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Modal Logic of Multivalued Frames over Inversely Well-Ordered Sets*

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Abstract. Multivalued frames generalize Kripke frames via introducing a set of values for pairs of states. A set of values Q is supposed to be an inversely well-ordered set. The intended multimodal language is interpreted in models based on multivalued frames over Q. Goldblatt-Thomason theorems for certain classes of Q-frames are established. Normal Q-modal logics are introduced, and some completeness results are naturally given by adjusting the canonical model method. Makinson's classification theorem as well as some logical properties are established for normal Q-modal logics.

1 Introduction

Multimodal logic is a name for a bunch of modal logics which formalize reasoning about multiple modalities. The following passage from Dana Scott is often quoted when the significance of multimodal logic is concerned:

> Here is what I consider one of the biggest mistakes of all in modal logic: concentration on a system with just one modal operator. The only way to have any philosophically significant results in deontic logic or epistemic logic is to combine those operators with: tense operators (otherwise how can you formulate principles of change?); the logical operators (otherwise how can you compare the relative with the absolute?); the operators like historical or physical necessity (otherwise how can you relate the agent to his environment?); and so on and so on. ([11], p. 161)

Scott emphasizes many faces of a philosophical modality. For instance, the notion of belief should be combined with tense if the change or update of an agent's belief is concerned. Although the combination is needed for some philosophical purposes, the exact way of combination which represents the intrinsic corelation between modalities has not been well-explored yet.

A direct response to Scott's remark is the development of multimodal logic in the study of reasoning in multiagent systems. Multimodal logics have been widely studied in the literature (e.g., [2, 3]). For example, the multiagent epistemic logic

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requires a knowledge operator K_a for each agent a (e.g., [13]). A typical example is the multimodal logic S5 as the standard multiagent epistemic logic. In an idealistic multiagent model, agents are usually assumed to be independent, i.e., every agent makes reasoning in situations without disturbance from other agents. However, if agents are organized or structured in a certain way, the corresponding modalities in logic must be connected in the same way. A sort of combination is shown in propositional dynamic logic where each program π is assigned with a modality $[\pi]$, and the composition, choice and repetition of programs are presented by interaction axioms between modalities (e.g., [9]).

From the semantic perspective, relational semantics for modal logic has been well-explored. A Kripke frame ('K-frame' for short) for a monomodal language is a pair (W, R) where W is a non-empty set of states, and R is a binary relation on W. Each modal formula $\Box\varphi$ is true at a state w if and only if φ is true at all R-accessible states of w (e.g., [1, 2]). An accessibility relation R is indeed a *bivalent* function $R : W \times W \to \{0, 1\}$. In the present work, we generalize Kripke frames by changing the set $\{0, 1\}$ into a set of values Q and obtain *multivalued* frames. This leads to a general framework for the investigation of multimodal logics. One can impose additional structure on Q, and study the modal logics of these special class of frames.

One should mention that two sorts of semantics for many-valued modal logic are given by Fitting [6, 7]. Let T be a finite distributive lattice the elements of which are accounted as values. Every formula φ will take a value $V(w, \varphi)$ at the state w in a model. In Fitting's second version, a model is a triple (W, R, V) where $W \neq \varnothing$ and R is a multivalued relation on W, i.e., a function $R : W \times W \to T$. The value $V(w, \Box\varphi)$ is defined as $\bigwedge\{R(w, u) \Rightarrow v(u, \varphi) : u \in W\}$ where $\Rightarrow$ is a relative pseudo-complement implication. Fitting's many-valued modal logic is a monomodal many-valued logic, and the value set of formulas coincides with that of relations in a frame.

In the present work, we assumes that the set of values Q is an inversely well-ordered (or dually well-ordered) set, i.e., every nonempty subset of Q has a maximal element. Formally, a partial Q-valued frame is a pair (W, σ) where $W \neq \varnothing$ is a set of states and $\sigma : W \times W \to Q$ is a partial function. If σ is a total function, one obtains total Q-valued frames. In general, $\sigma(w, u) = a$ (if exists) means that u is accessible from w by a. Each value $a \in Q$ can be interpreted in practical scenarios as an agent, and hence the relation $R^\sigma_a = \{\langle w, u\rangle : \sigma(w, u)$ exists and it is above a in $Q\}$ is used to interpret the modality $[a]$. A modal formula $[a]\varphi$ states that a *makes sure* that φ. In general, since the values in Q are ordered by $\leq$, one could require that $[b]\varphi$ holds if $a \leq b$ and $[a]\varphi$ holds. The agent with higher level has the ability to achieve what agents with lower level can do. Concrete scenarios where such frames can be used for modeling are not discussed in the present work. What we shall present contains a general framework for the study of multimodal logic and some related normal modal logics. This framework certainly differs from Fitting's many-valued modal logic.

This article is structured as follows. Section 2 gives the modal language and semantics where partial Q-valued frames are introduced. Section 3 introduces normal Q-modal logics and proves the completeness of the minimal normal Q-modal logic. Section 4 defines some model constructions and proves some preservation results. Section 5 presents a Goldblatt-Thomason theorem for characterizing the modal definability of certain classes of partial Q-frames by the duality between partial Q-valued frames and modal Q-algebras. Section 6 makes more observations on normal Q-modal logics and proves some general results. Section 7 gives concluding remarks.

2 Language and Semantics

The cardinal of a set X is denoted by $|X|$. In the present work, we assume the axiom of choice, and hence admit the well-ordering theorem, i.e., for every set X there exists a binary relation which well-orders X. An *inversely well-ordered set* ('*i.w.o* set' for short) is a pair $(Q, \leq)$ such that $\geq$ well-orders the nonempty set Q, i.e., $\leq$ is a linear order on Q such that every subset $\varnothing \neq X \subseteq Q$ has a maximal element $\bigvee X$. The minimal element of X is denoted by $\bigwedge X$ if it exists. Every *i.w.o* set Q has the top element 1. If Q is finite, every $a \in Q \setminus \{1\}$ has a unique proper successor a^*. A *downset* in an *i.w.o* set Q is a subset $X \subseteq Q$ such that $a \leq b \in X$ implies $a \in X$. An *upset* in Q is a subset $X \subseteq Q$ such that $a \in X$ and $a \leq b$ imply $b \in X$. Let $\downarrow X$ and $\uparrow X$ be the downset and upset in Q generated by X respectively.

Definition 1. Let Q be an *i.w.o* set. The multimodal language $\mathcal{L}_M(Q)$ consists of a denumerable set of propositional variables $\mathbb{P} = \{p_i : i < \omega\}$, connectives $\bot$ and $\to$, and unary modal operators $\{[a] : a \in Q\}$. The set of formulas $Fm(Q)$ is defined inductively as follows:

$$Fm(Q) \ni \varphi ::= p \mid \bot \mid (\varphi_1 \to \varphi_2) \mid [a]\varphi$$

where $p \in \mathbb{P}$ and $a \in Q$. Connectives $\top, \neg, \wedge, \vee$ and $\leftrightarrow$ are defined as usual. For every $a \in Q$, one defines $\langle a \rangle \varphi := \neg[a]\neg\varphi$. The *complexity* of a formula $\varphi \in Fm(Q)$, denoted by $\delta(\varphi)$, is defined inductively as follows:

$$\delta(p) = 0 = \delta(\bot); \delta(\varphi \to \psi) = max\{\delta(\varphi), \delta(\psi)\} + 1; \delta([a]\varphi) = \delta(\varphi) + 1.$$

A *substitution* is a function $s : \mathbb{P} \to Fm(Q)$. For every formula $\varphi \in Fm(Q)$, let φ^s be obtained from φ by the substitution s.

Definition 2. Let Q be an *i.w.o* set. A *multivalued frame over* Q ('Q-frame' for short) is a pair $\mathbb{F} = (W, \sigma)$ where $W \neq \varnothing$ is a set of states, and $\sigma : W \times W \to Q$ is a partial function from $W \times W$ to Q. An Q-frame $\mathbb{F} = (W, \sigma)$ is called *total*, if σ is total, i.e., every pair in $W \times W$ is defined.

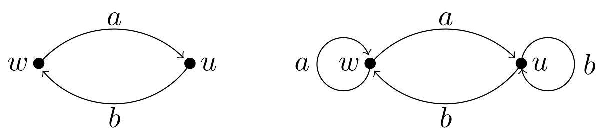
Let $\mathbb{F} = (W, \sigma)$ be a Q-frame. The notation $\sigma(w, u)!$ means that $\sigma(w, u)$ exists in Q. One writes $\sigma(w, u)! \geq a$ if $\sigma(w, u)!$ and $\sigma(w, u) \geq a$. For every $a \in Q$, the binary relation R_a^σ on W is defined as follows:

$$wR_a^\sigma u \text{ if and only if } \sigma(w,u)! \geq a.$$

Let $R_a^\sigma(w) = \{u \in W : \sigma(w,u)! \geq a\}$. Let $\mathcal{F}_Q$ be the class of all Q-frames.

A *valuation* in a Q-frame $\mathbb{F} = (W, \sigma)$ is a function $V : \mathbb{P} \to \mathcal{P}(W)$ from $\mathbb{P}$ to the powerset of W. A *Q-model* is a triple $\mathbb{M} = (W, \sigma, V)$ where (W, σ) is a Q-frame and V is a valuation in (W, σ).

