E(p) := Px$;
- $ST_x^E(\neg\varphi) := \neg ST_x^E(\varphi)$;
- $ST_x^E(\varphi \wedge \psi) := ST_x^E(\varphi) \wedge ST_x^E(\psi)$;
- $ST_x^E(\varphi \vee \psi) := ST_x^E(\varphi) \vee ST_x^E(\psi)$;
- $ST_x^E(\varphi \to \psi) := ST_x^E(\varphi) \to ST_x^E(\psi)$;
- $ST_x^E(\Box\varphi) := \forall y(Rxy \wedge (\bigwedge_{(v,w)\in E} \neg(x = v \wedge y = w)) \to ST_y^E(\varphi))$;
- $ST_x^E(\Diamond\varphi) := \exists y(Rxy \wedge (\bigwedge_{(v,w)\in E} \neg(x = v \wedge y = w)) \wedge ST_y^E(\varphi))$;
- $ST_x^E(\blacksquare\varphi) := \forall y\forall z(Ryz \wedge (\bigwedge_{(v,w)\in E} \neg(y = v \wedge z = w)) \to ST_x^{E\cup\{(y,z)\}}(\varphi))$;
- $ST_x^E(\blacklozenge\varphi) := \exists y\exists z(Ryz \wedge (\bigwedge_{(v,w)\in E} \neg(y = v \wedge z = w)) \wedge ST_x^{E\cup\{(y,z)\}}(\varphi))$;
- $ST_x^E(\Box^S\varphi) := \forall y(Rxy \wedge (\bigwedge_{(\mathbf{i}_{k0},\mathbf{i}_{k1})\in S} \neg(x = i_{k0} \wedge y = i_{k1})) \to ST_y^E(\varphi))$;
- $ST_x^E(\Diamond^S\varphi) := \exists y(Rxy \wedge (\bigwedge_{(\mathbf{i}_{k0},\mathbf{i}_{k1})\in S} \neg(x = i_{k0} \wedge y = i_{k1})) \wedge ST_y^E(\varphi))$;
- $ST_x^E((\Box^S)^{-1}\varphi) := \forall y(Ryx \wedge (\bigwedge_{(\mathbf{i}_{k0},\mathbf{i}_{k1})\in S} \neg(y = i_{k0} \wedge x = i_{k1})) \to ST_y^E(\varphi))$;
- $ST_x^E((\Diamond^S)^{-1}\varphi) := \exists y(Ryx \wedge (\bigwedge_{(\mathbf{i}_{k0},\mathbf{i}_{k1})\in S} \neg(y = i_{k0} \wedge x = i_{k1})) \wedge ST_y^E(\varphi))$.
- $ST_x^E(\mathrm{A}\varphi) := \forall y ST_y^E(\varphi)$;
- $ST_x^E(\mathrm{E}\varphi) := \exists y ST_y^E(\varphi)$;
- $ST_x^E(\forall\mathbf{i}\varphi) := \forall i ST_x^E(\varphi)$;
- $ST_x^E(\exists\mathbf{i}\varphi) := \exists i ST_x^E(\varphi)$.

It is easy to see that this translation is correct:

Proposition 3. *For any pointed model* $(\mathbb{M}, w)$ *and sabotage modal formula* φ*,*

$$(\mathbb{M}, w) \Vdash \varphi \quad \textit{iff} \quad \mathbb{M} \vDash ST_x^{\varnothing}(\varphi)[w].$$

For inequalities, quasi-inequalities, mega-inequalities, universally quantified inequalities and quasi-universally quantified inequalities, the standard translation is given in a global way:

Definition 4.

- $ST(\varphi \leq^{S}_{S'} \psi) := \forall x(ST^{S}_{x}(\varphi) \rightarrow ST^{S'}_{x}(\psi))$;
- $ST(\varphi_1 \leq^{S_1}_{S'_1} \psi_1 \& \cdots \& \varphi_n \leq^{S_n}_{S'_n} \psi_n \Rightarrow \varphi \leq^{S}_{S'} \psi) :=$
$$ST(\varphi_1 \leq^{S_1}_{S'_1} \psi_1) \wedge \cdots \wedge ST(\varphi_n \leq^{S_n}_{S'_n} \psi_n) \rightarrow ST(\varphi \leq^{S}_{S'} \psi);$$
- $ST(\text{Mega}_1 \ \& \ \text{Mega}_2) := ST(\text{Mega}_1) \wedge ST(\text{Mega}_2)$;
- $ST(\forall \mathbf{i} \forall \mathbf{j}(\mathbf{i} \leq^{S}_{S} \Diamond^{S} \mathbf{j} \Rightarrow \text{Mega})) :=$
$$\forall i \forall j (Rij \wedge (\bigwedge_{(v,w) \in S} \neg(i = v \wedge j = w)) \rightarrow ST(\text{Mega}));$$
- $ST(\forall \mathbf{i}_1 \cdots \forall \mathbf{i}_n \text{Ineq}) := \forall i_1 \cdots \forall i_n ST(\text{Ineq})$;
- $ST(\text{UQIneq}_1 \& \cdots \& \text{UQIneq}_n \Rightarrow \text{UQIneq}) :=$
$$ST(\text{UQIneq}_1) \wedge \cdots \wedge ST(\text{UQIneq}_n) \rightarrow ST(\text{UQIneq}).$$

Proposition 4. *For any model* $\mathbb{M}$ *and inequality* Ineq*, quasi-inequality* Quasi*, mega-inequality* Mega*, universally quantified inequality* UQIneq*, quasi-universally quantified inequality* QUQIneq*,*

$$\begin{array}{rcl} \mathbb{M} \Vdash \text{Ineq} & \textit{iff} & \mathbb{M} \vDash ST(\text{Ineq}); \\ \mathbb{M} \Vdash \text{Quasi} & \textit{iff} & \mathbb{M} \vDash ST(\text{Quasi}); \\ \mathbb{M} \Vdash \text{Mega} & \textit{iff} & \mathbb{M} \vDash ST(\text{Mega}); \\ \mathbb{M} \Vdash \text{UQIneq} & \textit{iff} & \mathbb{M} \vDash ST(\text{UQIneq}); \\ \mathbb{M} \Vdash \text{QUQIneq} & \textit{iff} & \mathbb{M} \vDash ST(\text{QUQIneq}). \end{array}$$

5 Sahlqvist Inequalities

In the present section, since we will use the algorithm ALBA^{SML} which is based on the classsification of nodes in the signed generation trees of sabotage modal formulas, we will use the unified correspondence style definition ([10, 15]) to define Sahlqvist inequalities. We will collect all the necessary preliminaries on Sahlqvist formulas/inequalities.

Definition 5 (Order-type of propositional variables, [9, p. 346])**.** For an n-tuple $(p_1, \ldots, p_n)$ of propositional variables, an order-type ε of $(p_1, \ldots, p_n)$ is an element in $\{1, \partial\}^n$. In the order-type ε, we say that p_i has order-type 1 if $\varepsilon_i = 1$, and denote $\varepsilon(p_i) = 1$ or $\varepsilon(i) = 1$; we say that p_i has order-type ∂ if $\varepsilon_i = \partial$, and denote $\varepsilon(p_i) = \partial$ or $\varepsilon(i) = \partial$.

Definition 6 (Signed generation tree, [10, Def. 4]). The *positive* (resp. *negative*) *generation tree* of any given formula φ is defined by first labelling the root of the generation tree of φ with $+$ (resp. $-$) and then labelling the children nodes as follows:

- Assign the same sign to the children nodes of any node labelled with $\vee, \wedge, \Box$, $\Diamond$, $\blacksquare$, $\blacklozenge$, $\Box^S$, $\Diamond^S$, $(\Box^S)^{-1}$, $(\Diamond^S)^{-1}$, A, E, $\forall\mathbf{i}$, $\exists\mathbf{i}$;
- Assign the opposite sign to the child node of any node labelled with $\neg$;
- Assign the opposite sign to the first child node and the same sign to the second child node of any node labelled with $\rightarrow$.

Nodes in signed generation trees are *positive* (resp. *negative*) if they are signed $+$ (resp. $-$).

Example 1. The positive generation tree of $+\blacklozenge(p\wedge\neg\blacksquare q) \rightarrow \Box q$ is given in Figure 1.

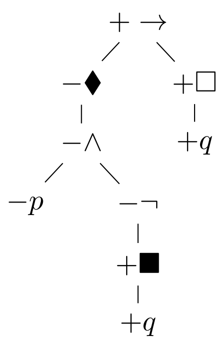

Figure 1: Positive generation tree for $\blacklozenge(p \wedge \neg\blacksquare q) \rightarrow \Box q$

Signed generation trees will be used in the inequalities $\varphi \leq \psi$, where the positive generation tree $+\varphi$ and the negative generation tree $-\psi$ will be considered. We will also say that an inequality $\varphi \leq \psi$ is *uniform* in a variable p_i if all occurrences of p_i in $+\varphi$ and $-\psi$ have the same sign, and that $\varphi \leq \psi$ is ε-*uniform* in an array $\vec{p}$ if $\varphi \leq \psi$ is uniform in p_i, occurring with the sign indicated by ε (i.e., p_i has the sign $+$ if $\varepsilon(p_i) = 1$, and has the sign $-$ if $\varepsilon(p_i) = \partial$), for each propositional variable p_i in $\vec{p}$.

For any given formula $\varphi(p_1, \ldots, p_n)$, any order-type ε over n, and any $1 \leq i \leq n$, an ε-*critical node* in a signed generation tree $*\varphi$ (where $* \in \{+, -\}$) is a leaf node $+p_i$ when $\varepsilon_i = 1$ or $-p_i$ when $\varepsilon_i = \partial$. An ε-*critical branch* in a signed generation tree is a branch from an ε-critical node. The ε-critical occurrences are intended to be those which the algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ will solve for. We say that $+\varphi$ (resp. $-\varphi$) *agrees with* ε, and write $\varepsilon(+\varphi)$ (resp. $\varepsilon(-\varphi)$), if every leaf node in the signed generation tree of $+\varphi$ (resp. $-\varphi$) is ε-critical.

