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A Qualitative Logical Analysis of Probabilistic Causal Models*

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Abstract. Uncertainty occurs frequently in the process of causal reasoning. Although the traditional structural equation model is very successful in the reasoning of deterministic causal structure, yet it is not designed to characterize probabilistic reasoning. Recently there have been proposals aiming to account for probabilistic causal reasoning by adding quantitative probabilistic expressions in the causal language. Contrast with the quantitative approach, this paper will present a qualitative model of probabilistic causal reasoning, which represents uncertainty of variables in terms of doxastic relations. The formal language based on this framework is able to express a qualitative notion of independence among causal variables, which can be used to analyse the co-relation between causality and probability.

1 Introduction

1.1 Toward a logical analysis of probabilistic causal reasoning

In recent years, a lot of effort has been put in the development of formal models for causal reasoning with counterfactuals. The most well-known model is the structural equation causal model developed in [8]. The logical inquiry of causality based on the structural equation approach also gained a lot of popularity since [3, 6, 9].

A structural equation causal model includes a signature $\mathcal{S}$ and a set of structural equations $\mathcal{F}$. The signature $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \Sigma)$ where $\mathcal{U}$ is the set of exogenous variables, $\mathcal{V}$ is the set of endogenous variables, and Σ is the range of those variables. The set $\mathcal{F} = \{f_X : X \in \mathcal{V}\}$ contains a structural function f_X for each endogenous variable $X \in \mathcal{V}$, and f_X maps any assignment to $\mathcal{U} \cup \mathcal{V} \setminus \{X\}$ to a possible value of X in Σ. Thus, each structural f_X describes how the value of X is determined by the setting of other variables.[1] As we can see from its definition, the classical structural equation causal model is deterministic and unable to deal with uncertainty and probability.

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[1]In many studies of structural equation models, $\mathcal{F}$ is assumed to be *recursive*, which means there is no sequence $X_1, \ldots, X_n$ such that for each $0 < k < n$ the value of X_{k+1} is dependent on X_k according to $\mathcal{F}$, and the value of X_1 is also dependent on X_n. In other words, being recursive means being non-cyclic.

On the other hand, there has been extensive study about the probabilistic features of causal Bayesian network. However, as argued by Pearl in [10], a pure causal Bayesian network is insufficient to provide a complete account of causal reasoning for counterfactuals (the causal Bayesian network is at a lower level than structural equation models in the so called "causal hierarchy").

Causal reasoning with counterfactuals in a non-deterministic or probabilistic situation is very natural in the real life. For instance, it is natural to infer from "It is very probable that the the heat is off" and "Had the heat been off, then it is very probable that the water pipes would been frozen this winter" to "It is very probable that the water pipes would been frozen this winter". Therefore it calls for a logical analysis of probabilistic causal reasoning with counterfactuals.

1.2 Toward a qualitative analysis of probabilistic causal reasoning

In [7], Ibeling and Icard synthesize the previous study of deterministic causal models and probabilistic logic, and present a logical framework for probabilistic causal reasoning. They propose a semantics based on structural equation models extended with an additional part P, which is a probabilistic distribution over the valuation of exogenous variables.

The probability of propositions in such a model is defined in the following way: the probability of ϕ in a model $(\mathcal{F}, P)$ is defined as the probability of the setting of exogenous variables supports ϕ under $\mathcal{F}$.[2] This can be seen as a "Laplacian" interpretation of probability: though the causal rules in the world is deterministic, there are still some unknown exogenous factors, which lead to the ignorance of their actual causal effects. Thus the uncertainty of a proposition (including any uncertainty of counterfactuals) can be reduced into the uncertainty of exogenous variables.

The logical framework proposed in [7] is quantitative. Its language needs to express the value of the probability assigned to a proposition. In addition, it has to include all the polynomial terms of the form $t + t$, $t \times t$, $-t$ in the object language. For instance, $\mathbb{P}(\phi) + \mathbb{P}(\psi) + 1$ is a polynomial term in the object language expressing the sum of the probability of ϕ and the probability of ψ plus 1.