Example 1. Consider the following Q-frames $\mathbb{F}_1$ and $\mathbb{F}_2$:



$\mathbb{F}_1$ $\mathbb{F}_2$

Here $\mathbb{F}_1$ is not total since $\sigma(w,w)$ and $\sigma(u,u)$ are undefined. But $\mathbb{F}_2$ is total.

Definition 3. Let $\mathbb{F} = (W, \sigma)$ be a Q-frame, $\mathbb{M} = (W, \sigma, V)$ a Q-model and $w \in W$. For every $\varphi \in Fm(Q)$, the *satisfaction relation* $\mathbb{M}, w \models \varphi$ is defined inductively as follows:

(1) $\mathbb{M}, w \models p$ if and only if $w \in V(p)$.
(2) $\mathbb{M}, w \not\models \bot$.
(3) $\mathbb{M}, w \models \varphi \to \psi$ if and only if $\mathbb{M}, w \not\models \varphi$ or $\mathbb{M}, w \models \psi$.
(4) $\mathbb{M}, w \models [a]\varphi$ if and only if $\mathbb{M}, u \models \varphi$ for all u such that $\sigma(w,u)! \geq a$.

Let $V(\varphi) = \{w \in W : \mathbb{M}, w \models \varphi\}$. A formula φ is *true* in $\mathbb{M}$, notation $\mathbb{M} \models \varphi$, if $V(\varphi) = W$. A formula φ is *valid* at w in $\mathbb{F} = (W, \sigma)$, notation $\mathbb{F}, w \models \varphi$, if $\mathbb{F}, V, w \models \varphi$ for every valuation V in $\mathbb{F}$. A formula φ is *valid* in $\mathbb{F}$, notation $\mathbb{F} \models \varphi$, if $\mathbb{F}, w \models \varphi$ for every $w \in W$. A formula φ is *valid* in a class of Q-frames $\mathcal{K}$, notation $\mathcal{K} \models \varphi$, if $\mathbb{F} \models \varphi$ for every $\mathbb{F} \in \mathcal{K}$.

Let $\Gamma \subseteq Fm(Q)$ be a set of formulas. Let $\mathbb{S} \models \Gamma$ stand for that $\mathbb{S} \models \varphi$ for all $\varphi \in \Gamma$. The class of all Q-frames defined by Γ is denoted by $\mathrm{Fr}_Q(\Gamma) = \{\mathbb{F} : \mathbb{F} \models \Gamma\}$. If $\Gamma = \{\varphi\}$, one writes $\mathrm{Fr}_Q(\varphi)$. The *modal theory* of a class of Q-frames $\mathcal{K}$ is defined as the set $\mathrm{Th}(\mathcal{K}) = \{\varphi \in Fm(Q) : \mathcal{K} \models \varphi\}$. We say that $\mathcal{K}$ is *modally Q-definable*, if $\mathcal{K} = \mathrm{Fr}_Q(\mathrm{Th}(\mathcal{K}))$.

Example 2. One can easily show that $[a]p \leftrightarrow [b]p$ is not valid in $\mathcal{F}_Q$ if $a \neq b$. Assume $a \neq b$. Without loss of generality, let $a > b$. Let $\mathbb{M} = (W, \sigma, V)$ be the Q-model where $W = \{w, u, v\}$, $\sigma(w,u) = a$ and $\sigma(w,v) = b$, and $V(p) = \{u\}$. Then $\mathbb{M}, w \models [a]p$ and $\mathbb{M}, w \not\models [b]p$. It follows that $\mathbb{M} \not\models [a]p \leftrightarrow [b]p$. However, by $b \leq a$, one has $\mathcal{F}_Q \models [b]p \to [a]p$. For every Q-frame $\mathbb{F} = (W, \sigma)$, $\mathbb{F} \models [a]p \to [b]p$ if and only if $\forall w, u \in W(\sigma(w,u)! \geq b \Rightarrow \sigma(w,u)! \geq a)$.

3 Normal Q-modal Logics and Completeness

In this section, we introduce normal Q-modal logics. As expected, the canonical method is applied in showing the completeness of the minimal normal Q-modal logic.

Definition 4. A *normal Q-modal logic* is a set of formulas $L \subseteq Fm(Q)$ such that L contains the following formulas:

(Tau) All instances of classical propositional tautologies.
(K_a) $[a](p \to q) \to ([a]p \to [a]q)$.
(C_Q) $[a]p \to [b]p$, where $a \leq b$ in Q.

and L is closed under the following rules:

(MP) if $\varphi, \varphi \to \psi \in L$, then $\psi \in L$.
(Gen) if $\varphi \in L$, then $[a]\varphi \in L$.
(Sub) if $\varphi \in L$, then $\varphi^s \in L$ for every substitution s.

A formula φ is a *theorem* of L, notation $\vdash_L \varphi$, if $\varphi \in L$.

For every family of normal Q-modal logics $\{L_i : i \in I\}$, $\bigcap_{i\in I} L_i$ is a normal Q-modal logic. The *minimal* normal Q-modal logic is denoted by K_Q. Let $\bigoplus_{i\in I} L_i$ be the smallest normal Q-modal logic containing $\bigcup_{i\in I} L_i$. For every set of formulas Σ, let $\mathrm{K}_Q \oplus \Sigma = \bigcap\{L : \Sigma \subseteq L\}$, the minimal normal Q-modal logic containing Σ. If $\Sigma = \{\varphi\}$, we write $\mathrm{K}_Q \oplus \varphi$ instead of $\mathrm{K}_Q \oplus \{\varphi\}$. For every normal Q-modal logic L, let $\mathrm{NExt}(L)$ be the set of all normal Q-modal logics containing L.

Remark 1. If $|Q| = 1$, K_Q is exactly the standard monomodal logic with a single modality $\Box$ (e.g., [2]). Notions for basic normal modal logic can be applied. If $|Q| > 1$, one obtains multimodal logics with respect to Q. Moreover, the following hold for every normal Q-modal logic L:

(1) $[a]\top \leftrightarrow \top \in L$ and $\langle a\rangle\bot \leftrightarrow \bot \in L$.
(2) $[a](\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow ([a]\varphi_1 \wedge \ldots [a]\varphi_n) \in L$.
(3) $\langle a\rangle(\varphi_1 \vee \ldots \vee \varphi_n) \leftrightarrow (\langle a\rangle\varphi_1 \vee \ldots \langle a\rangle\varphi_n) \in L$.
(4) $[a]\varphi \wedge \langle a\rangle\psi \to \langle a\rangle(\varphi \wedge \psi) \in L$.
(5) if $\varphi \to \psi \in L$, then $[a]\varphi \to [a]\psi \in L$ and $\langle a\rangle\varphi \to \langle a\rangle\psi \in L$.

Let L be a normal Q-modal logic. A formula φ is a *L-consequence* of a set of formulas Γ, notation $\Gamma \vdash_L \varphi$, if $\varphi \in L$ or there exist $\psi_1, \ldots, \psi_n \in \Gamma$ with $\psi_1 \wedge \ldots \wedge \psi_n \to \varphi \in L$. A set of formulas Γ is *L-consistent*, if $\Gamma \nvdash_L \bot$; and Γ is *maximal L-consistent*, if Γ is L-consistent and $\subseteq$-maximal. One obtains the deduction theorem and Lindenbaum-Tarski lemma for L: (i) $\Gamma, \varphi \vdash_L \psi$ if and only if $\Gamma \vdash_L \varphi \to \psi$; (ii) if Γ is L-consistent, there is a maximal L-consistent set Σ with $\Gamma \subseteq \Sigma$.

A normal Q-modal logic L is *complete*, if $L = \mathrm{Th}(\mathrm{Fr}_Q(L))$. One can obtain some completeness results using the canonical method.

Definition 5. Let W^L be the set of all maximal L-consistent sets of formulas. For every $a \in Q$, one defines $R_a^L \subseteq W^L \times W^L$ as follows:

$$\Sigma R_a^L \Theta \text{ if and only if } \varphi \in \Theta \text{ for all } [a]\varphi \in \Sigma.$$

For every pair $\langle \Sigma, \Theta \rangle \in W^L \times W^L$, one defines $X_Q^L(\Sigma, \Theta) = \{a \in Q : \Sigma R_a^L \Theta\}$. The *canonical Q-model* for L is defined as $\mathbb{M}^L = (W^L, \sigma^L, V^L)$ where

$$\sigma^L(\Sigma, \Theta) = \begin{cases} \bigvee X_Q^L(\Sigma, \Theta) & \text{if } X_Q^L(\Sigma, \Theta) \neq \varnothing. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

and $V^L(p) = \{\Sigma \in W^L : p \in \Sigma\}$ for every $p \in \mathbb{P}$. The *canonical Q-frame* for L is defined as $\mathbb{F}^L = (W^L, \sigma^L)$.

Lemma 1. *For every $\Sigma, \Theta \in W^L$ and $a \in Q$, the following hold:*

(1) $X_Q^L(\Sigma, \Theta)$ is a downset in Q.
(2) if $X_Q^L(\Sigma, \Theta) \neq \varnothing$, then $X_Q^L(\Sigma, \Theta) = \downarrow \bigvee X_Q(\Sigma, \Theta)$.
(3) $a \in X_Q^L(\Sigma, \Theta)$ if and only if $\sigma^L(\Sigma, \Theta)! \geq a$.