We will also use the notation $+\psi \prec *\varphi$ (resp. $-\psi \prec *\varphi$) to indicate that an occurence of a subformula ψ inherits the positive (resp. negative) sign from the signed

Outer	Inner
$+$ $\vee$ $\wedge$ $\Diamond$ $\blacklozenge$ $\neg$	$+$ $\wedge$ $\Box$ $\blacksquare$ $\neg$
$-$ $\wedge$ $\vee$ $\Box$ $\blacksquare$ $\neg$ $\rightarrow$	$-$ $\vee$ $\Diamond$ $\blacklozenge$ $\neg$

Table 1: Outer and Inner nodes.

generation tree $*\varphi$, where $* \in \{+, -\}$. We will write $\varepsilon(\gamma) \prec *\varphi$ (resp. $\varepsilon^\partial(\gamma) \prec *\varphi$) to indicate that the signed generation subtree γ, with the sign inherited from $*\varphi$, agrees with ε (resp. with ε^∂). We say that a propositional variable p is *positive* (resp. *negative*) in φ if $+p \prec +\varphi$ (resp. $-p \prec +\varphi$).

Definition 7 ([10, Def. 5])**.** Nodes in signed generation trees are called *outer nodes* and *inner nodes*, according to Table 1.

A branch in a signed generation tree is called a *excellent branch* if it is the concatenation of two paths P_1 and P_2, one of which might be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) of inner nodes only, and P_2 consists (apart from variable nodes) of outer nodes only.

Definition 8 (Sahlqvist inequalities, [10, Def. 6])**.** For any order-type ε, the signed generation tree $*\varphi$ of a formula $\varphi(p_1, \ldots, p_n)$ is ε-*Sahlqvist* if for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is excellent. An inequality $\varphi \leq \psi$ is ε-*Sahlqvist* if the signed generation trees $+\varphi$ and $-\psi$ are ε-Sahlqvist. An inequality $\varphi \leq \psi$ is *Sahlqvist* if it is ε-Sahlqvist for some ε.

Example 2. Here we give an example of a Sahlqvist inequality for the order-type $\varepsilon = (1, 1)$, where the outer nodes are marked with O, and the inner nodes are marked with I, and ε-critical branches are ended with leaf nodes marked with C.

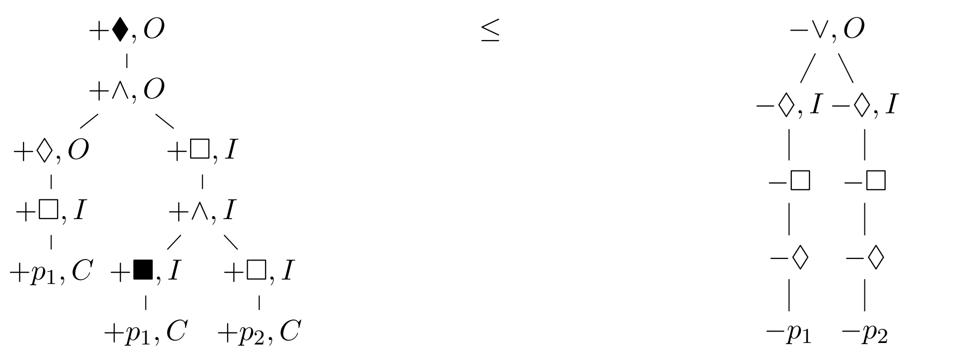

Figure 2: (1,1)-Sahlqvist inequality $\blacklozenge(\Diamond\Box p_1 \wedge \Box(\blacksquare p_1 \wedge \Box p_2)) \leq \Diamond\Box\Diamond p_1 \vee \Diamond\Box\Diamond p_2$

6 The Algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ for the Sabotage Modal Language

In the present section, we define the correspondence algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ for sabotage modal logic, in the style of [8, 9]. The algorithm goes in four steps.

1. **Preprocessing and first approximation**:
   In the generation tree of $+\varphi$ and $-\psi$[1],
   (a) Apply the distribution rules:
      i. Push down $+\Diamond, +\blacklozenge, -\neg, +\wedge, -\rightarrow$ by distributing them over nodes labelled with $+\vee$ which are outer nodes, and
      ii. Push down $-\Box, -\blacksquare, +\neg, -\vee, -\rightarrow$ by distributing them over nodes labelled with $-\wedge$ which are outer nodes.

   (b) Apply the splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma}\ (\wedge\text{-Spl.-1}) \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}\ (\vee\text{-Spl.-1})$$

   (c) Apply the monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)}\ (\text{Mon.}) \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}\ (\text{Ant.})$$

   for $\beta(p)$ positive in p and $\alpha(p)$ negative in p.

   We denote by $\mathrm{Preprocess}(\varphi \leq \psi)$ the finite set $\{\varphi_i \leq \psi_i\}_{i \in I}$ of inequalities obtained after the exhaustive application of the previous rules. Then we apply the following rule to every inequality in $\mathrm{Preprocess}(\varphi \leq \psi)$:

$$\frac{\varphi_i \leq \psi_i}{\mathbf{i}_0 \leq \varphi_i \quad \psi_i \leq \neg \mathbf{i}_1}\ (\text{First-Appr.})$$

   Here, $\mathbf{i}_0$ and $\mathbf{i}_1$ are special fresh nominals. Now we get a set of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}_{i \in I}$.
2. **The reduction stage**:
   In this stage, for each $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg \mathbf{i}_1\}$, we first add superscripts and subscripts $\varnothing$ to the two $\leq$s, and then apply the following rules to prepare for eliminating all the proposition variables in $\{\mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i, \psi_i \leq^{\varnothing}_{\varnothing} \neg \mathbf{i}_1\}$:
   (a) **Substage 1: decomposing the outer part**
   In the current substage, the following rules are applied to decompose the outer part of the Sahlqvist signed formula:

[1] The discussion below relies on the definition of signed generation tree in Section 5. In what follows, we identify a formula with its signed generation tree.

i. Splitting rules:

$$\frac{\alpha \leq^{S}_{S'} \beta \wedge \gamma}{\alpha \leq^{S}_{S'} \beta \quad \alpha \leq^{S}_{S'} \gamma} \ (\wedge\text{-Spl.-2}) \qquad \frac{\alpha \vee \beta \leq^{S}_{S'} \gamma}{\alpha \leq^{S}_{S'} \gamma \quad \beta \leq^{S}_{S'} \gamma} \ (\vee\text{-Spl.-2})$$

ii. Approximation rules:

$$\frac{\mathbf{i} \leq^{S}_{S'} \Diamond\alpha}{\mathbf{j} \leq^{S'}_{S'} \alpha \quad \mathbf{i} \leq^{S}_{S'} \Diamond^{S'}\mathbf{j}} \ (\Diamond\text{-Appr.}) \qquad \frac{\Box\alpha \leq^{S}_{S'} \neg\mathbf{i}}{\alpha \leq^{S}_{S} \neg\mathbf{j} \quad \Box^{S}\neg\mathbf{j} \leq^{S}_{S'} \neg\mathbf{i}} \ (\Box\text{-Appr.})$$

$$\frac{\mathbf{i} \leq^{S}_{S'} \blacklozenge\alpha}{\mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'}\mathbf{i}_{m1} \qquad \mathbf{i} \leq^{S}_{S'\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}} \alpha} \ (\blacklozenge\text{-Appr.})$$

$$\frac{\blacksquare\alpha \leq^{S}_{S'} \neg\mathbf{i}}{\mathbf{i}_{m0} \leq^{S}_{S} \Diamond^{S}\mathbf{i}_{m1} \qquad \alpha \leq^{S\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}}_{S'} \neg\mathbf{i}} \ (\blacksquare\text{-Appr.})$$

$$\frac{\alpha \rightarrow \beta \leq^{S}_{S'} \neg\mathbf{i}}{\mathbf{j} \leq^{S}_{S} \alpha \qquad \beta \leq^{S}_{S} \neg\mathbf{k} \qquad \mathbf{j} \rightarrow \neg\mathbf{k} \leq^{S}_{S'} \neg\mathbf{i}} \ (\rightarrow\text{-Appr.})$$

The nominals introduced by the approximation rules must not occur in the system before applying the rule.

iii. Residuation rules:

$$\frac{\mathbf{i} \leq^{S}_{S'} \neg\alpha}{\alpha \leq^{S'}_{S} \neg\mathbf{i}} \ (\text{Nom-}\neg\text{-Res.}) \qquad \frac{\neg\alpha \leq^{S}_{S'} \neg\mathbf{i}}{\mathbf{i} \leq^{S'}_{S} \alpha} \ (\text{CoNom-}\neg\text{-Res.})$$

(b) **Substage 2: decomposing the inner part**

In the current substage, the following rules are applied to decompose the inner part of the Sahlqvist signed formula:

i. Splitting rules:

$$\frac{\alpha \leq^{S}_{S'} \beta \wedge \gamma}{\alpha \leq^{S}_{S'} \beta \quad \alpha \leq^{S}_{S'} \gamma} \ (\wedge\text{-Spl.-2}) \qquad \frac{\alpha \vee \beta \leq^{S}_{S'} \gamma}{\alpha \leq^{S}_{S'} \gamma \quad \beta \leq^{S}_{S'} \gamma} \ (\vee\text{-Spl.-2})$$

ii. Residuation rules:

$$\frac{\alpha \leq^{S}_{S'} \neg\beta}{\beta \leq^{S'}_{S} \neg\alpha} \ (\neg\text{-Res.-1}) \qquad \frac{\neg\alpha \leq^{S}_{S'} \beta}{\neg\beta \leq^{S'}_{S} \alpha} \ (\neg\text{-Res.-2})$$

$$\frac{\Diamond\alpha \leq^{S}_{S'} \beta}{\alpha \leq^{S}_{S'} (\Box^{S})^{-1}\beta} \ (\Diamond\text{-Res.}) \qquad \frac{\alpha \leq^{S}_{S'} \Box\beta}{(\Diamond^{S'})^{-1}\alpha \leq^{S}_{S'} \beta} \ (\Box\text{-Res.})$$

$$\frac{\alpha \leq^{S}_{S'} \blacksquare\beta}{\forall \mathbf{i}_{m0}\forall \mathbf{i}_{m1}(\mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'}\mathbf{i}_{m1} \ \Rightarrow\ \alpha \leq^{S}_{S'\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}} \beta)}\ (\blacksquare\text{-Res.})$$

$$\frac{\blacklozenge\alpha \leq^{S}_{S'} \beta}{\forall \mathbf{i}_{m0}\forall \mathbf{i}_{m1}(\mathbf{i}_{m0} \leq^{S}_{S} \Diamond^{S}\mathbf{i}_{m1} \ \Rightarrow\ \alpha \leq^{S\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}}_{S'} \beta)}\ (\blacklozenge\text{-Res.})$$