In this paper, on the contrary, I want to propose a qualitative logical framework instead of a quantitative framework, whose language does not involve any specific value of probability. In the procedure of causal reasoning, an agent does not necessarily involve any precise value of probability. For instance in the water pipe example above, to make the reasoning go through, it is sufficient to qualitatively know that the probability of certain propositions is almost 1, without knowing any quantity.

[2] Formally $P(\phi) := P(\{\overrightarrow{u} \in \Sigma \mid \mathcal{F}, \overrightarrow{u} \models \phi\})$.

2 Qualitative Representation of Probabilistic Space

Similar to the quantitative account of probabilistic causal reasoning, my qualitative model consists of two parts: one part is the qualitative representation of causal effect, the other is the qualitative representation of the probabilistic distribution over variables. I also adopt the Laplacian treatment of uncertainty as [7] which reduces the uncertainty of causal effects into the uncertainty of the value of variables. Therefore the causal effect is still deterministic and can be represented by structural equations in the classical way as in [6, 10]. So we only need to looking for a qualitative representation of probabilistic distribution.

In [2], Baltag and Smets adopts the Popper-Renyi theory of conditional probabilities originated from [11, 12] (later developed by [4, 5]), and proposed a qualitative representation of conditional probabilistic space.

Definition 1 (Discrete Conditional Probabilistic Space)**.** A pair (S, μ) with S is a finite set of states and $\mu : \wp(S) \times \wp(S) \to [0, 1]$ satisfying:[3]

- $\mu(A|A) = 1$
- $\mu(A \cup B|C) = \mu(A|C) + \mu(B|C)$, if $A \cap B = \emptyset, C \neq \emptyset$
- $\mu(A \cap B|C) = \mu(A|B \cap C) \cdot \mu(B|C)$

μ is known as the discrete Popper function. Since S is assumed to be finite, the function μ is completely characterized by the behavior on pairs of states $(s, t)_\mu$ (reads the *priority degree* of s with respect to t) defined by:

$$(s, t)_\mu := \mu(\{s\}|\{s, t\})$$

According to [2], $\mu(A|B) = 1$ can be interpreted as A is almost certain given B, and $\mu(A|B) = 0$ can be interpreted as A is almost impossible given B.

Given a probabilistic distribution characterised by a discrete probabilistic space, it is natural to assume an agent has a conditional belief $B^P Q$ (reads "believe Q conditional on P") whenever the possibility of Q is almost certain given P. In other words, $B^P Q$ iff $\mu(Q|P) = 1$, as proposed in [2].

Inspired by [5], [2] presents a qualitative description of a discrete conditional probabilistic space which only concern those probabilities that are equal to 1 or 0 (which is enough for the evaluation of beliefs). The qualitative model is based on a notion called priority relation which is defined as:

$$s \leq_\mu t \quad \text{iff} \quad \mu(t|\{s, t\}) \neq 0 \tag{*}$$

[3] $\wp(S)$ is the power set of S.

Just as the priority degree completely characterizes a discrete Popper function μ on a finite space, the priority degree qualitatively characterizes the space in the sense that, for any pair of propositions, it tells whether the conditional probability is 1 or 0, as shown by the following theorem: ([1])

Theorem 1. *Let (S, μ) be a discrete conditional probabilistic space and $\leq_\mu$ be the priority relation derived from μ according to (*). For any $X, Y \in \wp(S)$, $\mu(X|Y) = 1$ iff all states in X is minimal with respect to $\leq_\mu$ in Y.*

Although the qualitative description only encodes a small part of the information carried by the quantitative model of probabilistic distribution, yet it is still powerful in the analysis of many important notions. For instance, it characterizes (conditional) belief in the following way: An agent believes P conditional on Q if and only if $\min_\leq P \subset Q$, where $\min_\leq P$ is the set of states that is minimal among the P-states with respect to $\leq_\mu$. Unconditional belief can be defined as belief conditional on tautology. Baltag and Smets show that such a notion of belief based on discrete conditional probabilistic space can be characterized by conditional doxastic logic (CDL) in [1].