Proof. (1) Assume $a \leq b$ and $b \in X_Q^L(\Sigma, \Theta)$. Then $\Sigma R_b^L \Theta$. Suppose $[a]\varphi \in \Sigma$. By (C_Q), $[a]\varphi \to [b]\varphi \in L$. Hence $[b]\varphi \in \Sigma$. By $\Sigma R_b^L \Theta$, one obtains $\varphi \in \Theta$. It follows that $a \in X_Q^L(\Sigma, \Theta)$.

(2) Assume $X_Q^L(\Sigma, \Theta) \neq \varnothing$. Then $\bigvee X_Q^L(\Sigma, \Theta)$ is the maximal element of $X_Q^L(\Sigma, \Theta)$. By (1), $X_Q^L(\Sigma, \Theta) = \downarrow \bigvee X_Q^L(\Sigma, \Theta)$.

(3) Assume $a \in X_Q^L(\Sigma, \Theta)$. Then $\sigma^L(\Sigma, \Theta)! = \bigvee X_Q^L(\Sigma, \Theta) \geq a$. Assume $\sigma^L(\Sigma, \Theta)! \geq a$. Then $a \leq \bigvee X_Q^L(\Sigma, \Theta)$. By (2), $a \in X_Q^L(\Sigma, \Theta)$. □

Lemma 2. *For every $\Sigma \in W^L$, if $[a]\varphi \notin \Sigma$, there exists $\Theta \in W^L$ with $a \in X_Q^L(\Sigma, \Theta)$ and $\varphi \notin \Theta$.*

Proof. Assume $[a]\varphi \notin \Sigma$. Let $\Gamma = \{\psi : [a]\psi \in \Sigma\} \cup \{\neg\varphi\}$. Assume that Γ is not L-consistent. Then $\Gamma \vdash_L \bot$. There exist $\psi_1, \ldots, \psi_n \in \Gamma$ with $(\psi_1 \wedge \ldots \wedge \psi_n) \to \varphi \in L$. By (Gen), (K_a) and (MP), $[a](\psi_1 \wedge \ldots \wedge \psi_n) \to [a]\varphi \in L$. By $[a](\psi_1 \wedge \ldots \wedge \psi_n) \leftrightarrow ([a]\psi_1 \wedge \ldots [a]\psi_n) \in L$, one obtains $[a]\varphi \in L$. Then $[a]\varphi \in \Sigma$, which contradicts the assumption. Hence Γ is L-consistent. Let $\Gamma \subseteq \Theta \in W^L$. Then $a \in X_Q^L(\Sigma, \Theta)$ and $\varphi \notin \Theta$. □

Lemma 3. *For every $\Sigma \in W^L$, $\mathbb{M}^L, \Sigma \models \varphi$ if and only if $\varphi \in \Sigma$.*

Proof. The proof proceeds by induction on the complexity $\delta(\varphi)$. The atomic and Boolean cases are obvious. Let $\varphi = [a]\psi$. Assume $[a]\psi \in \Sigma$. Suppose $\sigma^L(\Sigma, \Theta)! \geq a$. By Lemma 1, $a \in X_Q^L(\Sigma, \Theta)$. Then $\Sigma R_a^L \Theta$. Hence $\varphi \in \Theta$. By induction hypothesis, $\mathbb{M}^L, \Theta \models \varphi$. Hence $\mathbb{M}^L, \Sigma \models [a]\varphi$. Assume $[a]\psi \notin \Sigma$. By Lemma 2, there exists $\Theta \in W^L$ with $a \in X_Q^L(\Sigma, \Theta)$ and $\varphi \notin \Theta$. Then $X_Q^L(\Sigma, \Theta) \neq \varnothing$ and $\sigma^L(\Sigma, \Theta)! = \bigvee X_Q^L(\Sigma, \Theta) \geq a$. By induction hypothesis, $\mathbb{M}^L, \Theta \not\models \varphi$. Hence $\mathbb{M}^L, \Sigma \not\models [a]\varphi$. □

Theorem 1. K_Q *is complete.*

Proof. Clearly $\mathrm{Fr}(\mathrm{K}_Q) = \mathcal{F}_Q$. Obviously $\mathrm{K}_Q \subseteq \mathrm{Th}(\mathcal{F}_Q)$. Assume $\varphi \notin \mathrm{K}_Q$. Then $\{\neg\varphi\}$ is K_Q-consistent. Let $\Sigma \in W^{\mathrm{K}_Q}$ with $\neg\varphi \in \Sigma$. By Lemma 3, $\mathbb{M}^L, \Sigma \not\models \varphi$. Hence $\varphi \notin \mathrm{Th}(\mathcal{F}_Q)$. □

4 Model Constructions and Preservation Results

As far as modal Q-definability of Q-frames concerned, one can define some interesting properties of Q-frames. For example, let Q be finite. For every $a < 1$, consider the property $\Phi(a)$: for every state w there exist u with $\sigma(w, u) = a$ and no v with $\sigma(w, v) > a$. Clearly $\Phi(a)$ is defined by the formula $\langle a \rangle \top \wedge [a^*]\bot$ where a^* is the successor of a in Q. For more general results on the modal Q-definability of frames in $\mathcal{L}_M(Q)$, one needs some preservation results on Q-frames.

Definition 6. The *disjoint union* of a family of Q-frames $\{\mathbb{F}_i = (W_i, \sigma_i) : i \in I\}$ is defined as $\biguplus_{i \in I} \mathbb{F}_i = (W, \sigma)$ where $W = \bigcup_{i \in I}(W_i \times \{i\})$ and $\sigma : W \times W \to Q$ is defined as follows:

$$\sigma(\langle w, i\rangle, \langle u, j\rangle) = \begin{cases} \sigma_i(w, u) & \text{if } w, u \in W_i \text{ for some } i \in I \text{ and } i = j. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The *disjoint union* of a family of Q-models $\{\mathbb{M}_i = (\mathbb{F}_i, V_i) : i \in I\}$ is defined as $\biguplus_{i \in I} \mathbb{M}_i = (\biguplus_{i \in I} \mathbb{F}_i, V)$ where $V(p) = \bigcup_{i \in I}(V_i(p) \times \{i\})$ for all $p \in \mathbb{P}$.

Proposition 1. *Let* $\{\mathbb{M}_i = (\mathbb{F}_i, V_i) : i \in I\}$ *be a family of disjoint Q-models where* $\mathbb{F}_i = (W_i, \sigma_i)$ *with* $i \in I$. *For every* $i \in I$, $w \in W_i$ *and* $\varphi \in Fm(Q)$, *(1)* $\mathbb{M}_i, w \models \varphi$ *if and only if* $\biguplus_{i \in I} \mathbb{M}_i, \langle w, i\rangle \models \varphi$; *and (2)* $\biguplus_{i \in I} \mathbb{F}_i \models \varphi$ *if and only if* $\mathbb{F}_i \models \varphi$ *for all* $i \in I$.

Proof. One obtains (1) immediately by induction on the complexity $\delta(\varphi)$. The proof is omitted. Obviously (2) follows from (1). □

By Proposition 1, every modally Q-definable class of partial Q-frames is closed under taking disjoint unions. It follows that the class of all total Q-frames is *not* modally Q-definable since it is certainly not closed under taking disjoint unions. The disjoint union of more than two total Q-frames (Q-models) must be partial and not total.

Definition 7. Let $\mathbb{F} = (W, \sigma)$ be a Q-frame. For every $\varnothing \neq X \subseteq W$, the *subframe* of $\mathbb{F}$ generated by X is the Q-frame $\mathbb{F}_X = (W_X, \sigma_X)$ where (i) W_X is the smallest subset of W containing X such that $\sigma(w, u)!$ implies $u \in W_X$ whenever $w \in W_X$ and $u \in W$; and (ii) $\sigma_X = \sigma \cap (W_X \times W_X)$. If $X = \{v\}$, one writes $\mathbb{F}_v = (W_v, \sigma_v)$. A Q-frame $\mathbb{F}' = (W', \sigma')$ is a *generated subframe* of $\mathbb{F}$, if $\mathbb{F}' = \mathbb{F}_X$ for some $\varnothing \neq X \subseteq W$. A Q-model $\mathbb{M}' = (\mathbb{F}', V')$ is a *generated submodel* of a Q-model $\mathbb{M} = (\mathbb{F}, V)$, if $\mathbb{F}$ is a generated subframe of $\mathbb{F}'$ and $V(p) = V'(p) \cap W$ for every $p \in \mathbb{P}$. One uses $\mathbb{S} \rightarrowtail \mathbb{S}'$ to denote that $\mathbb{S}$ is isomorphic to a generated substructure of $\mathbb{S}'$.

Proposition 2. *Let* $\mathbb{M} = (\mathbb{F}, V)$ *and* $\mathbb{M}' = (\mathbb{F}', V')$ *be Q-models. Assume* $\mathbb{M} \rightarrowtail \mathbb{M}'$. *For every* $w \in W$ *and formula* $\varphi \in Fm(Q)$, (1) $\mathbb{M}, w \models \varphi$ *if and only if* $\mathbb{M}', w \models \varphi$; *and* (2) *if* $\mathbb{F}' \models \varphi$, *then* $\mathbb{F} \models \varphi$.