The nominals introduced by the residuation rules must not occur in the system before applying the rule.

iii. Second splitting rules (Second-Spl.):

$$\frac{\forall \mathbf{i}_{m0}\forall \mathbf{i}_{m1}(\mathbf{i}_{m0} \leq^{S}_{S} \Diamond^{S}\mathbf{i}_{m1} \Rightarrow \ \mathrm{Mega}_1 \,\&\, \mathrm{Mega}_2)}{\forall \mathbf{i}_{m0}\forall \mathbf{i}_{m1}(\mathbf{i}_{m0} \leq^{S}_{S} \Diamond^{S}\mathbf{i}_{m1} \Rightarrow \mathrm{Mega}_1) \quad \forall \mathbf{i}_{m0}\forall \mathbf{i}_{m1}(\mathbf{i}_{m0} \leq^{S}_{S} \Diamond^{S}\mathbf{i}_{m1} \Rightarrow \mathrm{Mega}_2)}$$

Here Mega_1 and Mega_2 denote mega-inequalities.

(c) **Substage 3: preparing for the Ackermann rules**

In this substage, we prepare for eliminating the propositional variables by the Ackermann rules, with the help of the following packing rules:

Packing rules:

- Pack.-1:

$$\frac{\forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}(\mathbf{i}_{m_k0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k1} \Rightarrow \cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(\mathbf{i}_{m_00} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_01} \ \Rightarrow\ \alpha \leq^{S}_{S'} p)\cdots)}{\exists \mathbf{i}_{m_k0}\exists \mathbf{i}_{m_k1}\cdots \exists \mathbf{i}_{m_00}\exists \mathbf{i}_{m_01}(A(\mathbf{i}_{m_k0} \to \Diamond^{S_k}\mathbf{i}_{m_k1}) \wedge \cdots \wedge A(\mathbf{i}_{m_00} \to \Diamond^{S_0}\mathbf{i}_{m_01}) \wedge \alpha) \leq^{\varnothing}_{\varnothing} p}$$

  where α is pure and does not contain contextual connectives $\Box, \Diamond, \blacksquare, \blacklozenge$.

- Pack.-2:

$$\frac{\forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}(\mathbf{i}_{m_k0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k1} \Rightarrow \cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(\mathbf{i}_{m_00} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_01} \ \Rightarrow\ p \leq^{S}_{S'} \beta)\cdots)}{p \leq^{\varnothing}_{\varnothing} \forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}\cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(A(\mathbf{i}_{m_k0} \to \Diamond^{S_k}\mathbf{i}_{m_k1}) \wedge \cdots \wedge A(\mathbf{i}_{m_00} \to \Diamond^{S_0}\mathbf{i}_{m_01}) \to \beta)}$$

  where β is pure and does not contain contextual connectives $\Box, \Diamond, \blacksquare, \blacklozenge$.

- Pack.-3:

$$\frac{\forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}(\mathbf{i}_{m_k0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k1} \Rightarrow \cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(\mathbf{i}_{m_00} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_01} \ \Rightarrow\ \alpha \leq^{S}_{S'} \gamma)\cdots)}{\forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}\cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(\top \leq^{\varnothing}_{S'} A(\mathbf{i}_{m_k0} \to \Diamond^{S_k}\mathbf{i}_{m_k1}) \wedge \cdots \wedge A(\mathbf{i}_{m_00} \to \Diamond^{S_0}\mathbf{i}_{m_01}) \wedge \alpha \to \gamma)}$$

  where α is pure and does not contain contextual connectives $\Box, \Diamond, \blacksquare, \blacklozenge$.

- Pack.-4:

$$\frac{\forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}(\mathbf{i}_{m_k0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k1} \Rightarrow \cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(\mathbf{i}_{m_00} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_01} \ \Rightarrow\ \gamma \leq^{S}_{S'} \alpha)\cdots)}{\forall \mathbf{i}_{m_k0}\forall \mathbf{i}_{m_k1}\cdots \forall \mathbf{i}_{m_00}\forall \mathbf{i}_{m_01}(\top \leq^{\varnothing}_{S} A(\mathbf{i}_{m_k0} \to \Diamond^{S_k}\mathbf{i}_{m_k1}) \wedge \cdots \wedge A(\mathbf{i}_{m_00} \to \Diamond^{S_0}\mathbf{i}_{m_01}) \wedge \gamma \to \alpha)}$$

  where α is pure and does not contain contextual connectives $\Box, \Diamond, \blacksquare, \blacklozenge$.

(d) **Substage 4: the Ackermann stage**

In this substage, we compute the minimal/maximal valuation for propositional variables and use the Ackermann rules to eliminate all the propositional variables. These two rules are the core of ALBA, since their application eliminates proposition variables. In fact, all the preceding steps are aimed at reaching a shape in which the rules can be applied. Notice that an important feature of these rules is that they are executed on the whole set of (universally quantified) inequalities, and not on a single inequality.

The right-handed Ackermann rule:

The system $\begin{cases} \alpha_1 \leq^{\varnothing}_{\varnothing} p \\ \vdots \\ \alpha_n \leq^{\varnothing}_{\varnothing} p \\ \forall \vec{\mathbf{i}_1}(\beta_1 \leq^{T_1}_{T'_1} \gamma_1) \\ \vdots \\ \forall \vec{\mathbf{i}_m}(\beta_m \leq^{T_m}_{T'_m} \gamma_m) \end{cases}$ is replaced by:

$$\begin{cases} \forall \vec{\mathbf{i}_1}(\beta_1((\alpha_1 \vee \cdots \vee \alpha_n)/p) \leq^{T_1}_{T'_1} \gamma_1((\alpha_1 \vee \cdots \vee \alpha_n)/p)) \\ \vdots \\ \forall \vec{\mathbf{i}_m}(\beta_m((\alpha_1 \vee \cdots \vee \alpha_n)/p) \leq^{T_m}_{T'_m} \gamma_m((\alpha_1 \vee \cdots \vee \alpha_n)/p)) \end{cases}$$

where: (i). $p, \vec{\mathbf{i}_1}, \ldots, \vec{\mathbf{i}_m}$ do not occur in $\alpha_1, \ldots, \alpha_n$;

(ii). Each β_i is positive in p, and each γ_i negative in p, for $1 \leq i \leq m$;

(iii). Each α_i is pure and contains no contextual modalities $\Box, \Diamond, \blacksquare, \blacklozenge$.

The left-handed Ackermann rule:

The system $\begin{cases} p \leq^{\varnothing}_{\varnothing} \alpha_1 \\ \vdots \\ p \leq^{\varnothing}_{\varnothing} \alpha_n \\ \forall \vec{\mathbf{i}_1}(\beta_1 \leq^{T_1}_{T'_1} \gamma_1) \\ \vdots \\ \forall \vec{\mathbf{i}_m}(\beta_m \leq^{T_m}_{T'_m} \gamma_m) \end{cases}$ is replaced by:

$$\begin{cases} \forall \vec{\mathbf{i}_1}(\beta_1((\alpha_1 \wedge \cdots \wedge \alpha_n)/p) \leq^{T_1}_{T'_1} \gamma_1((\alpha_1 \wedge \cdots \wedge \alpha_n)/p)) \\ \vdots \\ \forall \vec{\mathbf{i}_m}(\beta_m((\alpha_1 \wedge \cdots \wedge \alpha_n)/p) \leq^{T_m}_{T'_m} \gamma_m((\alpha_1 \wedge \cdots \wedge \alpha_n)/p)) \end{cases}$$

where: (i). $p, \vec{\mathbf{i}_1}, \ldots, \vec{\mathbf{i}_m}$ do not occur in $\alpha_1, \ldots, \alpha_n$;

(ii). Each β_i is negative in p, and each γ_i positive in p, for $1 \leq i \leq m$.

(iii). Each α_i is pure and contains no contextual modalities $\Box, \Diamond, \blacksquare, \blacklozenge$.

3. **Output**: If in the previous stage, for some $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg\mathbf{i}_1\}$, the algorithm gets stuck, i.e. some proposition variables cannot be eliminated by the

application of the reduction rules, then the algorithm halts and output "failure". Otherwise, each initial tuple $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg\mathbf{i}_1\}$ of inequalities after the first approximation has been reduced to a set of pure (universally quantified) inequalities $\mathrm{Reduce}(\varphi_i \leq \psi_i)$, and then the output is a set of quasi-(universally quantified) inequalities $\{\&\mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1 : \varphi_i \leq \psi_i \in \mathrm{Preprocess}(\varphi \leq \psi)\}$, where & is the big meta-conjunction in quasi-inequalities. Then the algorithm use the standard translation to transform the quasi-(universally quantified) inequalities into first-order formulas.

7 Soundness of $\mathrm{ALBA}^{\mathrm{SML}}$

In the present section, we will prove the soundness of the algorithm $\mathrm{ALBA}^{\mathrm{SML}}$ with respect to Kripke frames. The basic proof structure is similar to [9, 18].