3 The Qualitative Logical Framework for Probabilistic Causal Reasoning

3.1 Embedding the plausibility relation into a causal model

In order to provide a qualitative account of probabilistic causal reasoning, I will embed the qualitative representation of probability proposed in [2] into a causal structure described by structural equation functions of variables.

Given a set of causal variables $\mathcal{U} \cup \mathcal{V}$, the probabilistic distribution over variables can be represented by a probabilistic distribution over assignments. An assignment $\mathcal{A}$ to $\mathcal{U} \cup \mathcal{V}$ is a function from $\mathcal{U} \cup \mathcal{V}$ to Σ, and for each $X \in \mathcal{U} \cup \mathcal{V}$, $\mathcal{A}(X)$ refers to the value assigned by $\mathcal{A}$. The probabilistic distribution over the value of variables can be defined in terms of a probabilistic space (S, μ) whose states are all possible assignments to $\mathcal{U} \cup \mathcal{V}$, and the probability of $X = x$ conditional on $Y = y$ can be defined as the value of $\mu(\{\mathcal{A} \in S|\mathcal{A}(X) = x\} \mid \{\mathcal{A} \in S|\mathcal{A}(Y) = y\})$ in a conditional probabilistic space. Particularly, in a causal scenario, S are those assignments that comply with the causal rules. For convenience, I denote the set of all possible assignments that complies with a set of structural functions $\mathcal{F}$ by $W^{\mathcal{F}}$, which is formally defined as $\{\mathcal{A} \in \Sigma^{\mathcal{U} \cup \mathcal{V}} \mid \forall X \in \mathcal{V}, \mathcal{A}(X) = f_X((\mathcal{A})^{-X})\}$.[4]

Then, by applying the approach in [2], such a conditional probabilistic space $(W^{\mathcal{F}}, \mu)$ can be qualitatively represented by a plausibility model $\langle W^{\mathcal{F}}, \leq_\mu \rangle$ where the priority relation qualitatively represents μ by the bi-condition: $\mu(X|Y) = 1$ iff all states in X is minimal with respect to $\leq_\mu$ in Y.

[4] I denote the sub-assignment of $\mathcal{A}$ to $\mathcal{U} \cup \mathcal{V} \setminus \{X\}$ by $(\mathcal{A})^{-X}$, if $\mathcal{A}$ is an assignment to $\mathcal{U} \cup \mathcal{V}$.

Therefore I propose the following model as a combination of the structural equation causal model and plausibility model.

Definition 2. A causal plausibility model is a tuple $M = \langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A} \rangle$

- $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \Sigma)$ is the signature where $\mathcal{U}$ is the set of exogenous variables, $\mathcal{V}$ is the set of endogenous variables, Σ is the range of those variables.
- $\mathcal{F}$ is the set of structural functions, which is assumed to be recursive.
- $\leq$ is a total order over $W^{\mathcal{F}}$ (namely all possible assignments that complies with $\mathcal{F}$).
- $\mathcal{A}$ is an assignment to $\mathcal{U} \cup \mathcal{V}$ that complies with $\mathcal{F}$, namely $\mathcal{A} \in W^{\mathcal{F}}$

3.2 Syntax and semantics based on the new model

The formal language to talk about a causal plausibility model based on the signature $\mathcal{S}$ is defined as below.