Proof. One obtains (1) immediately by induction on the complexity $\delta(\varphi)$. The proof is omitted. Obviously (2) follows from (1). □

Definition 8. Let $\mathbb{F} = (W, \sigma)$ and $\mathbb{F}' = (W', \sigma')$ be Q-frames. A function $\eta : W \to W'$ is a *bounded morphism* from $\mathbb{F}$ to $\mathbb{F}'$, if the following conditions hold for all $w, u \in W$, $u' \in W'$ and $a \in Q$:

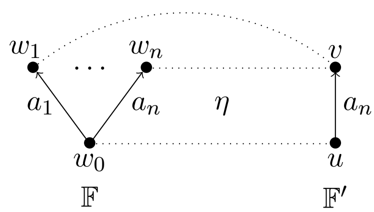
(1) if $\sigma(w, u)! \geq a$, then $\sigma(\eta(w), \eta(u))! \geq a$.
(2) if $\sigma'(\eta(w), u')! \geq a$, there exists $u \in W$ with $\sigma(w, u)! \geq a$ and $\eta(u) = u'$.

For Q-models $\mathbb{M} = (\mathbb{F}, V)$ and $\mathbb{M}' = (\mathbb{F}', V')$, a function $\eta : W \to W'$ is a *bounded morphism* from $\mathbb{M}$ to $\mathbb{M}'$, if η is a bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$ and the following condition holds for all $w \in W$ and $p \in \mathbb{P}$:

(3) $w \in V(p)$ if and only if $\eta(w) \in V'(p)$.

A Q-frame $\mathbb{F}'$ is called a *bounded morphic image* of $\mathbb{F}$, notation $\mathbb{F} \twoheadrightarrow \mathbb{F}'$, if there exists a surjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$.

Example 3. Let $\{a_1, \ldots, a_n\} \subseteq Q$ with $a_i \leq a_j$ for $1 \leq i \leq j \leq n$. Let $\mathbb{F} = (W, \sigma)$ and $\mathbb{F}' = (W', \sigma')$ be Q-frames where (i) $W = \{w_i : i \leq n\}$ and $\sigma(w_0, w_i) = a_i$; (ii) $W' = \{u, v\}$ and $\sigma'(u, v) = a_n$.



Let $\eta : W \to W'$ be the function with $\eta(w_0) = u$ and $\eta(w_i) = v$ for all $1 \leq i \leq n$. Note that $\sigma'(u, v) = a_n \geq a_i$ for all $1 \leq i \leq n$. It is quite easy to observe that η is a surjective bounded morphism from $\mathbb{F}$ to $\mathbb{F}'$.

Proposition 3. *Let $\mathbb{F} = (W, \sigma)$ and $\mathbb{F}' = (W', \sigma')$ be Q-frames, and $\mathbb{M} = (\mathbb{F}, V)$ and $\mathbb{M}' = (\mathbb{F}', V')$ be Q-models. Assume that $\eta : W \to W'$ is a bounded morphism from $\mathbb{M}$ to $\mathbb{M}'$. For every $w \in W$ and formula $\varphi \in Fm(Q)$, (1) $\mathbb{M}, w \models \varphi$ if and only if $\mathbb{M}', \eta(w) \models \varphi$; and (2) if η is surjective and $\mathbb{F} \models \varphi$, then $\mathbb{F}' \models \varphi$.*

Proof. One obtains (1) by induction on the complexity $\delta(\varphi)$. The proof is omitted. For (2), let η be surjective and $\mathbb{F}' \not\models \varphi$. Let $\mathbb{F}', V', w' \not\models \varphi$ for some valuation V' and w' in $\mathbb{F}'$. Let V be the valuation in $\mathbb{F}$ with $V(p) = \{u \in W : \eta(u) \in V'(p)\}$ for each $p \in \mathbb{P}$. Since η is surjective, there exists $w \in W$ with $\eta(w) = w'$. By (1), $\mathbb{F}, V, w \not\models \varphi$. Hence $\mathbb{F} \not\models \varphi$. □

Let $\mathbb{F} = (W, \sigma)$ be a Q-frame. The unary operation $\Diamond_a^\sigma$ on $\mathcal{P}(W)$ is defined by setting $\Diamond_a^\sigma Y = \{w \in W \mid R_a^\sigma(w) \cap Y \neq \varnothing\}$. For every $Y \subseteq W$, let $\overline{Y} = W \setminus Y$. One defines $\Box_a^\sigma Y = \overline{\Diamond_a^\sigma \overline{Y}} = \{w \in W \mid R_a^\sigma(w) \subseteq Y\}$.

Lemma 4. *Let $\mathbb{F} = (W, \sigma)$ be a Q-frame. For every $Y \subseteq W$ and $a, b \in Q$, if $a \leq b$, then $\Diamond_b^\sigma Y \subseteq \Diamond_a^\sigma Y$ and $\Box_a^\sigma Y \subseteq \Box_b^\sigma Y$.*

Proof. Assume $a \leq b$ and $w \in \Diamond_b^\sigma Y$. Let $u \in R_b^\sigma(w) \cap Y$. Then $u \in R_a^\sigma(w)$. Then $u \in \Diamond_a^\sigma Y$. Hence $\Diamond_b^\sigma Y \subseteq \Diamond_a^\sigma Y$. Similarly $\Box_a^\sigma Y \subseteq \Box_b^\sigma Y$. □

Let $\mathbb{F} = (W, \sigma)$ be a Q-frame and W^{ue} be the set of ultrafilters over W. A subset $T \subseteq \mathcal{P}(W)$ has the *finite intersection property* (FIP), if $Y_1, \ldots, Y_n \in T$ imply $Y_1 \cap \ldots \cap Y_n \neq \varnothing$. If T has the FIP, there exists $u \in W^{ue}$ with $T \subseteq u$.

Definition 9. Let $\mathbb{F} = (W, \sigma)$ be a Q-frame. For every $u, v \in W^{ue}$, let

$$X_Q^{ue}(u, v) = \{a \in Q : (\forall Y \in v) \Diamond_a^\sigma Y \in u\}.$$

The *ultrafilter extension* of $\mathbb{F}$ is defined as the Q-frame $\mathbb{F}^{ue} = (W^{ue}, \sigma^{ue})$ where $\sigma^{ue} : W^{ue} \times W^{ue} \to Q$ is defined as follows:

$$\sigma^{ue}(u, v) = \begin{cases} \bigvee X_Q^{ue}(u, v) & \text{if } X_Q^{ue}(u, v) \neq \varnothing. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The *ultrafilter extension* of a partial Q-model $\mathbb{M} = (\mathbb{F}, V)$ is defined as $\mathbb{M}^{ue} = (\mathbb{F}^{ue}, V^{ue})$ where $V^{ue}(p) = \{u \in W^{ue} : V(p) \in u\}$ for every $p \in \mathbb{P}$.

Proposition 4. *Let $\mathbb{F} = (W, \sigma)$ be a Q-frame and $\mathbb{M} = (\mathbb{F}, V)$ be a Q-model. For every $u \in W^{ue}$ and formula $\varphi \in Fm(Q)$,* (1) *$V(\varphi) \in u$ if and only if $\mathbb{M}^{ue}, u \models \varphi$; and* (2) *if $\mathbb{F}^{ue} \models \varphi$, then $\mathbb{F} \models \varphi$.*

Proof. (1) The proof proceeds by induction on $\delta(\varphi)$. Atomic and Boolean cases are obvious. Let $\varphi = \langle a \rangle \psi$. Assume $\mathbb{M}^{ue}, u \models \langle a \rangle \psi$. There exists $v \in W^{ue}$ with $\sigma^{ue}(u, v)! \geq a$ and $\mathbb{M}^{ue}, v \models \psi$. By induction hypothesis, $V(\psi) \in v$. Let

$\sigma^{ue}(u,v) = b \geq a$. Then $b \in X_Q^{ue}(u,v)$. Then $\Diamond_b^\sigma V(\psi) \in u$. By Lemma 4, $V(\langle a\rangle\psi) = \Diamond_a^\sigma V(\psi) \supseteq \Diamond_b^\sigma V(\psi)$. Hence $V(\langle a\rangle\psi) \in u$. Assume $V(\langle a\rangle\psi) \in u$. Let $v' = \{Y : \Box_a^\sigma Y \in u\}$. Clearly v' is closed under finite intersection. Suppose $Y \in v'$. Then $\Box_a^\sigma Y \in u$. Hence $\varnothing \neq \Box_a^\sigma Y \cap V(\langle a\rangle\psi) \in u$. Let $s \in \Box_a^\sigma Y \cap V(\langle a\rangle\psi) = \Box_a^\sigma Y \cap \Diamond_a^\sigma V(\psi) \subseteq \Diamond_a^\sigma(Y \cap V(\psi))$. Then there exists $t \in Y \cap V(\psi)$, i.e., $Y \cap V(\psi) \neq \varnothing$. Hence $v' \cup \{V(\psi)\}$ has the FIP. Let $v \in W^{ue}$ and $v' \cup \{V(\psi)\} \subseteq v$. By induction hypothesis, $\mathbb{M}^{ue}, v \models \psi$. Now we show $a \in X_Q^{ue}(u,v)$. Suppose not. Then $\Diamond_a^\sigma Z \notin u$ for some $Z \in v$. Then $\Box_a^\sigma \overline{Z} \in u$. Then $\overline{Z} \in v' \subseteq v$ which contradicts $Z \in v$. Hence $a \in X_Q^{ue}(u,v)$. Then $\sigma^{ue}(u,v)! \geq a$. Hence $\mathbb{M}^{ue}, u \models \langle a\rangle\psi$.