Theorem 5 (Soundness). *If* $\mathrm{ALBA}^{\mathrm{SML}}$ *runs successfully on* $\varphi \leq \psi$ *and outputs* $\mathrm{FO}(\varphi \leq \psi)$*, then for any Kripke frame* $\mathbb{F} = (W, R_0)$*,*

$$\mathbb{F} \Vdash \varphi \leq \psi \quad \textit{iff} \quad \mathbb{F} \models \mathrm{FO}(\varphi \leq \psi).$$

Proof. The proof goes similarly to [9, Thm. 8.1]. Let $\varphi_i \leq \psi_i, 1 \leq i \leq n$ denote the inequalities produced by preprocessing $\varphi \leq \psi$ after Stage 1, and $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \neg\mathbf{i}_1\}$ denote the inequalities after the first-approximation rule, $\mathrm{Reduce}(\varphi_i \leq \psi_i)$ denote the set of pure (universally quantified) inequalities after Stage 2, and $\mathrm{FO}(\varphi \leq \psi)$ denote the standard translation of the quasi-(universally quantified) inequalities into first-order formulas, then we have the following chain of equivalences:

It suffices to show the equivalence from (1) to (5) given below:

$$\mathbb{F} \Vdash \varphi \leq \psi \tag{1}$$

$$\mathbb{F} \Vdash \varphi_i \leq \psi_i, \text{ for all } 1 \leq i \leq n \tag{2}$$

$$\mathbb{F} \Vdash (\mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i \;\&\; \psi_i \leq^{\varnothing}_{\varnothing} \neg\mathbf{i}_1) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1 \text{ for all } 1 \leq i \leq n \tag{3}$$

$$\mathbb{F} \Vdash \mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1 \text{ for all } 1 \leq i \leq n \tag{4}$$

$$\mathbb{F} \Vdash \mathrm{FO}(\varphi \leq \psi) \tag{5}$$

- The equivalence between (1) and (2) follows from Proposition 6;
- the equivalence between (2) and (3) follows from Proposition 7;
- the equivalence between (3) and (4) follows from Propositions 8, 9, 10, 11;
- the equivalence between (4) and (5) follows from Proposition 4. □

In the remainder of this subsection, we prove the soundness of the rules in Stage 1, 2 and 3.

Proposition 6 (Soundness of the rules in Stage 1)**.** *For the distribution rules, the splitting rules and the monotone and antitone variable-elimination rules, they are sound in both directions in $\mathbb{F}$, i.e., it is sound from the premise to the conclusion and the other way round.*

Proof. For the soundness of the distribution rules, it follows from the fact that the following equivalences are valid in $\mathbb{F}$:

- $\Diamond(\alpha \vee \beta) \leftrightarrow \Diamond\alpha \vee \Diamond\beta$;
- $\blacklozenge(\alpha \vee \beta) \leftrightarrow \blacklozenge\alpha \vee \blacklozenge\beta$;
- $\neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta$;
- $(\alpha \vee \beta) \wedge \gamma \leftrightarrow (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$;
- $\alpha \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$;
- $((\alpha \vee \beta) \to \gamma) \leftrightarrow ((\alpha \to \gamma) \wedge (\beta \to \gamma))$;
- $\Box(\alpha \wedge \beta) \leftrightarrow \Box\alpha \wedge \Box\beta$;
- $\blacksquare(\alpha \wedge \beta) \leftrightarrow \blacksquare\alpha \wedge \blacksquare\beta$;
- $\neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta$;
- $(\alpha \wedge \beta) \vee \gamma \leftrightarrow (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$;
- $\alpha \vee (\beta \wedge \gamma) \leftrightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$;
- $(\alpha \to \beta \wedge \gamma) \leftrightarrow (\alpha \to \beta) \wedge (\alpha \to \gamma)$.

For the soundness of the splitting rules, it follows from the following fact:

$$\mathbb{F} \Vdash \alpha \leq \beta \wedge \gamma \quad \text{iff} \quad (\mathbb{F} \Vdash \alpha \leq \beta \text{ and } \mathbb{F} \Vdash \alpha \leq \gamma);$$

$$\mathbb{F} \Vdash \alpha \vee \beta \leq \gamma \quad \text{iff} \quad (\mathbb{F} \Vdash \alpha \leq \gamma \text{ and } \mathbb{F} \Vdash \beta \leq \gamma).$$

For the soundness of the monotone and antitone variable elimination rules, we show the soundness for the first rule. Suppose $\alpha(p)$ is negative in p and β is positive in p.

If $\mathbb{F} \Vdash \alpha(p) \leq \beta(p)$, then for all valuations V, $(\mathbb{F}, V) \Vdash \alpha(p) \leq \beta(p)$, thus for the valuation $V^p_\varnothing$ such that $V^p_\varnothing$ is the same as V except that $V^p_\varnothing(p) = \varnothing$, $(\mathbb{F}, V^p_\varnothing) \Vdash \alpha(p) \leq \beta(p)$, therefore $(\mathbb{F}, V^p_\varnothing) \Vdash \alpha(\bot) \leq \beta(\bot)$, thus $(\mathbb{F}, V) \Vdash \alpha(\bot) \leq \beta(\bot)$, so $\mathbb{F} \Vdash \alpha(\bot) \leq \beta(\bot)$.

For the other direction, suppose $\mathbb{F} \vDash \alpha(\bot) \leq \beta(\bot)$, then by the fact that $\alpha(p)$ is negative in p and $\beta(p)$ is positive in p, we have that $\mathbb{F} \vDash \alpha(p) \leq \alpha(\bot)$ and $\mathbb{F} \vDash \beta(\bot) \leq \beta(p)$, therefore $\mathbb{F} \vDash \alpha(p) \leq \beta(p)$.

The soundness of the other rule is similar. □

Proposition 7. (2) *and* (3) *are equivalent, i.e. the first-approximation rule is sound in* $\mathbb{F}$.

Proof. (2) $\Rightarrow$ (3): Suppose $\mathbb{F} \Vdash \varphi_i \leq \psi_i$. Then for any valuation V on $\mathbb{F}$, if $(\mathbb{F}, V) \Vdash \mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i$ and $(\mathbb{F}, V) \Vdash \psi_i \leq^{\varnothing}_{\varnothing} \neg\mathbf{i}_1$, then $(\mathbb{F}, V), V(\mathbf{i}_0) \Vdash \varphi_i$ and $(\mathbb{F}, V), V(\mathbf{i}_1) \nVdash \psi_i$, so by $\mathbb{F} \Vdash \varphi_i \leq \psi_i$ we have $(\mathbb{F}, V), V(\mathbf{i}_0) \Vdash \psi_i$, so $\mathbf{i}_0 \neq \mathbf{i}_1$, so $(\mathbb{F}, V) \Vdash \mathbf{i}_0 \leq \neg\mathbf{i}_1$.

(3) $\Rightarrow$ (2): Suppose $\mathbb{F} \Vdash (\mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i \ \&\ \psi_i \leq^{\varnothing}_{\varnothing} \neg\mathbf{i}_1) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1$. Then if $\mathbb{F} \nVdash \varphi_i \leq \psi_i$, then there is a valuation V on $\mathbb{F}$ and a $w \in W$ such that $(\mathbb{F}, V), w \Vdash \varphi_i$ and $(\mathbb{F}, V), w \nVdash \psi_i$. Then by taking $V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}$ to be the valuation which is the same as V except that $V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}(\mathbf{i}_0) = V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}(\mathbf{i}_1) = \{w\}$, then since $\mathbf{i}_0, \mathbf{i}_1$ do not occur in φ_i and ψ_i, we have that $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}), w \Vdash \varphi_i$ and $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}), w \nVdash \psi_i$, therefore $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}) \Vdash \mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i$ and $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}) \Vdash \psi_i \leq^{\varnothing}_{\varnothing} \neg\mathbf{i}_1$, by $\mathbb{F} \Vdash (\mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i \ \&\ \psi_i \leq^{\varnothing}_{\varnothing} \neg\mathbf{i}_1) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{i}_1$, we have that $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}) \Vdash \mathbf{i}_0 \leq \neg\mathbf{i}_1$, so $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}), w \Vdash \mathbf{i}_0$ implies that $(\mathbb{F}, V^{\mathbf{i}_0,\mathbf{i}_1}_{w,w}), w \Vdash \neg\mathbf{i}_1$, a contradiction. So $\mathbb{F} \Vdash \varphi_i \leq \psi_i$. □

The next step is to show the soundness of Stage 2, for which it suffices to show the soundness of each rule in each substage.

Proposition 8. *The splitting rules, the approximation rules for $\Diamond, \Box, \blacklozenge, \blacksquare, \rightarrow$, the residuation rules for $\neg$ in Substage 1 are sound in $\mathbb{F}$.*

Proof. By Lemma 1, 2, 3, 4 and 5 below. □

Lemma 1. *The splitting rules in Substage* 1 *and Substage* 2 *are sound in $\mathbb{F}$.*

Proof. For the soundness of the splitting rules, it follows from the fact that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$,

- $(\mathbb{F}, V) \Vdash \alpha \leq^{S}_{S'} \beta \wedge \gamma$ iff $((\mathbb{F}, V) \Vdash \alpha \leq^{S}_{S'} \beta$ and $(\mathbb{F}, V) \Vdash \alpha \leq^{S}_{S'} \gamma)$,
- $(\mathbb{F}, V) \Vdash \alpha \vee \beta \leq^{S}_{S'} \gamma$ iff $((\mathbb{F}, V) \Vdash \alpha \leq^{S}_{S'} \gamma$ and $(\mathbb{F}, V) \Vdash \beta \leq^{S}_{S'} \gamma)$. □

Lemma 2. *The approximation rules for $\Diamond, \Box$ in Substage 1 are sound in $\mathbb{F}$.*

Proof. We prove for $\Diamond$, the case for $\Box$ is similar. For the soundness of the approximation rule for $\Diamond$, it suffices to show that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$,

1. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond\alpha$, then there is a valuation $V^{\mathbf{j}}$ such that $V^{\mathbf{j}}$ is the same as V except $V^{\mathbf{j}}(\mathbf{j})$, and $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond^{S'}\mathbf{j}$ and $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{j} \leq^{S'}_{S'} \alpha$;
2. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond^{S'}\mathbf{j}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \leq^{S'}_{S'} \alpha$, then $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond\alpha$.