Definition 3 (Language for Qualitative Probabilistic Causal Reasoning)**.** Let $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \Sigma)$, formulas φ of the language $\mathcal{L}(\mathcal{S})$ are given by[5]

$$\varphi ::= X{=}x \mid \neg\varphi \mid \varphi \wedge \varphi \mid B^{\psi}\phi \mid [\overrightarrow{V} = \overrightarrow{v}]\varphi$$

where $X \in \mathcal{U} \cup \mathcal{V}$, $x \in \Sigma$ and $\overrightarrow{V} = \overrightarrow{v}$ is a sequence of the form $V_1 = v_1, \ldots, V_n = v_n$ where $\overrightarrow{V} \in \mathcal{V}$.[6]

The language $\mathcal{L}(\mathcal{S})$ is a combination of the language of conditional doxastic logic and the logic for causal reasoning. It not only contains the doxastic operator B (which stands for belief) but also has the intervention operator $[\overrightarrow{X} = \overrightarrow{x}]$ which stands for the antecedent of a counterfactual.

I will define the semantics of counterfactual based on the extended model under the classical interventionist interpretation: a counterfactual $[\overrightarrow{X} = \overrightarrow{x}]\phi$ holds on a model M whenever ϕ holds on the model $M_{\overrightarrow{X}=\overrightarrow{x}}$ which results from setting the value of $\overrightarrow{X}$ to $\overrightarrow{x}$.

Therefore, I will first define the notion of intervention on the causal plausibility model, as below.

Definition 4 (Intervention)**.** Let $M = \langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A} \rangle$ be a causal plausibility model; let $\overrightarrow{X}{=}\overrightarrow{x}$ be a (possibly partial) assignment on the endogenous variables. The causal plausibility model $M_{\overrightarrow{X}=\overrightarrow{x}} = \langle \mathcal{S}, \mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}, \leq_{\overrightarrow{X}=\overrightarrow{x}} \mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}} \rangle$, resulting from an inter-

[5] $B\phi$ is seen as the abbreviation of $B^{\top}\phi$.

[6] For convenience, I will write both $V_1 = v_1, \ldots, V_n = v_n$ and $V_1 = v_1 \wedge \cdots \wedge V_n = v_n$ as $\overrightarrow{V} = \overrightarrow{v}$.

vention setting the values of $\overrightarrow{X}$ to $\overrightarrow{x}$ (where $\overrightarrow{X}$ are endogenous variables), is defined as follows:

- the functions in $\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}} = \{f'_V \mid V \in \mathcal{V}\}$ are such that: for each V *not in* $\overrightarrow{X}$, the function f'_V is exactly as f_V, and for each $V{=}X_i \in \overrightarrow{X}$, the function f'_{X_i} is a *constant* function returning the value $x_i \in \overrightarrow{x}$ regardless of the values of all other variables.
- $\mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}}$ is the unique solution to $\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}$ whose assignment to exogenous variables is identical with $\mathcal{A}$.[7] Formally, $\mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}}(Y)$ is the unique assignment that satisfies the following equations:

$$\mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}}(Y) = \begin{cases} \mathcal{A}(Y) & \text{if } Y \in \mathcal{U} \\ f'_Y((\mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}})^{-Y}) & \text{if } Y \in \mathcal{V}. \end{cases}$$

- $\leq_{\overrightarrow{X}=\overrightarrow{x}} \subset W^{\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}} \times W^{\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}}$ is the unique assignment satisfying $\mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}} \leq_{\overrightarrow{X}=\overrightarrow{x}} \mathcal{A}'^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}}$ whenever $\mathcal{A} \leq \mathcal{A}^{\mathcal{F}}$. [8]

This definition of causal model results from an intervention is exactly in line with the traditional definition of intervention developed in [6] and [3].

Based on the definition of intervention on causal plausibility models, the semantics of $\mathcal{L}(\mathcal{S})$ is given as below:

Definition 5. Semantics of $\mathcal{L}(\mathcal{S})$

- $\langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A}\rangle \models X = x$ iff $\mathcal{A}(X) = x$
- $\langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A}\rangle \models B^{\psi}\phi$ iff $\min_{\leq} ||\psi|| \subset ||\phi||$, where $||\phi|| := \{\mathcal{A}' \in W^{\mathcal{F}} \mid \langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A}'\rangle \models \phi\}$
- $\langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A}\rangle \models [\overrightarrow{X} = \overrightarrow{x}]\phi$ iff $\langle \mathcal{S}, \mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}, \leq_{\overrightarrow{X}=\overrightarrow{x}} \mathcal{A}^{\mathcal{F}}_{\overrightarrow{X}=\overrightarrow{x}}\rangle \models \phi$
- the Boolean connectives are defined in the usual way.