(2) Assume $\mathbb{F} \not\models \varphi$. There exists a valuation V in $\mathbb{F}$ and $w \in W$ with $w \notin V(\varphi)$. Let $\pi(w) = \{Y \subseteq W : w \in Y\}$. Clearly $\pi(w) \in W^{ue}$. Then $V(\varphi) \notin \pi(w)$. By (1), $\mathbb{M}^{ue}, \pi(w) \not\models \varphi$. Hence $\mathbb{F}^{ue} \not\models \varphi$. □

A class of Q-frames $\mathcal{K}$ *reflects ultrafilter extensions*, if $\mathbb{F}^{ue} \in \mathcal{K}$ implies $\mathbb{F} \in \mathcal{K}$. By Proposition 4, every modally Q-definable class of Q-frames reflects ultrafilter extensions.

5 Goldblatt-Thomason Theorems

The Goldblatt-Thomason theorem for modal logic (e.g., [2, 8]) applies the Birkhoff's variety theorem in universal algebra to modal logic. In this section, we first show such theorem holds for finite transitive Q-frames. Then we introduce modal Q-algebras for normal Q-modal logics, and establish a Goldblatt-Thomason theorem for inversely well-ordered modal logic via the duality between modal Q-algebras and Q-frames.

Definition 10. A Q-frame $\mathbb{F} = (W,\sigma)$ is *transitive*, if $\sigma(w,u)!$ and $\sigma(u,v)!$ imply $\sigma(w,v)!$ for all $w,u,v \in W$. Let $\mathcal{T}_Q^{<\omega}$ be the class of all finite transitive Q-frames. A transitive Q-frame $\mathbb{F} = (W,\sigma)$ is *rooted*, if there exists $w \in W$ with $\sigma(w,u)!$ for all $u \neq w$ in W. Such a state is called the *root* of $\mathbb{F}$.

Remark 2. Let Q be finite and $\bigwedge Q = a$. The class of all transitive Q-frames is defined by the formula $[a]p \to [a][a]p$. This is shown as follows. Let $\mathbb{F} = (W,\sigma)$ be a Q-frame. Assume that $\mathbb{F}$ is transitive. Let $\mathbb{M} = (\mathbb{F}, V)$ be a model. Suppose $\mathbb{M}, w \models [a]p$, $\sigma(w,u)! \geq a$ and $\sigma(u,v)! \geq a$. By the transitivity, $\sigma(w,v)! \geq a$. Then $\mathbb{M}, v \models p$. Hence $\mathbb{F} \models [a]p \to [a][a]p$. Now assume $\mathbb{F} \models [a]p \to [a][a]p$. Suppose $\sigma(w,u)!$ and $\sigma(u,v)!$. Let V be a valuation in $\mathbb{F}$ with $V(p) = R_a^\sigma(w)$. Then $\mathbb{M}, w \models [a]p$. Hence $\mathbb{M}, w \models [a][a]p$. Then $\mathbb{M}, v \models p$. Then $v \in R_a^\sigma(w)$, i.e., $\sigma(w,v)! \geq a$.

Let Q be finite. A Goldblatt-Thomason theorem within the class $\mathcal{T}_Q^{<\omega}$ can be established by the Jankov-Fine formula for a rooted finite transitive frame as in basic modal logic (e.g., [2, pp. 143–144]). Let $\mathbb{F} = (W,\sigma)$ be a finite transitive Q-frame with root $w = w_0$ and $W = \{w_0, \ldots, w_n\}$. Each index $i \leq n$ is associated with a variable p_i. Let $\bigwedge Q = a$ and $[a]^+\varphi = \varphi \wedge [a]\varphi$ for every formula φ. The *Jankov-Fine* formula $\varphi_{\mathbb{F},w}$ is defined as the conjunction of the following formulas:

$$p_0 \tag{I}$$

$$[a] \bigvee_{i \leq n} p_i \tag{II}$$

$$\bigwedge_{i \leq j \leq n} [a]^+(p_i \to \neg p_j) \tag{III}$$

$$\bigwedge_{w_j \in R_a^\sigma(w_i)} [a]^+(p_i \to \langle a \rangle p_j) \tag{IV}$$

$$\bigwedge_{w_j \notin R_a^\sigma(w_i)} [a]^+(p_i \to \neg\langle a \rangle p_j) \tag{V}$$

For every transitive Q-frame $\mathbb{G} = (G, \tau)$ and $v \in G$, let $\mathbb{G}_v = (G', \tau')$ be the Q-frame generated by v.

Lemma 5. *Let Q be finite. For every transitive Q-frame $\mathbb{G} = (G, \tau)$ and $v \in G$, there exists a valuation U in $\mathbb{G}$ with $\mathbb{G}, U, v \models \varphi_{\mathbb{F},w}$ if and only if there exists a surjective bounded morphism η from $\mathbb{G}_v$ to $\mathbb{F}$ with $\eta(v) = w$.*

Proof. Let $\bigwedge Q = a$ and $\mathbb{G}_v = (G', \tau')$. Assume that $\eta : G' \to W$ is a surjective bounded morphism from $\mathbb{G}_v$ to $\mathbb{F}$ with $\eta(v) = w$. Let V be a valuation in $\mathbb{F}$ with $V(p_i) = \{w_i\}$ for $i \leq n$. Let U be a valuation in $\mathbb{G}_v$ with $U(p_i) = \{x \in G' \mid \eta(x) = w_i\}$ for $i \leq n$, and $U(q) = V(q) = \varnothing$ for all $q \notin \{p_0, \ldots, p_n\}$. Clearly $x \in U(p_i)$ if and only if $\eta(x) \in V(p_i)$. Then η is a surjective bounded morphism from $(\mathbb{F}, V)$ to $(\mathbb{G}_v, U)$. Obviously $\mathbb{F}, V, w \models \varphi_{\mathbb{F},w}$. By Proposition 3, $\mathbb{G}_v, U, v \models \varphi_{\mathbb{F},w}$. By Proposition 2, $\mathbb{G}, U, v \models \varphi_{\mathbb{F},w}$. Now assume $\mathbb{G}, U, v \models \varphi_{\mathbb{F},w}$. By Proposition 2, $\mathbb{G}_v, U, v \models \varphi_{\mathbb{F},w}$. Let $\eta : G' \to W$ be defined by setting: $\eta(x) = w_i$ if and only if $x \in U(p_i)$. Now one shows that η is a surjective bounded morphism.

(1) Clearly $\eta(v) = w$ since $v \in U(p_0)$. For every $i > 0$, one has $\sigma(w_i, w_0) \geq a$. By the formula (IV) is true at v in $(\mathbb{G}_v, U)$, one has $\mathbb{G}_v, U, v \models \langle a \rangle p_i$. Hence there exists $x \in R_a^\sigma(v)$ with $x \in U(p_i)$, i.e., $\eta(x) = w_i$.

(2) Assume $x R_a^{\tau'} y$. Let $\eta(x) = w_i$ and $\eta(y) = w_j$. Then $x \in U(p_i)$ and $y \in U(p_j)$. Hence $\mathbb{G}_v, U, x \models \langle a \rangle p_j$. For a contradiction, suppose $w_j \notin R_a^\sigma(w_i)$. Since (V) is true at v in $(\mathbb{G}_v, U)$ and $x \in U(p_i)$, one has $\mathbb{G}_v, U, x \models \neg\langle a \rangle p_j$ which contradicts $\mathbb{G}_v, U, x \models \langle a \rangle p_j$.

(3) Assume $\eta(x) R_a^\sigma w_j$. Let $\eta(x) = w_i$. Then $x \in U(p_i)$. Since (IV) is true at v in $(\mathbb{G}_v, U)$, one has $\mathbb{G}_v, U, x \models \langle a \rangle p_j$. Then there exists $y \in G'$ with $x R_a^{\tau'} y$ and $y \in V(p_j)$, i.e., $\eta(y) = w_j$. □

Let Q be finite. We say that a class of finite transitive Q-frames $\mathcal{K}$ is *modally Q-definable* within $\mathcal{T}_Q^{<\omega}$, if $\mathcal{K} = \mathrm{Fr}_Q(\mathrm{Th}(\mathcal{K})) \cap \mathcal{T}_Q^{<\omega}$. Then we obtain the following Goldblatt-Thomason theorem.

Theorem 2. *Let Q be finite. A class of finite transitive Q-frames $\mathcal{K}$ is* modally Q-definable *within $\mathcal{T}_Q^{<\omega}$ if and only if $\mathcal{K}$ is closed under taking disjoint unions, generated subframes and bounded morphic images.*

Proof. The left-to-right direction follows from Proposition 1, Proposition 2 and Proposition 3. Conversely, assume that $\mathcal{K}$ is closed under taking disjoint unions, generated subframes and bounded morphic images. We show $\mathcal{K} = \mathrm{Fr}_Q(\mathrm{Th}(\mathcal{K})) \cap \mathcal{T}_Q^{<\omega}$. Obviously $\mathcal{K} \subseteq \mathrm{Fr}_Q(\mathrm{Th}(\mathcal{K})) \cap \mathcal{T}_Q^{<\omega}$. Assume that $\mathbb{F} \in \mathcal{T}_Q^{<\omega}$ and $\mathbb{F} \models \mathrm{Th}(\mathcal{K})$. Let $\mathbb{F} = (W, \sigma)$. One has two cases:

Case 1. Suppose $\mathbb{F}$ has root w. Obviously $\neg\varphi_{\mathbb{F},w} \notin Th(\mathcal{K})$. There exists $\mathbb{G} \in \mathcal{K}$ with $\mathbb{G} \not\models \neg\varphi_{\mathbb{F},w}$. By Lemma 5, there exists v in $\mathbb{G}$ such that $\mathbb{F}$ is a bounded morphic image of $\mathbb{G}_v$. Since $\mathbb{G} \in \mathcal{K}$, one has $\mathbb{G}_v \in \mathcal{K}$ and so $\mathbb{F} \in \mathcal{K}$.