For item 1, if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond\alpha$, then $(W, (R_0 \setminus S'), V), V(\mathbf{i}) \Vdash \Diamond\alpha$, therefore there exists a $w \in W$ such that $(V(\mathbf{i}), w) \in (R_0 \setminus S')$ and $(W, (R_0 \setminus S'), V), w \Vdash \alpha$. Now take $V^{\mathbf{j}}$ such that $V^{\mathbf{j}}$ is the same as V except that $V^{\mathbf{j}}(\mathbf{j}) = \{w\}$, then $(V^{\mathbf{j}}(\mathbf{i}), V^{\mathbf{j}}(\mathbf{j})) \in (R_0 \setminus S')$, so $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond^{S'}\mathbf{j}$ and $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{j} \leq^{S'}_{S'} \alpha$.

For item 2, suppose $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond^{S'}\mathbf{j}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \leq^{S'}_{S'} \alpha$. Then $(V(\mathbf{i}), V(\mathbf{j})) \in (R_0 \setminus S')$ and $(W, (R_0 \setminus S'), V), V(\mathbf{j}) \Vdash \alpha$, so $(W, (R_0 \setminus S'), V), V(\mathbf{i}) \Vdash \Diamond\alpha$, therefore $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \Diamond\alpha$. □

Lemma 3. *The approximation rules for* $\blacklozenge, \blacksquare$ *in Substage 1 are sound in* $\mathbb{F}$.

Proof. We prove for $\blacklozenge$, the case for $\blacksquare$ is similar. For the soundness of the approximation rule for $\blacklozenge$, it suffices to show that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$,

1. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \blacklozenge\alpha$, then there is a valuation $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}$ such that $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}$ is the same as V except $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}_{m0})$ and $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}_{m1})$, and $(\mathbb{F}, V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}) \Vdash \mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'}\mathbf{i}_{m1}$ and $(\mathbb{F}, V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}) \Vdash \mathbf{i} \leq^{S}_{S'\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}} \alpha$;
2. if $(\mathbb{F}, V) \Vdash \mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'}\mathbf{i}_{m1}$ and $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}} \alpha$, then $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \blacklozenge\alpha$.

For item 1, if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \blacklozenge\alpha$, then $(W, (R_0 \setminus S'), V), V(\mathbf{i}) \Vdash \blacklozenge\alpha$, therefore there are $(w, v) \in (R_0 \setminus S')$ such that $(W, ((R_0 \setminus S') \setminus \{(w, v)\}), V), V(\mathbf{i}) \Vdash \alpha$. Now take $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}$ such that $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}$ is the same as V except $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}_{m0}) = \{w\}$ and $V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}_{m1}) = \{v\}$, then by $(w, v) \in (R_0 \setminus S')$, we have that $(W, R_0, V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}) \Vdash \mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'}\mathbf{i}_{m1}$, and from $(W, ((R_0 \setminus S') \setminus \{(w, v)\}), V), V(\mathbf{i}) \Vdash \alpha$ we have that $(W, ((R_0 \setminus S') \setminus \{(V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}_{m0}), V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}_{m1}))\}), V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}), V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}(\mathbf{i}) \Vdash \alpha$, so $(\mathbb{F}, V^{\mathbf{i}_{m0},\mathbf{i}_{m1}}) \Vdash \mathbf{i} \leq^{S}_{S'\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}} \alpha$.

For item 2, if $(\mathbb{F}, V) \Vdash \mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'}\mathbf{i}_{m1}$ and $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'\cup\{(\mathbf{i}_{m0},\mathbf{i}_{m1})\}} \alpha$, then $(V(\mathbf{i}_{m0}), V(\mathbf{i}_{m1})) \in (R_0 \setminus S')$, and $(W, (R_0 \setminus (S' \cup \{(\mathbf{i}_{m0}, \mathbf{i}_{m1})\})), V), V(\mathbf{i}) \Vdash \alpha$, so $(W, (R_0 \setminus S'), V), V(\mathbf{i}) \Vdash \blacklozenge\alpha$, therefore $(\mathbb{F}, V) \Vdash \mathbf{i} \leq^{S}_{S'} \blacklozenge\alpha$. □

Lemma 4. *The approximation rule for* $\rightarrow$ *in Substage* 1 *is sound in* $\mathbb{F}$.

Proof. For the soundness of the approximation rule for $\rightarrow$, it suffices to show that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$,

1. if $(\mathbb{F}, V) \Vdash \alpha \rightarrow \beta \leq^{S}_{S'} \neg\mathbf{i}$, then there is a valuation $V^{\mathbf{j},\mathbf{k}}$ such that $V^{\mathbf{j},\mathbf{k}}$ is the same as V except $V^{\mathbf{j},\mathbf{k}}(\mathbf{j})$ and $V^{\mathbf{j},\mathbf{k}}(\mathbf{k})$, and $(\mathbb{F}, V^{\mathbf{j},\mathbf{k}}) \Vdash \mathbf{j} \leq^{S}_{S} \alpha$, $(\mathbb{F}, V^{\mathbf{j},\mathbf{k}}) \Vdash \beta \leq^{S}_{S} \neg\mathbf{k}$ and $(\mathbb{F}, V^{\mathbf{j},\mathbf{k}}) \Vdash \mathbf{j} \rightarrow \neg\mathbf{k} \leq^{S}_{S'} \neg\mathbf{i}$;
2. if $(\mathbb{F}, V) \Vdash \mathbf{j} \leq^{S}_{S} \alpha$, $(\mathbb{F}, V) \Vdash \beta \leq^{S}_{S} \neg\mathbf{k}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \rightarrow \neg\mathbf{k} \leq^{S}_{S'} \neg\mathbf{i}$, then $(\mathbb{F}, V) \Vdash \alpha \rightarrow \beta \leq^{S}_{S'} \neg\mathbf{i}$.

   For item 1, if $(\mathbb{F}, V) \Vdash \alpha \rightarrow \beta \leq^{S}_{S'} \neg\mathbf{i}$, then $(W, (R_0 \setminus S), V), V(\mathbf{i}) \Vdash \alpha$ and $(W, (R_0 \setminus S), V), V(\mathbf{i}) \Vdash \neg\beta$. Now take $V^{\mathbf{j},\mathbf{k}}$ such that $V^{\mathbf{j},\mathbf{k}}$ is the same as V except that $V^{\mathbf{j},\mathbf{k}}(\mathbf{j}) = V^{\mathbf{j},\mathbf{k}}(\mathbf{k}) = V(\mathbf{i})$, we have that $(W, (R_0 \setminus S), V^{\mathbf{j},\mathbf{k}}), V^{\mathbf{j},\mathbf{k}}(\mathbf{j}) \Vdash \alpha$ and $(W, (R_0 \setminus S), V^{\mathbf{j},\mathbf{k}}), V^{\mathbf{j},\mathbf{k}}(\mathbf{k}) \Vdash \neg\beta$, so $(\mathbb{F}, V^{\mathbf{j},\mathbf{k}}) \Vdash \mathbf{j} \leq^{S}_{S} \alpha$, $(\mathbb{F}, V^{\mathbf{j},\mathbf{k}}) \Vdash \beta \leq^{S}_{S} \neg\mathbf{k}$. Since $V^{\mathbf{j},\mathbf{k}}(\mathbf{j}) = V^{\mathbf{j},\mathbf{k}}(\mathbf{k}) = V^{\mathbf{j},\mathbf{k}}(\mathbf{i}) = V(\mathbf{i})$, it is easy to see that $V^{\mathbf{j},\mathbf{k}}(\mathbf{j} \rightarrow \neg\mathbf{k}) = V^{\mathbf{j},\mathbf{k}}(\neg\mathbf{i})$, so $(\mathbb{F}, V^{\mathbf{j},\mathbf{k}}) \Vdash \mathbf{j} \rightarrow \neg\mathbf{k} \leq^{S}_{S'} \neg\mathbf{i}$.

   For item 2, if $(\mathbb{F}, V) \Vdash \mathbf{j} \leq^{S}_{S} \alpha$, $(\mathbb{F}, V) \Vdash \beta \leq^{S}_{S} \neg\mathbf{k}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \rightarrow \neg\mathbf{k} \leq^{S}_{S'} \neg\mathbf{i}$, then $V(\mathbf{j} \rightarrow \neg\mathbf{k}) \subseteq V(\neg\mathbf{i})$, so $V(\mathbf{i}) \subseteq V(\mathbf{j} \wedge \mathbf{k})$, since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are nominals, there interpretations are singletons, so $V(\mathbf{i}) = V(\mathbf{j}) = V(\mathbf{k})$. Now from $(\mathbb{F}, V) \Vdash \mathbf{j} \leq^{S}_{S} \alpha$ we have that $(W, (R_0 \setminus S), V), V(\mathbf{j}) \Vdash \alpha$, and from $(\mathbb{F}, V) \Vdash \beta \leq_S^S \neg\mathbf{k}$ we have that $(W, (R_0 \setminus S), V), V(\mathbf{k}) \Vdash \neg\beta$, so $(W, (R_0 \setminus S), V), V(\mathbf{i}) \Vdash \alpha$ and $(W, (R_0 \setminus S), V), V(\mathbf{i}) \Vdash \neg\beta$, so $(\mathbb{F}, V) \Vdash \alpha \to \beta \leq_{S'}^S \neg\mathbf{i}$. □

Lemma 5. *The residuation rules for* $\neg$ *in Substage* 1 *and* 2 *are sound in* $\mathbb{F}$.