Given the semantics above, a logic for qualitative probabilistic causal reasoning can be derived from the syntax and semantics above. I denote all the valid formulas of $\mathcal{L}(\mathcal{S})$ by $L_{CP}(\mathcal{S})$. In this paper, I will not discuss the axiomatization of $L_{CP}(\mathcal{S})$. The idea of building the axiom system seems to be a combination of all the axioms in conditional doxastic logic as in [2] and the logic of counterfactuals in [6]. However, such a logic system is not just a simple sum of the axioms of CDL and the logic of counterfactuals. For instance, $\overrightarrow{X}{=}\overrightarrow{x} \wedge \phi \to B^{\overrightarrow{X}=\overrightarrow{x}}\phi$ (if $\{\overrightarrow{X}\} = \mathcal{U} \cup \mathcal{V}$) and $B[\overrightarrow{X}{=}\overrightarrow{x}]\phi \to [\overrightarrow{X}{=}\overrightarrow{x}]B\phi$ are both valid according to my semantics while it cannot

[7] Since $\mathcal{F}$ is recursive, $\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}$ is also recursive. Thus $\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}$ has unique solution with respect to each setting of exogenous variables.

[8] $\leq_{\overrightarrow{X}=\overrightarrow{x}}$ is well-defined because $\mathcal{F}$ is recursive, so that there is a bijective mapping from $W^{\mathcal{F}}$ and $W^{\mathcal{F}_{\overrightarrow{X}=\overrightarrow{x}}}$.

be derived simply using the axioms of CDL and the logic of counterfactuals. Therefore the axiomatization for $L_{CP}(\mathcal{S})$ is still an open question worth investigation in the future.

4 Further Discussion

4.1 Qualitative independence

In probability theory, the quantitative representation of (conditional) probability gives rise to a quantitative notion of (conditional) independence. Let μ be a probabilistic distribution and X, Y be two variables, the independence between X and Y can be defined in terms of the equation between unconditional probability and conditional probability, that is: X is independent of Y with respect to μ whenever for any value x, y, $\mu(X{=}x){=}\mu(X{=}x \mid Y{=}y)$, as far as $\mu(Y{=}y) \neq 0$.

Similarly, based on my causal plausibility model, the qualitative independence can be defined in terms of the equivalence between unconditional belief and conditional belief, that is: X is qualitatively independent of Y with respect to M whenever for any value x, y, $M \models B(X{=}x) \Leftrightarrow M \models B^{Y=y}(X{=}x)$, as far as $M \models \neg B^{Y=y}\bot$.

We can also generalize this idea to conditional independence. X is probabilistically independent of Y conditional on Z with respect to μ, can be defined by: For any x, y, z, $\mu(X{=}x \mid Y{=}y, Z{=}z) = \mu(X{=}x \mid Z{=}z)$ as far as $\mu(Y{=}y, Z{=}z) \neq 0$. Therefore, X is qualitatively independent of Y conditional on Z with respect to M can be defined by: $M \models B^{Y=y,Z=z}(X{=}x) \leftrightarrow B^{Y=y}(X{=}x)$ as far as $M \models \neg B^{Y=y,Z=z}\bot$.

Therefore, the syntax and semantics given in this paper enable us to derive a qualitative notion of independence in the same way as the quantitative notion of independence is derived from the probability.