Case 2. Suppose $\mathbb{F}$ is not rooted. Clearly $\mathbb{F}$ is a bounded morphic image of the disjoint union $\biguplus_{x\in W} \mathbb{F}_x$. By the proof of Case 1, one has $\mathbb{F}_x \in \mathcal{K}$ for every $x \in W$. It follows that $\mathbb{F} \in \mathcal{K}$. $\square$

Now, for a more general Glodblatt-Thomason theorem, let us introduce modal Q-algebras for an arbitrary *i.w.o* set Q and give some duality results.

Definition 11. Let Q be a *i.o.w* set. An algebra $\mathfrak{B} = (B, +, -, 0, \Diamond_a)_{a\in Q}$ is a *modal Q-algebra* ('Q-MA' for short), if $(B, +, -, 0)$ is a Boolean algebra and $\Diamond_a$ is a unary operator on B satisfying the following conditions:

(1) $\Diamond_a 0 = 0$.
(2) $\Diamond_a(x + y) = \Diamond_a x + \Diamond_a y$.
(3) $\Diamond_b x \leq \Diamond_a x$ if $a \leq b$.

One defines $x \cdot y = -(-x + -y)$, $\Box_a x := -\Diamond_a - x$ and $1 := -0$. One writes $(B, \Diamond_a)_{a\in Q}$ as a Q-MA where B is supposed to be Boolean.

Fact 3. Let $(B, \Diamond_a)_{a\in Q}$ be a Q-MA and $x, y \in B$. The following hold:

(1) if $x \leq y$, then $\Diamond_a x \leq \Diamond_a y$ and $\Box_a x \leq \Box_a y$.
(2) $\Box_a 1 = 1$ and $\Box_a(x \cdot y) = \Box_a x \cdot \Box_a y$.
(3) if $a \leq b$, then $\Box_a x \leq \Box_b x$.

Basic notions in Q-MA are defined as usual (e.g., [2]). Let $\mathfrak{B} = (B, \Diamond_a)_{a\in Q}$ and $\mathfrak{B}' = (B', \Diamond'_a)_{a\in Q}$ be Q-MAs. One writes (i) $\mathfrak{B} \cong \mathfrak{B}'$, if $\mathfrak{B}$ is isomorphic to $\mathfrak{B}'$; (ii) $\mathfrak{B} \rightarrowtail \mathfrak{B}'$, if $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{B}'$; and (iii) $\mathfrak{B} \twoheadrightarrow \mathfrak{B}'$, if $\mathfrak{B}'$ is a homomorphic image of $\mathfrak{B}$. The product of a family of Q-MAs $\{\mathfrak{B}_i = (B_i, \Diamond^i_a)_{a\in Q} : i \in I\}$ is denoted by $\prod_{i\in I} \mathfrak{B}_i = (\prod_{i\in I} B_i, \prod_{i\in I} \Diamond^i_a)_{a\in Q}$ where $\prod_{i\in I} \Diamond^i_a$ is the operation $\prod_{i\in I} \Diamond^i_a(x)(j) = \Diamond^j_a(x(j))$ with $x \in \prod_{i\in I} B_i$. Let **H**, **S** and **P** be class operations of homomorphism, subalgebra and product respectively. A class of Q-MAs $\mathcal{C}$ is a *variety*, if $\mathcal{C} = \mathbf{HSP}\mathcal{C}$.

Let $\mathbb{F} = (W, \sigma)$ be a Q-frame. The *complex algebra* of $\mathbb{F}$ is defined as $\mathbb{F}^+ = (\mathcal{P}(W), \cup, \overline{(.)}, \varnothing, \Diamond_a^\sigma)_{a \in Q}$. It is quite easy to observe that $\mathbb{F}^+$ is a Q-MA. For every class of Q-frames $\mathcal{K}$, let $\mathbf{Cm}\mathcal{K} = \{\mathbb{F}^+ : \mathbb{F} \in \mathcal{K}\}$.

Definition 12. Let $\mathfrak{B} = (B, \Diamond_a)_{a \in Q}$ be a finite Q-MA. An element $x \in B$ is called an *atom*, if $x \neq 0$, and $0 \leq y < x$ implies $y = 0$. Let W_B be the set of all atoms in $\mathfrak{B}$. For every $x, y \in W_B$, one defines

$$S_Q(x, y) = \{a \in Q : \forall z \in B(x \leq \Box_a z \Rightarrow y \leq z)\}.$$

Let $\sigma_B : W_B \times W_B \to Q$ be the partial function defined as follows:

$$\sigma_B(x, y) = \begin{cases} \bigvee S_Q(x, y) & \text{if } S_Q(x, y) \neq \varnothing. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The Q-frame $\mathfrak{B}_\bullet = (W_B, \sigma_B)$ is called the *dual* of $\mathfrak{B}$.

Proposition 5. *$\mathfrak{B} \cong (\mathfrak{B}_\bullet)^+$ for every finite Q-MA $\mathfrak{B}$.*

Proof. Let $\mathfrak{B} = (B, \Diamond_a)_{a \in Q}$. One defines $\eta : B \to \mathcal{P}(W_B)$ by setting

$$\eta(x) = \{z \in W_B : z \leq x\}.$$

It suffices to show that η is an isomorphism. Clearly η is a bijective Boolean homomorphism. It suffices to show $\eta(\Box_a x) = \Box_a^{\sigma_B} \eta(x)$. Assume $y \in \eta(\Box_a x)$. Then $y \in W_B$ and $y \leq \Box_a x$. Suppose $\sigma_B(y, z)! \geq a$. Then $b = \bigvee S_Q(y, z) \geq a$. It follows $\Box_a x \leq \Box_b x$. Then $y \leq \Box_b x$. By $b \in S_Q(y, z)$, we have $z \leq x$, i.e., $z \in \eta(x)$. Hence $y \in \Box_a^{\sigma_B} \eta(x)$. Now assume $y \in \Box_a^{\sigma_B} \eta(x)$. Then

$$\forall z \in W_B(\bigvee S_Q(y, z) \geq a \Rightarrow z \leq x). \tag{†}$$

Let $v = \bigwedge\{u \in B : y \leq \Box_a u\}$. Then $\Box_a v = \Box_a \bigwedge\{u \in B : y \leq \Box_a u\} = \bigwedge\{\Box_a u \in B : y \leq \Box_a u\}$. Clearly $S_Q(y, z)$ is a downset. Then $\bigvee S_Q(y, z) \geq a$ if and only if $a \in S_Q(y, z)$, which is equivalent to $\forall u \in B(y \leq \Box_a u \Rightarrow z \leq u)$, and also equivalent to $z \leq v$. By (†), one gets

$$\forall z \in W_B(z \leq v \Rightarrow z \leq x). \tag{‡}$$

By (‡), $v \leq x$. Then $\Box_a v \leq \Box_a x$. Clearly $y \leq \Box_a v$. Hence $y \leq \Box_a x$. □

Definition 13. Let $\mathfrak{B} = (B, \Diamond_a)_{a \in Q}$ be a Q-MA. Let B^{uf} be the set of all ultrafilters in B. For every $u, v \in B^{\text{uf}}$, one defines

$$X_Q^{\text{uf}}(u, v) = \{a \in Q : (\forall x \in v)\Diamond_a x \in u\}.$$

The partial function $\sigma^{\mathrm{uf}} : B^{\mathrm{uf}} \times B^{\mathrm{uf}} \to Q$ is defined by:

$$\sigma^{\mathrm{uf}}(u, v) = \begin{cases} \bigvee X_Q^{\mathrm{uf}}(u, v) & \text{if } X_Q^{\mathrm{uf}}(u, v) \neq \varnothing. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The Q-frame $\mathfrak{B}_+ = (B^{\mathrm{uf}}, \sigma^{\mathrm{uf}})$ is called the *ultrafilter frame* of $\mathfrak{B}$.

Obviously, for every Q-frame $\mathbb{F}$, $(\mathbb{F}^+)_+ = \mathbb{F}^{ue}$. Moreover, the following Jónsson-Tarski representation theorem holds for Q-MAs.

Theorem 4. $\mathfrak{B} \rightarrowtail (\mathfrak{B}_+)^+$ *for every* Q*-MA* $\mathfrak{B}$.