Proof. It is easy to see that the residuation rules for $\neg$ in Substage 1 are special cases of the residuation rules for $\neg$ in Substage 2 (modulo double negation elimination). Now we only prove it for the residuation rule in Substage 2 where negation symbols occur on the right-hand side of the inequalities, the other rule is similar.

For the soundness of the residuation rule for $\neg$, it suffices to show that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$, $(\mathbb{F}, V) \Vdash \alpha \leq_{S'}^S \neg\beta$ iff $(\mathbb{F}, V) \Vdash \beta \leq_S^{S'} \neg\alpha$. Indeed, it follows from the following equivalence:

$$
\begin{aligned}
& (\mathbb{F}, V) \Vdash \alpha \leq_{S'}^S \neg\beta \\
\text{iff} \quad & \text{for all } w \in W, \text{ if } (W, (R_0 \setminus S), V), w \Vdash \alpha, \text{ then } (W, (R_0 \setminus S'), V), w \nVdash \beta \\
\text{iff} \quad & \text{for all } w \in W, \text{ if } (W, (R_0 \setminus S'), V), w \Vdash \beta, \text{ then } (W, (R_0 \setminus S), V), w \nVdash \alpha \\
\text{iff} \quad & (\mathbb{F}, V) \Vdash \beta \leq_S^{S'} \neg\alpha.
\end{aligned}
$$
□

Proposition 9. *The splitting rules, the residuation rules for* $\neg, \Diamond, \Box, \blacklozenge, \blacksquare$, *the second splitting rule in Substage* 2 *are sound in* $\mathbb{F}$.

Proof. By Lemma 1, 5, 6, 7 and 8. □

Lemma 6. *The residuation rules for* $\Diamond, \Box$ *in Substage* 2 *are sound in* $\mathbb{F}$.

Proof. We prove it for $\Diamond$, and the rule for $\Box$ is similar.

To show the soundness of the residuation rule for $\Diamond$ in Substage 2, it suffices to show that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$, $(\mathbb{F}, V) \Vdash \Diamond\alpha \leq_{S'}^S \beta$ iff $(\mathbb{F}, V) \Vdash \alpha \leq_{S'}^S (\Box^S)^{-1}\beta$.

$\Rightarrow$: if $(\mathbb{F}, V) \Vdash \Diamond\alpha \leq_{S'}^S \beta$, then for all $w \in W$, if $(W, (R_0 \setminus S), V), w \Vdash \Diamond\alpha$, then $(W, (R_0 \setminus S'), V), w \Vdash \beta$. Our aim is to show that for all $v \in W$, if $(W, (R_0 \setminus S), V), v \Vdash \alpha$, then $(W, (R_0 \setminus S'), V), v \Vdash (\Box^S)^{-1}\beta$.

Consider any $v \in W$ such that $(W, (R_0 \setminus S), V), v \Vdash \alpha$. Then for any $u \in W$ such that $(u, v) \in (R_0 \setminus S)$, $(W, (R_0 \setminus S), V), u \Vdash \Diamond\alpha$. Since $(\mathbb{F}, V) \Vdash \Diamond\alpha \leq_{S'}^S \beta$, we have that $(W, (R_0 \setminus S'), V), u \Vdash \beta$, so for any $u \in W$ such that $(v, u) \in (R_0 \setminus S)^{-1}$, $(W, (R_0 \setminus S'), V), u \Vdash \beta$, so $(W, (R_0 \setminus S'), V), v \Vdash (\Box^S)^{-1}\beta$.

$\Leftarrow$: if $(\mathbb{F}, V) \Vdash \alpha \leq_{S'}^S (\Box^S)^{-1}\beta$, then for all $w \in W$, if $(W, (R_0 \setminus S), V), w \Vdash \alpha$, then $(W, (R_0 \setminus S'), V), w \Vdash (\Box^S)^{-1}\beta$. Our aim is to show that for all $v \in W$, if $(W, (R_0 \setminus S), V), v \Vdash \Diamond\alpha$, then $(W, (R_0 \setminus S'), V), v \Vdash \beta$.

Now assume that $(W, (R_0 \setminus S), V), v \Vdash \Diamond\alpha$. Then there is a $u \in W$ such that $(v, u) \in (R_0 \setminus S)$ and $(W, (R_0 \setminus S), V), u \Vdash \alpha$. By $(\mathbb{F}, V) \Vdash \alpha \leq_{S'}^S (\Box^S)^{-1}\beta$, we have that $(W, (R_0 \setminus S'), V), u \Vdash (\Box^S)^{-1}\beta$. Therefore, for $v \in W$, we have $(u, v) \in (R_0 \setminus S)^{-1}$, thus $(W, (R_0 \setminus S'), V), v \Vdash \beta$. □

Lemma 7. *The residuation rules for* $\blacklozenge, \blacksquare$ *in Substage* 2 *are sound in* $\mathbb{F}$.

Proof. We prove it for $\blacksquare$, and the rule for $\blacklozenge$ is similar.

For the residuation rule for $\blacksquare$, it suffices to show that for any Kripke frame $\mathbb{F} = (W, R_0)$, any valuation V on $\mathbb{F}$, $(\mathbb{F}, V) \Vdash \alpha \leq^S_{S'} \blacksquare\beta$ iff $(\mathbb{F}, V) \Vdash \forall \mathbf{i}_{m0} \forall \mathbf{i}_{m1} (\mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'} \mathbf{i}_{m1} \Rightarrow \alpha \leq^S_{S' \cup \{(\mathbf{i}_{m0}, \mathbf{i}_{m1})\}} \beta)$. Indeed:

$$(\mathbb{F}, V) \Vdash \forall \mathbf{i}_{m0} \forall \mathbf{i}_{m1} (\mathbf{i}_{m0} \leq^{S'}_{S'} \Diamond^{S'} \mathbf{i}_{m1} \Rightarrow \alpha \leq^S_{S' \cup \{(\mathbf{i}_{m0}, \mathbf{i}_{m1})\}} \beta)$$

iff for all $w, v \in W$, if $(w, v) \in (R_0 \setminus S')$,
then $(W, R_0, V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}) \Vdash \alpha \leq^S_{S' \cup \{(\mathbf{i}_{m0}, \mathbf{i}_{m1})\}} \beta$,
where $V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}$ is the same as V except that: $V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}(\mathbf{i}_{m0}) = \{w\}$,
$V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}(\mathbf{i}_{m1}) = \{v\}$.

iff for all $u, v, w \in W$, if $(w, v) \in (R_0 \setminus S')$
and $(W, (R_0 \setminus S), V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}), u \Vdash \alpha$,
then $(W, (R_0 \setminus (S' \cup \{(\mathbf{i}_{m0}, \mathbf{i}_{m1})\})), V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}), u \Vdash \beta$

iff for all $u \in W$, if $(W, (R_0 \setminus S), V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}), u \Vdash \alpha$,
then for all $v, w \in W$, if $(w, v) \in (R_0 \setminus S')$, then:
$(W, (R_0 \setminus (S' \cup \{(\mathbf{i}_{m0}, \mathbf{i}_{m1})\})), V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}), u \Vdash \beta$

iff for all $u \in W$, if $(W, (R_0 \setminus S), V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}), u \Vdash \alpha$, then:
$(W, (R_0 \setminus S'), V^{\mathbf{i}_{m0}, \mathbf{i}_{m1}}_{w,v}), u \Vdash \blacksquare\beta$

iff for all $u \in W$, if $(W, (R_0 \setminus S), V), u \Vdash \alpha$, then:
$(W, (R_0 \setminus S'), V), u \Vdash \blacksquare\beta$
(since $\mathbf{i}_{m0}$ and $\mathbf{i}_{m1}$ do not occur in α and β)

iff $(\mathbb{F}, V) \Vdash \alpha \leq^S_{S'} \blacksquare\beta$. □

Lemma 8. *The second splitting rule in Substage* 2 *is sound in* $\mathbb{F}$.

Proof. It follows immediately from the meta-equivalence that $\forall x \forall y (\alpha \to \beta \wedge \gamma) \leftrightarrow \forall x \forall y (\alpha \to \beta) \wedge \forall x \forall y (\alpha \to \gamma)$. □

Proposition 10. *The packing rules in Substage* 3 *are sound in* $\mathbb{F}$.

Proof. We only prove the soundness of the first packing rule, the others are similar.

It is easy to see that in the mega-inequality of the premise and in the conclusion, contextual connectives $\Box, \Diamond, \blacksquare, \blacklozenge$ do not occur, so we can ignore the superscripts and subscripts in the inequalities occuring in the rule.