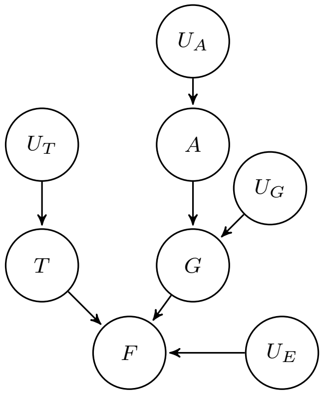
4.2 Strengthen the connection between causality and probability in the model

The matching between causal structure and probability is an important aspect in the studies of causality. For instance the Markovian relativity and the “d-separation” criteria developed in [13] are core concepts in the study of causal Bayesian networks.

So far as I define the causal plausibility model in Section 3, there is only a weak connection between causality and probability: that is, only states that comply with the causal rules are considered as possible. It guarantees some obvious connection between causality and probability, for instance if $\overrightarrow{X}$ are all those variables other than Y, then $f_Y(\overrightarrow{X}{=}\overrightarrow{x}){=}y$ implies $B^{\overrightarrow{X}=\overrightarrow{x}}Y{=}y$ (which means the probability of $Y{=}y$ conditional on $\overrightarrow{X}{=}\overrightarrow{x}$ is 1). However such a connection is still weak that is unable to derive certain features that are intuitively right in the process of causal reasoning, as in the following example.

Example 1. John needs to catch up a flight at the airport tomorrow morning. If John gets up late or there is a traffic jam on the road to the airport next morning, then it is very probable that John will miss the flight. If John does not set any alarm clock in advance, then John probably gets up late.

The causal structure of this example can be graphically represented as below:



- f_T: $T=1$ iff $U_T=1$
- f_A: $A=1$ iff $U_A=1$
- f_G: $G=1$ iff $A=0$ and $U_G=1$
- f_F: $F=1$ iff $T=1 \vee G=1$ and $U_E=1$

In this scenario, the following statements are intuitively right:

- (i) Given that John gets up early or not, the probability/belief of John missing the flight is independent of whether John sets up any alarm clock in advance.
- (ii) The probability/belief of there is a traffic jam on the road is independent of whether John sets up any alarm clock in advance.

As the qualitative independence can be expressed by $\mathcal{L}(\mathcal{S})$, if the causal reasoning of an agent about this scenario is properly modelled by a causal plausibility model M, then this properties should be reflected by the formulas of this language. For instance, according to the analysis in section 4.1, property (i) can be expressed by the truth of the following formula:

$$\bigwedge_{f,g,a\in\Sigma}(B^{G=g,A=a}F=f \leftrightarrow B^{G=g}F=f)$$

From the perspective of causal Bayesian nets, feature (i) and (ii) is guaranteed by the d-separation criteria: if assume the probabilistic distribution to be Markovian relative to the causal graph, then A and F must be independent conditional on G as they are d-separated by $\{G\}$. For the same reason, T and A are independent as they are d-separated by $\emptyset$. Unfortunately, Definition 4 is unable to guarantee such kind of properties in a qualitative sense: for instance, it could be the case that for some $\leq$ over $W^{\mathcal{F}}$, those most plausible $G{=}0 \wedge A{=}1$-worlds are all $F{=}1$ worlds while some most plausible $G{=}0$-worlds are $F{=}0$-worlds, so that $M \models B^{G=0,A=1}F{=}1$ but $M \models \neg B^{G=0}F{=}1$. This means an agent changes her mind about F by learning from A even if the information about G is given. This is not inconsistent, of course, if we take $\leq$ as some qualitative representation of the subjective belief about the external world. However if we want the agent to be more rational, then the possibility of such kind of belief revision should be ruled out.

In order to make $\leq$ more objective, $\leq$ should not be too arbitrary. Therefore some proper restrictions should be imposed on $\leq$ with respect to $\mathcal{F}$. For instance, a natural restriction is that all exogenous variables should be independent.