Proof. Let $\mathfrak{B} = (B, \Diamond_a)_{a \in Q}$ and $\mathfrak{B}_+ = (B^{\mathrm{uf}}, \sigma^{\mathrm{uf}})$. Let $U : B \to \mathcal{P}(B^{\mathrm{uf}})$ be the map $U(x) = \{u \in B^{\mathrm{uf}} : x \in u\}$ for $x \in B$. Clearly U is a Boolean homomorphism and injective. We show $U(\Diamond_a x) = \Diamond_a^{\sigma^{\mathrm{uf}}} U(x)$. Assume $u \in U(\Diamond_a x)$. Then $\Diamond_a x \in u$. Clearly $\{y \in B : \Box_a y \in u\} \cup \{x\}$ has the finite meet property (i.e., every meet of finitely many elements in this set is nonzero), and so it is extended to an ultrafilter v. Then $a \in X_Q^{\mathrm{uf}}(u, v)$ and $\sigma^{\mathrm{uf}}(u, v)! \geq a$. Hence $u \in \Diamond_a^{\sigma^{\mathrm{uf}}} U(x)$. Now assume $u \in \Diamond_a^{\sigma^{\mathrm{uf}}} U(x)$. There exists $v \in B^{\mathrm{uf}}$ with $\sigma^{\mathrm{uf}}(u, v)! \geq a$ and $v \in U(x)$. Clearly $X_Q^{\mathrm{uf}}(u, v)$ is a downset and so $a \in X_Q^{\mathrm{uf}}(u, v)$. By $v \in U(x)$, $x \in v$ and so $\Diamond_a x \in u$, i.e., $u \in U(\Diamond_a x)$. □

Lemma 6. *Let* $\{\mathbb{F}_i = (W_i, \sigma_i) : i \in I\}$ *be a family of disjoint* Q*-frames. Then* $(\biguplus_i \mathbb{F}_i)^+ \cong \prod_i \mathbb{F}_i^+$.

Proof. One defines $g : \mathcal{P}(\biguplus_{i \in I} W_i) \to \prod_{i \in I} \mathcal{P}(W_i)$ by setting, for all $X \subseteq \biguplus_{i \in I} \mathcal{P}(W_i)$, $g(X)(i) = X \cap W_i$. Then g is an isomorphism. Here we verify $g(\Diamond_a^{\biguplus_{i \in I} \sigma_i}(X)) = \prod_{i \in I} \Diamond_a^{\sigma_i}(g(X))$. By the definition, $g(\Diamond_a^{\biguplus_{i \in I} \sigma_i}(X))(j) = \Diamond_a^{\biguplus_{i \in I} \sigma_i}(X) \cap W_j = \Diamond_a^{\sigma_j}(X \cap W_j)$ and $\prod_{i \in I} \Diamond_a^{\sigma_i}(g(X))(j) = \Diamond_a^{\sigma_j}(g(X)(j)) = \Diamond_a^{\sigma_j}(X \cap W_j)$. For all $j \in I$, $g(\Diamond_a^{\biguplus_{i \in I} \sigma_i}(X))(j) = \prod_{i \in I} \Diamond_a^{\sigma_i}(g(X))(j)$. □

Let $\eta : W \to W'$ be a function. The *dual* of η is the function $\eta^+ : \mathcal{P}(W') \to \mathcal{P}(W)$ defined by $\eta^+(X') = \eta^{-1}(X')$. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be Q-MAs and $g : B \to B'$ be a function. The *dual* of g is the function $g_+ : B'^{\mathrm{uf}} \to B^{\mathrm{uf}}$ defined by $g_+(u') = g^{-1}(u')$. Since $u' \in B'^{\mathrm{uf}}$, one gets $g^{-1}(u') \in B^{\mathrm{uf}}$.

Lemma 7. *Let* $\mathbb{F} = (W, \sigma)$ *and* $\mathbb{G} = (W', \sigma')$ *be* Q*-frames, and* $\eta : W \to W'$ *be a bounded morphism from* $\mathbb{F}$ *to* $\mathbb{G}$*. Then (1)* η^+ *is a homomorphism from* $\mathbb{G}^+$ *to* $\mathbb{F}^+$*; (2) if* η *is injective, then* η^+ *is surjective; and (3) if* η *is surjective, then* η^+ *is injective.*

Proof. Clearly (2) and (3) hold by the definition. For (1), the dual η^+ is a Boolean homomorphism. It suffices to show $\eta^+(\Diamond_a^{\sigma'} X') = \Diamond_a^{\sigma} \eta^+(X')$ for every $a \in Q$ and $X' \subseteq W'$. Assume $y \in \eta^+(\Diamond_a^{\sigma'} X')$. Then $\eta(y) \in \Diamond_a^{\sigma'} X'$. There exists $x' \in X'$ with $\sigma'(\eta(y), x')! \geq a$. Since η is a bounded morphism, there exists

$x \in W$ with $\sigma(y,x)! \geq a$ and $\eta(x) = x'$. Then $y \in \Diamond_a^\sigma \eta^+(X')$. Conversely, assume $y \in \Diamond_a^\sigma \eta^+(X')$. There exists $x \in \eta^+(X')$ with $\sigma(y,x)! \geq a$. Since η is a bounded morphism, one obtains $\eta(x) \in X'$ with $\sigma'(\eta(y), \eta(x))! \geq a$. Then $\eta(y) \in \Diamond_a^{\sigma'} X'$. Hence $y \in \eta^+(\Diamond_a^{\sigma'} X')$. □

Lemma 8. *Let* $\mathfrak{B} = (B, \Diamond_a)_{a \in Q}$ *and* $\mathfrak{B}' = (B', \Diamond'_a)_{a \in Q}$ *be Q-MAs, and* $g : B \to B'$ *be a homomorphism. Then (1)* g_+ *is a bounded morphism from* $\mathfrak{B}'_+$ *to* $\mathfrak{B}_+$*; (2) if* g *is injective, then* g_+ *is surjective; and (3) if* g *is surjective, then* g_+ *is injective.*

Proof. Clearly (2) and (3) hold by the definition. For (1), let $\mathfrak{B}_+ = (B^{\mathrm{uf}}, \sigma^{\mathrm{uf}})$ and $\mathfrak{B}'_+ = (B'^{\mathrm{uf}}, \sigma'^{\mathrm{uf}})$. Let $u, v \in B'^{\mathrm{uf}}$ and $\sigma'^{\mathrm{uf}}(u,v)! \geq a$. Let $\Delta_1 = \{b \in Q : (\forall x' \in v)\Diamond'_b x' \in u\}$ and $\Delta_2 = \{b \in Q : (\forall x \in g_+(v))\Diamond_b x \in g_+(u)\}$. Then $\sigma'^{\mathrm{uf}}(u,v) = \bigvee \Delta_1$ and $\sigma^{\mathrm{uf}}(g_+(u), g_+(v)) = \bigvee \Delta_2$. It suffices to show $\Delta_1 \subseteq \Delta_2$ which yields $\sigma^{\mathrm{uf}}(g_+(u), g_+(v)) = \bigvee \Delta_2 \geq \bigvee \Delta_1 \geq a$. Assume $b \in \Delta_1$. Let $x \in g_+(v)$. Then $g(x) \in v$. By $b \in \Delta_1$, $\Diamond'_b g(x) = g(\Diamond_b x) \in u$. Then $\Diamond_b x \in g_+(u)$. Hence $b \in \Delta_2$. It follows that $\Delta_1 \subseteq \Delta_2$.

Assume $u \in B'^{\mathrm{uf}}$, $v' \in B^{\mathrm{uf}}$ and $\sigma^{\mathrm{uf}}(g_+(u), v') \geq a$. It suffices to find $v \in B'^{\mathrm{uf}}$ such that $\sigma'^{\mathrm{uf}}(u,v) \geq a$ and $g_+(v) = v'$. Let $v_1 = \{g(x) : x \in v'\}$ and $v_2 = \{y \in B' : \Box'_a y \in u\}$. One can easily show that $v_1 \cup v_2$ has the finite meet property. Then there exists $v \in B'^{\mathrm{uf}}$ with $v_1 \cup v_2 \subseteq v$. Now one shows that v is a required ultrafilter as follows: (i) $\sigma'^{\mathrm{uf}}(u,v) \geq a$. It suffices to show $a \in \Delta_1$. For a contradiction, suppose $x \in v$ and $\Diamond'_a x \notin u$. Then $-\Diamond'_a x = -\Diamond'_a - -x \in u$. Then $-x \in v_2 \subseteq v$ which contradicts $x \in v$. (ii) $g_+(v) = v'$. If $x \in v'$, then $g(x) \in v_1 \subseteq v$ and so $x \in g_+(v)$. Suppose $x \notin v'$. Then $-x \in v'$ and $g(-x) \in v_1 \subseteq v$. Hence $-x \in g_+(v)$, i.e., $x \notin g_+(v)$. □

Lemma 9. *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *be Q-frames,* $\mathfrak{B}$ *and* $\mathfrak{C}$ *be Q-MAs. Then (1) if* $\mathbb{F} \rightarrowtail \mathbb{G}$*, then* $\mathbb{G}^+ \twoheadrightarrow \mathbb{F}^+$*; (2) if* $\mathbb{F} \twoheadrightarrow \mathbb{G}$*, then* $\mathbb{G}^+ \rightarrowtail \mathbb{F}^+$*; (3) if* $\mathfrak{B} \rightarrowtail \mathfrak{C}$*, then* $\mathfrak{C}_+ \twoheadrightarrow \mathfrak{B}_+$*; and (4) if* $\mathfrak{B} \twoheadrightarrow \mathfrak{C}$*, then* $\mathfrak{C}_+ \rightarrowtail \mathfrak{B}_+$*.*