We first define the following mega-inequalities and formulas:

$$\text{Mega}_0 := \forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1} (\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0} \mathbf{i}_{m_0 1} \Rightarrow \alpha \leq^S_{S'} p)$$

$$\text{Mega}_n := \forall \mathbf{i}_{m_n 0} \forall \mathbf{i}_{m_n 1} (\mathbf{i}_{m_n 0} \leq^{S_n}_{S_n} \Diamond^{S_n} \mathbf{i}_{m_n 1} \Rightarrow \text{Mega}_{n-1})$$

$$\varphi_0 := \exists \mathbf{i}_{m_0 0} \exists \mathbf{i}_{m_0 1} (A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}) \wedge \alpha)$$

$$\varphi_n := \exists \mathbf{i}_{m_n 0} \exists \mathbf{i}_{m_n 1} (A(\mathbf{i}_{m_n 0} \to \Diamond^{S_n} \mathbf{i}_{m_n 1}) \land \varphi_{n-1})$$

Then we can prove by induction on k that for any Kripke frame $\mathbb{F} = (W, R_0)$ and any valuation V on it,

$$\begin{aligned}
& (\mathbb{F}, V) \Vdash \mathrm{Mega}_n \\
\text{iff} \quad & (\mathbb{F}, V) \Vdash \forall \mathbf{i}_{m_n 0} \forall \mathbf{i}_{m_n 1} (\mathbf{i}_{m_n 0} \leq^{S_n}_{S_n} \Diamond^{S_n} \mathbf{i}_{m_n 1} \\
& \quad \Rightarrow \cdots \forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1} (\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0} \mathbf{i}_{m_0 1} \Rightarrow \alpha \leq^{S}_{S'} p) \cdots) \\
\text{iff} \quad & (\mathbb{F}, V) \Vdash \exists \mathbf{i}_{m_n 0} \exists \mathbf{i}_{m_n 1} \cdots \exists \mathbf{i}_{m_0 0} \exists \mathbf{i}_{m_0 1} \\
& \quad (A(\mathbf{i}_{m_n 0} \to \Diamond^{S_n} \mathbf{i}_{m_n 1}) \land \cdots \land A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}) \land \alpha) \leq^{\varnothing}_{\varnothing} p, \\
\text{iff} \quad & (\mathbb{F}, V) \Vdash \varphi_n \leq^{\varnothing}_{\varnothing} p.
\end{aligned}$$

(i) When $k = 0$, for any Kripke frame $\mathbb{F} = (W, R_0)$ and any valuation V on it,

$$\begin{aligned}
& (\mathbb{F}, V) \Vdash \mathrm{Mega}_0 \\
\text{iff} \quad & (\mathbb{F}, V) \Vdash \forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1} (\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0} \mathbf{i}_{m_0 1} \Rightarrow \alpha \leq^{S}_{S'} p) \\
\text{iff} \quad & \text{for all } w_{m_0 0}, w_{m_0 1} \in W, \\
& \text{if } (w_{m_0 0}, w_{m_0 1}) \in (R_0 \setminus S_0), \text{ then } (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}) \Vdash \alpha \leq^{S}_{S'} p \\
\text{iff} \quad & \text{for all } w_{m_0 0}, w_{m_0 1} \in W, \\
& \text{if } (w_{m_0 0}, w_{m_0 1}) \in (R_0 \setminus S_0), \text{ then } (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}) \Vdash \alpha \leq^{\varnothing}_{\varnothing} p \\
\text{iff} \quad & \text{for all } w_{m_0 0}, w_{m_0 1}, v \in W, \\
& \text{if } (w_{m_0 0}, w_{m_0 1}) \in (R_0 \setminus S_0), (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}), v \Vdash \alpha \\
& \text{then } (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}), v \Vdash p \\
\text{iff} \quad & \text{for all } w_{m_0 0}, w_{m_0 1}, v \in W, \\
& \text{if } (w_{m_0 0}, w_{m_0 1}) \in (R_0 \setminus S_0), (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}), v \Vdash \alpha \\
& \text{then } (\mathbb{F}, V), v \Vdash p \\
\text{iff} \quad & \text{for all } w_{m_0 0}, w_{m_0 1}, v \in W, \\
& \text{if } (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}) \Vdash A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}), (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}), v \Vdash \alpha \\
& \text{then } (\mathbb{F}, V), v \Vdash p \\
\text{iff} \quad & \text{for all } w_{m_0 0}, w_{m_0 1}, v \in W, \\
& \text{if } (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}), v \Vdash A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}) \land \alpha \\
& \text{then } (\mathbb{F}, V), v \Vdash p \\
\text{iff} \quad & \text{for all } v \in W, \text{ if there exists } w_{m_0 0}, w_{m_0 1} \in W \text{ s.t.} \\
& (\mathbb{F}, V^{\mathbf{i}_{m_0 0}, \mathbf{i}_{m_0 1}}_{w_{m_0 0}, w_{m_0 1}}), v \Vdash A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}) \land \alpha, \text{ then } (\mathbb{F}, V), v \Vdash p \\
\text{iff} \quad & \text{for all } v \in W, \\
& \text{if } (\mathbb{F}, V), v \Vdash \exists \mathbf{i}_{m_0 0} \exists \mathbf{i}_{m_0 1} (A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}) \land \alpha) \\
& \text{then } (\mathbb{F}, V), v \Vdash p \\
\text{iff} \quad & (\mathbb{F}, V) \Vdash \exists \mathbf{i}_{m_0 0} \exists \mathbf{i}_{m_0 1} (A(\mathbf{i}_{m_0 0} \to \Diamond^{S_0} \mathbf{i}_{m_0 1}) \land \alpha) \leq^{\varnothing}_{\varnothing} p.
\end{aligned}$$

(ii) When $k = n$, by induction hypothesis, we have proved the equivalence between Mega_{n-1} and $\varphi_{n-1} \leq^{\varnothing}_{\varnothing} p$. Then for $k = n$, for any Kripke frame $\mathbb{F} = (W, R_0)$ and any valuation V on it,

$$
\begin{array}{ll}
& (\mathbb{F}, V) \Vdash \text{Mega}_n \\
\text{iff} & (\mathbb{F}, V) \Vdash \forall \mathbf{i}_{m_n 0} \forall \mathbf{i}_{m_n 1} (\mathbf{i}_{m_n 0} \leq^{S_n}_{S_n} \Diamond^{S_n} \mathbf{i}_{m_n 1} \ \Rightarrow \ \text{Mega}_{n-1}) \\
\text{iff} & (\mathbb{F}, V) \Vdash \forall \mathbf{i}_{m_n 0} \forall \mathbf{i}_{m_n 1} (\mathbf{i}_{m_n 0} \leq^{S_n}_{S_n} \Diamond^{S_n} \mathbf{i}_{m_n 1} \ \Rightarrow \ \varphi_{n-1} \leq^{\varnothing}_{\varnothing} p) \\
\text{iff} & (\mathbb{F}, V) \Vdash \exists \mathbf{i}_{m_n 0} \exists \mathbf{i}_{m_n 1} (A(\mathbf{i}_{m_n 0} \to \Diamond^{S_n} \mathbf{i}_{m_n 1}) \wedge \varphi_{n-1}) \leq^{\varnothing}_{\varnothing} p \\
\text{iff} & (\mathbb{F}, V) \Vdash \varphi_n \leq^{\varnothing}_{\varnothing} p.
\end{array}
$$

□

Proposition 11. *The Ackermann rules in Substage 4 are sound in* $\mathbb{F}$.

Proof. We only prove it for the right-handed Ackermann rule, the other rule is similar.

Without loss of generality we assume that $n = m = 1$. It suffices to show the following right-handed Ackermann lemma:

Lemma 9. *Assume* α *is pure and contains no contextual modalities* $\Box, \Diamond, \blacksquare, \blacklozenge$ *and does not contain nominals in* $\vec{\mathbf{i}}$, β *is positive in* p *and* γ *is negative in* p*, then for any Kripke frame* $\mathbb{F} = (W, R_0)$ *and any valuation* V *on it,*

$(\mathbb{F}, V) \Vdash \forall \vec{\mathbf{i}}(\beta(\alpha/p) \leq^{T}_{T'} \gamma(\alpha/p))$ *iff there exists a valuation* V^p *such that* $(\mathbb{F}, V^p) \Vdash \alpha \leq^{\varnothing}_{\varnothing} p$ *and* $(\mathbb{F}, V^p) \Vdash \forall \vec{\mathbf{i}}(\beta \leq^{T}_{T'} \gamma)$*, where* V^p *is the same as* V *except* $V^p(p)$.

Notice that α and p do not contain contextual modalities, so their valuation do not change when the context is different.

$\Rightarrow$: Take V^p such that V^p is the same as V except that $V^p(p) = V(\alpha)$. Since α does not contain p, it is easy to see that $V^p(\alpha) = V(\alpha) = V^p(p)$. Therefore $(\mathbb{F}, V^p) \Vdash \alpha \leq^{\varnothing}_{\varnothing} p$. Since the valuation of α and p do not change when the context is different, so for any $w \in W$,

$(W, (R_0 \setminus T), V^p), w \Vdash \beta$ iff $(W, (R_0 \setminus T), V), w \Vdash \beta(\alpha/p)$, and
$(W, (R_0 \setminus T'), V^p), w \Vdash \gamma$ iff $(W, (R_0 \setminus T'), V), w \Vdash \gamma(\alpha/p)$, so
from $(\mathbb{F}, V) \Vdash \forall \vec{\mathbf{i}}(\beta(\alpha/p) \leq^{T}_{T'} \gamma(\alpha/p))$ one can get $(\mathbb{F}, V^p) \Vdash \forall \vec{\mathbf{i}}(\beta \leq^{T}_{T'} \gamma)$.

$\Leftarrow$: This direction follows from the monotonicity of β in p and the antitonicity of γ in p, and that the valuation of α and p do not change when the context is different.

□

8 Success of $\mathrm{ALBA}^{\mathrm{SML}}$ on Sahlqvist Inequalities

In the present section, we show that $\mathrm{ALBA}^{\mathrm{SML}}$ succeeds on all Sahlqvist inequalities, in the style of [18]:

Theorem 12. $\mathrm{ALBA}^{\mathrm{SML}}$ *succeeds on all Sahlqvist inequalities.*

Definition 9 (Definite ε-Sahlqvist inequality)**.** Given an order type ε, $* \in \{-, +\}$, the signed generation tree $*\varphi$ of the term $\varphi(p_1, \ldots, p_n)$ is *definite* ε*-Sahlqvist* if there is no $+\vee, -\wedge$ occurring in the outer part on an ε-critical branch. An inequality $\varphi \leq \psi$ is definite ε-Sahlqvist if the trees $+\varphi$ and $-\psi$ are both definite ε-Sahlqvist.