Definition 6. The plausibility ordering in $\langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A} \rangle$ **keeps the independence of exogenous variables** when the following property holds: For any $\mathcal{A}_1, \mathcal{A}_2 \in W^{\mathcal{F}}$ and $\overrightarrow{U} \in \mathcal{U}, \overrightarrow{U^-} = \mathcal{U} \setminus \{\overrightarrow{U}\}$, if $\mathcal{A}_1 \leq \mathcal{A}_2$ $\mathcal{A}_1(\overrightarrow{U}^-) = \mathcal{A}_2(\overrightarrow{U}^-)$, then for any $\mathcal{A}'_1$ and $\mathcal{A}'_2$, $\mathcal{A}'_1(\overrightarrow{U}) = \mathcal{A}_1(\overrightarrow{U})$, $\mathcal{A}'_2(\overrightarrow{U}) = \mathcal{A}_2(\overrightarrow{U})$ and $\mathcal{A}'_1(\overrightarrow{U}^-) = \mathcal{A}'_2(\overrightarrow{U}^-)$ implies $\mathcal{A}'_1 \leq \mathcal{A}'_2$.

If we assume the causal plausibility models modelling Example 1 must keep the independence of exogenous variables, then it exactly has the desired dependence and independence. In order to illustrate how this assumption works, let us take independence between $F{=}1$ and $A{=}1$ conditional on $G{=}0$ as an example:

$M \models B^{G=0 \wedge A=1} F{=}1$ iff $\min_{\leq} ||G{=}0 \wedge A{=}1|| \subset ||F{=}1||$[9]
iff $\min_{\leq} ||G{=}0 \wedge A{=}1|| \subset ||G{=}0 \wedge A{=}1 \wedge F{=}1||$
iff $\min_{\leq} ||U_A{=}1|| \subset ||U_A{=}1 \wedge U_T{=}1 \wedge U_E{=}1||$
iff $\min_{\leq} ||U_A{=}1 \vee U_G{=}0|| \subset ||(U_A{=}1 \vee U_G{=}0) \wedge U_T{=}1 \wedge U_E{=}1||$ (by the assumption that M keeps the independence of exogenous variable)
iff $\min_{\leq} ||G{=}0|| \subset ||G{=}0 \wedge F{=}1||$ iff $\min_{\leq} ||G{=}0|| \subset ||F{=}1||$
iff $B^{G=0} F{=}1$

This result fits our intuition: given John does not get up late, then whether the alarm is set will be irrelevant to whether John misses the flight.

Therefore, based on the same syntax and semantics as in section 3.2, the logic with respect to causal plausibility models that keep the independence of exogenous variables is also useful in the qualitative analysis of dependence and independence.

5 Conclusion

In this paper, I proposed a qualitative model of probabilistic causal reasoning called causal plausibility model. Based on the method of [2], the plausibility relation in this model can be seen as a qualitative representation of the probabilistic distribution over causal variables. I also proposed a syntax and semantics to describe the model from a logical perspective. In addition, the formal language is able to express a qualitative notion of dependence among causal variables in terms of epistemic operators. I also showed that by imposing certain restrictions on the plausibility ordering of the causal plausibility model, the qualitative logical framework can also be used to analyse the dependence and independence among variables in a causal graph.

[9] $||\phi|| = \{\mathcal{A} \in W^{\mathcal{F}} | \langle \mathcal{S}, \mathcal{F}, \leq, \mathcal{A} \rangle \models \phi\}$.

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概率因果模型的质化逻辑分析

谢凯博

摘　要

因果推理往往需要处理不确定性。尽管传统的结构等式模型擅于处理确定因果结构中的推理，却不能直接用来刻画涉及概率的因果推理。当前一些学者试图将量化的概率表达式引入到关于因果性的形式语言中，从而描述概率因果推理。有别于这种量化研究路径，本文提出一种关于概率因果推理的质化模型，通过认知关系来表达变元的不确定性。此外，基于该模型的形式语言能够在质化的意义上表达变元间的独立关系，有助于进一步研究因果与概率之间的联系。

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