Lemma 10. *Let* $\{\varphi_i \leq \psi_i\}_{i\in I} = \mathrm{Preprocess}(\varphi \leq \psi)$ *obtained by exhaustive application of the rules in Stage 1 on an input* ε*-Sahlqvist inequality* $\varphi \leq \psi$*. Then each* $\varphi_i \leq \psi_i$ *is a definite* ε*-Sahlqvist inequality.*

Proof. It is easy to see that by applying the distribution rules, all occurrences of $+\vee$ and $-\wedge$ in the outer part of an ε-critical branch have been pushed up towards the root of the signed generation trees $+\varphi$ and $-\psi$. Then by exhaustively applying the splitting rules, all such $+\vee$ and $-\wedge$ are eliminated. Since by applying the distribution rules, the splitting rules and the monotone/antitone variable elimination rules do not change the ε-Sahlqvistness of a signed generation tree, in $\mathrm{Preprocess}(\varphi \leq \psi)$, each signed generation tree $+\varphi_i$ and $-\psi_i$ are ε-Sahlqvist, and since they do not have $+\vee$ and $-\wedge$ in the outer part in the ε-critical branches, they are definite. □

Definition 10 (Inner ε-Sahlqvist signed generation tree)**.** Given an order type ε, $* \in \{-,+\}$, the signed generation tree $*\varphi$ of the term $\varphi(p_1,\ldots,p_n)$ is *inner* ε*-Sahlqvist* if its outer part P_2 on an ε-critical branch is always empty, i.e. its ε-critical branches have inner nodes only.

Lemma 11. *Given inequalities* $\mathbf{i}_0 \leq^{\varnothing}_{\varnothing} \varphi_i$ *and* $\psi_i \leq^{\varnothing}_{\varnothing} \neg\mathbf{i}_1$ *obtained from Stage 1 where* $+\varphi_i$ *and* $-\psi_i$ *are definite* ε*-Sahlqvist, by applying the rules in Substage* 1 *of Stage* 2 *exhaustively, the inequalities that we get are in one of the following forms:*

1. *pure inequalities which does not have occurrences of propositional variables;*
2. *inequalities of the form* $\mathbf{i} \leq^{S}_{S'} \alpha$ *where* $+\alpha$ *is inner* ε*-Sahlqvist;*
3. *inequalities of the form* $\beta \leq^{S}_{S'} \neg\mathbf{i}$ *where* $-\beta$ *is inner* ε*-Sahlqvist.*

Proof. Indeed, the rules in the Substage 1 of Stage 2 deal with outer nodes in the signed generation trees $+\varphi_i$ and $-\psi_i$ except $+\vee$,$-\wedge$. For each rule, without loss of generality assume we start with an inequality of the form $\mathbf{i} \leq^{S}_{S'} \alpha$, then by applying the rules in Substage 1 of Stage 2, the inequalities we get are either a pure inequality without propositional variables, or an inequality where the left-hand side (resp. right-hand side) is $\mathbf{i}$ (resp. $\neg\mathbf{i}$), and the other side is a formula α' which is a subformula of α, such that α' has one root connective less than α. Indeed, if α' is on the left-hand side (resp. right-hand side) then $-\alpha'$ $(+\alpha')$ is definite ε-Sahlqvist.

By applying the rules in the Substage 1 of Stage 2 exhaustively, we can eliminate all the outer connectives in the critical branches, so for non-pure inequalities, they become of form 2 or form 3. □

Lemma 12. *Assume we have an inequality* $\mathbf{i} \leq^{S}_{S'} \alpha$ *or* $\beta \leq^{S}_{S'} \neg\mathbf{i}$ *where* $+\alpha$ *and* $-\beta$ *are inner* ε*-Sahlqvist, by applying the rules in Substage* 2 *of Stage* 2*, we have (mega-)inequalities (*k *can be 0 where a mega-inequality becomes an inequality) of the following form:*

1. $\forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1}(\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_0 1} \Rightarrow \cdots \forall \mathbf{i}_{m_k 0} \forall \mathbf{i}_{m_k 1}(\mathbf{i}_{m_k 0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k 1} \Rightarrow \alpha \leq^{S}_{S'} p) \cdots)$*, where* $\varepsilon(p) = 1$*,* α *is pure and does not contain contextual connectives* $\Box, \Diamond, \blacksquare, \blacklozenge$*;*
2. $\forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1}(\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_0 1} \Rightarrow \cdots \forall \mathbf{i}_{m_k 0} \forall \mathbf{i}_{m_k 1}(\mathbf{i}_{m_k 0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k 1} \Rightarrow p \leq^{S}_{S'} \beta) \cdots)$*, where* $\varepsilon(p) = \partial$*,* β *is pure and does not contain contextual connectives* $\Box, \Diamond, \blacksquare, \blacklozenge$*;*
3. $\forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1}(\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_0 1} \Rightarrow \cdots \forall \mathbf{i}_{m_k 0} \forall \mathbf{i}_{m_k 1}(\mathbf{i}_{m_k 0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k 1} \Rightarrow \alpha \leq^{S}_{S'} \gamma) \cdots)$*, where* α *is pure and does not contain contextual connectives* $\Box, \Diamond, \blacksquare, \blacklozenge$*, and* $+\gamma$ *is* ε^{∂}*-uniform;*
4. $\forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1}(\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_0 1} \Rightarrow \cdots \forall \mathbf{i}_{m_k 0} \forall \mathbf{i}_{m_k 1}(\mathbf{i}_{m_k 0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k 1} \Rightarrow \gamma \leq^{S}_{S'} \beta) \cdots)$*, where* β *is pure and does not contain contextual connectives* $\Box, \Diamond, \blacksquare, \blacklozenge$*, and* $-\gamma$ *is* ε^{∂}*-uniform.*

Proof. First of all, from the rules of the Substage 2 of Stage 2, it is easy to see that from the given inequality, what we will obtain would be a set of mega-inequalities, and by the second splitting rule those mega-inequalities are built up from inequalities by $\forall \mathbf{i} \forall \mathbf{j}(\mathbf{i} \leq^{S}_{S} \Diamond^{S}\mathbf{j} \Rightarrow \text{Mega})$, so we will have mega-inequalities of the form $\forall \mathbf{i}_{m_0 0} \forall \mathbf{i}_{m_0 1}(\mathbf{i}_{m_0 0} \leq^{S_0}_{S_0} \Diamond^{S_0}\mathbf{i}_{m_0 1} \Rightarrow \cdots \forall \mathbf{i}_{m_k 0} \forall \mathbf{i}_{m_k 1}(\mathbf{i}_{m_k 0} \leq^{S_k}_{S_k} \Diamond^{S_k}\mathbf{i}_{m_k 1} \Rightarrow \gamma \leq^{S}_{S'} \delta) \cdots)$. Now it suffices to check the shape of γ and δ. (From now on we call $\gamma \leq^{S}_{S'} \delta$ the *head* of the mega-inequality.)

Notice that for each input inequality, it is of the form $\mathbf{i} \leq^{S}_{S'} \alpha$ or $\beta \leq^{S}_{S'} \neg\mathbf{i}$, where $+\alpha$ and $-\beta$ are inner ε-Sahlqvist. By applying the splitting rules and the residuation rules in this substage, it is easy to see that the head of the (mega-)inequality will have one side of the inequality pure and have no contextual connectives $\Box, \Diamond, \blacksquare, \blacklozenge$, and the other side still inner ε-Sahlqvist. By applying these rules exhaustively, one will either have p as the non-pure side (with this p on a critical branch), or have an inner ε-Sahlqvist signed generation tree with no critical branch, i.e., ε^{∂}-uniform. □

Lemma 13. *Assume we have (mega-)inequalities of the form as described in Lemma* 12*. Then we can get inequalities of the following form:*

1. $\alpha \leq^{\varnothing}_{\varnothing} p$ *where* $\varepsilon(p) = 1$*,* α *is pure and do not contain contextual connectives* $\Box, \Diamond, \blacksquare, \blacklozenge$*;*
2. $p \leq^{\varnothing}_{\varnothing} \alpha$ *where* $\varepsilon(p) = \partial$*,* α *is pure and do not contain contextual connectives* $\Box, \Diamond, \blacksquare, \blacklozenge$*;*
3. $\forall \mathbf{i}_1 \cdots \forall \mathbf{i}_n(\top \leq^{\varnothing}_{S} \gamma)$ *where* $+\gamma$ *is* ε^{∂}*-uniform.*

Proof. From the shape of the mega-inequalities, we can see that for all the mega-inequalities we can apply the corresponding packing rules so that we can get the inequalities as described in the lemma. □

Lemma 14. *Assume we have inequalities of the form as described in Lemma* 13*, the Ackermann lemmas are applicable and therefore all propositional variables can be eliminated.*

Proof. Immediate observation from the requirements of the Ackermann lemmas. □

Proof of Theorem 12 Assume we have an ε-Sahlqvist inequality $\varphi \leq \psi$ as input. By Lemma 10, we get a set of definite ε-Sahlqvist inequalities. Then by Lemma 11, we get inequalities as described in Lemma 11. By Lemma 12, we get the mega-inequalities as described. Therefore by Lemma 13, we can apply the packing rules to get inequalities and universally quantified inequalities as described in the lemma. Finally by Lemma 14, the (universally quantified) inequalities are in the right shape to apply the Ackermann rules, and thus we can eliminate all the propositional variables and the algorithm succeeds on the input. □

9 Discussions and Further Directions

Future interesting questions include the following:

- Extending the Sahlqvist sabotage formulas to inductive sabotage formulas as well as to the language of sabotage modal logic with fixpoint operators;
- A Kracht-type theorem characterizing the first-order correspondents of Sahlqvist sabotage formulas;
- A Goldblatt-Thomason-type theorem characterizing the sabotage modally definable classes of Kripke frames;
- Extend results on sabotage modal logic to the class of relation changing modal logics.([3])

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破坏模态逻辑的算法对应理论

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摘　要

破坏模态逻辑是一种动态逻辑。它在静态模态逻辑的基础上加入了一个动态算子，解释成"在删掉一条边后，公式为真"。在本文中，我们试图解决一个开放问题，即给出破坏模态逻辑的 Sahlqvist 对应定理。我们定义破坏模态逻辑的 Sahlqvist 公式，并给出一个算法 $\mathrm{ALBA}^{\mathrm{SML}}$ 来计算破坏模态逻辑的 Sahlqvist 公式的一阶对应。

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