Proof. Straightforward by Lemma 7 and Lemma 8. □

Theorem 5. *Let* Q *be a* i.w.o *set and* $\mathcal{K}$ *be a class of Q-frames which is closed under taking ultrafilter extensions. Then* $\mathcal{K}$ *is modally Q-definable if and only if it is closed under taking disjoint unions, generated subframes, bounded morphic images, and reflects ultrafilter extensions.*

Proof. The left-to-right direction follows from Proposition 1, Proposition 2, Proposition 3 and Proposition 4. Assume that $\mathcal{K}$ satisfies the closure conditions. Clearly $\mathcal{K} \subseteq \mathrm{Fr}_Q(\mathrm{Th}(\mathcal{K}))$. Assume $\mathbb{F} \models \mathrm{Th}(\mathcal{K})$. By Birkhoff's theorem, $\mathbb{F}^+ \in \mathbf{HSPCm}\mathcal{K}$. Then there exist a family of Q-frames $\{\mathbb{G}_i\}_{i \in I}$ in $\mathcal{K}$ and a Q-MA $\mathfrak{B}$ such that $\mathbb{F}^+ \twoheadleftarrow \mathfrak{B} \rightarrowtail \prod_{i \in I} \mathbb{G}_i^+$. By Lemma 6, one obtains $\mathbb{F}^+ \twoheadleftarrow \mathfrak{B} \rightarrowtail (\biguplus_{i \in I} \mathbb{G}_i)^+$. By Lemma 9, one obtains $\mathbb{F}^{ue} = (\mathbb{F}^+)_+ \rightarrowtail \mathfrak{B}_+ \twoheadleftarrow ((\biguplus_{i \in I} \mathbb{G}_i)^+)_+ = (\biguplus_{i \in I} \mathbb{G}_i)^{ue}$. Since $\mathcal{K}$

is closed under taking disjoint unions, $\biguplus_i \mathbb{G}_i \in \mathcal{K}$. Furthermore, as $\mathcal{K}$ is closed under taking ultrafilter extensions, bounded morphic images and generated subframes, it follows that $(\biguplus_i \mathbb{G}_i)^{ue}$, $\mathfrak{B}_+$ and $\mathbb{F}^{ue}$ belong to $\mathcal{K}$. Then $\mathbb{F} \in \mathcal{K}$ since $\mathcal{K}$ reflects ultrafilter extensions. □

6 More Observations on Normal Q-modal Logics

In section 3, we introduce normal Q-modal logics and apply the canonical method to prove the completeness of the minimal normal Q-modal logic. In this section, we will make some particular observations on these logics.

A normal Q-modal logic is *canonical*, if $\mathbb{F}^L \in \mathrm{Fr}_Q(L)$. Obviously every canonical normal Q-modal logic is complete. As the Sahlqvist theorem for modal logic (e.g., [2]), one obtains Sahlqvist normal Q-modal logics which are elementary and complete. The statement of elementarity needs an appropriate first-order language for talking about properties of Q-frames. A choice is the first-order Q-frame language with identity $\mathcal{L}^1_Q$ consists of binary relational symbols $\{R_a : a \in Q\}$ where each R_a is interpreted as $\sigma(w,u)! \geq a$ in a Q-frame $\mathbb{F} = (W, \sigma)$.

Name	Formulas	First-order Correspondent
(D_a)	$\langle a\rangle\top$	$\forall x\exists y R_a xy$
(T_a)	$[a]p \to p$	$\forall x R_a xx$
(4_{abc})	$[a]p \to [b][c]p$	$\forall xyz(R_a xy \wedge R_b yz \to R_c xz)$
(B_{ab})	$p \to [a]\langle b\rangle p$	$\forall xy(R_a xy \to R_b yx)$
(5_{abc})	$\langle a\rangle p \to [b]\langle c\rangle p$	$\forall xy(R_a xy \wedge R_b xz \to R_c yz)$

Table 1: Some correspondence results

A formula $\varphi \in Fm(Q)$ corresponds a sentence α in $\mathcal{L}^1_Q$, if $\mathrm{Fr}_Q(\varphi)$ is defined by α. The correspondence between modal and first-order sentences in Table 1 can be shown immediately. One can define *Sahlqvist formulas* exactly as in e.g., [2]. Every Sahqvist formula φ in $Fm(Q)$ has a correspondent α_φ in $\mathcal{L}^1_Q$ that is computed automatedly. A *Sahlqvist Q-modal logic* is $\mathrm{K}_Q \oplus \Gamma$ where Γ is a set of Sahlqvist formulas. One can show as usual that every Sahlqvist Q-modal logic is canonical and hence complete (e.g., [2]).

Next we make observations on the modal logics of singleton Q-frames. In the standard normal monomodal logic, there are only two singleton frames the logics of which are Post complete (e.g., [4, 12]). This is a direct consequence of Makinson's classification theorem ([10]). In the setting of normal Q-modal logics with $|Q| > 1$ the situation is different. Let $\bullet$ be the Q-frame $(\{\bullet\}, \sigma)$ with $\sigma(\bullet, \bullet)$ undefined. Let $\circ_a$ be the Q-frame $(\{\circ\}, \sigma)$ with $\sigma(\circ, \circ) = a$. A normal Q-modal logic L is *consistent*, if $\mathrm{Fr}_Q(L) \neq \varnothing$.

Proposition 6. *Let $a, b \in Q$ and $\varnothing \neq X, Y \subseteq Q$. The following hold:*

(1) $Th(\bullet) = \mathrm{K}_Q \oplus \{[a]\bot : a \in Q\}$.
(2) if $a \neq b \in Q$*, then* $Th(\circ_a) \not\subseteq Th(\circ_b)$ *and* $Th(\circ_b) \not\subseteq Th(\circ_a)$.
(3) $Th(\circ_1) = \mathrm{K}_Q \oplus p \leftrightarrow [1]p$.
(4) $Th(\biguplus_{a \in X} \circ_a) = \bigcap_{a \in X} Th(\circ_a)$.

Proof. (1) Let $L_\bullet = \mathrm{K}_Q \oplus \{[a]\bot : a \in Q\}$. Clearly $\bullet \models L_\bullet$, i.e., $L_\bullet \subseteq Th(\bullet)$. Suppose $\varphi \notin L_\bullet$. Let Σ be a state in the canonical frame $\mathbb{F}^{L_\bullet} = (W^{L_\bullet}, \sigma^{L_\bullet})$ with $\varphi \notin \Sigma$. Clearly there is no $\Delta \in W^{L_\bullet}$ with $\sigma^{L_\bullet}(\Sigma, \Delta)!$. Then $\bullet$ is the subframe of $\mathbb{F}^{L_\bullet}$ generated by Σ. Hence $\bullet \not\models \varphi$. It follows that $L = Th(\bullet)$.

(2) Assume $a \neq b$. Then $a < b$ or $b < a$. Without loss of generality, suppose $a < b$. Then $\langle b \rangle \top \in Th(\circ_b) \setminus Th(\circ_a)$ and $[b]\bot \in Th(\circ_a) \setminus Th(\circ_b)$.

(3) Let $L_1 = \mathrm{K}_Q \oplus p \leftrightarrow [1]p$. Then $\circ_1 \models L_1$, i.e., $L_1 \subseteq Th(\circ_1)$. Obviously L_1 is canonical, i.e., its canonical frame $\mathbb{F}^{L_1} \models L_1$. Clearly $\mathbb{F}^{L_1}$ satisfies the conditions $\forall x R_1^\sigma xx$ and $\forall xy(R_1^\sigma xy \to x = y)$. This means that $\mathbb{F}^{L_1}$ consists of isolated copies of $\circ_1$. Hence $\circ_1$ is a generated subframe of $\mathbb{F}^{L_1}$.

(4) It follows from Proposition 1. □

Definition 14. Let $\mathbb{M} = (W, \sigma, V)$ be a Q-model. A subset $U \subseteq W$ is *definable* in $\mathbb{M}$, if there exists $\varphi \in Fm(Q)$ with $U = V(\varphi)$. A Q-model $\mathbb{M}' = (W, \sigma, V')$ with a valuation V' in W is called a *variant* of $\mathbb{M}$. We say that $\mathbb{M}'$ is a *definable variant* of $\mathbb{M}$, if $V'(p)$ is definable in $\mathbb{M}$ for each $p \in \mathbb{P}$.

Lemma 10. *Suppose* $\Gamma \subseteq Fm(Q)$ *is closed under substitution. If* $\mathbb{M} \models \Gamma$*, then* $\mathbb{M}' \models \Gamma$ *for every definable variant* $\mathbb{M}'$ *of* $\mathbb{M}$.

Proof. Let $\mathbb{M} = (W, \sigma, V)$ be a model and $\mathbb{M} \models \Gamma$. Let $\varphi \in Fm(Q)$ and φ' be the formula obtained from φ by substituting ψ_i for p_i in φ, and $\mathbb{M}' = (W, \sigma, V')$ be the Q-model where $V'(p_i) = V(\psi_i)$ with $1 \leq i \leq n$. By induction on the complexity $\delta(\varphi)$ one obtains that $w \in V(\varphi')$ if and only if $w \in V'(\varphi)$. One obtains $\mathbb{M}' \models \Gamma$ by the same proof as [5, Theorem 5]. □

Theorem 6. *Let* $L \in \mathrm{NExt}(\mathrm{K}_Q)$ *be consistent. The following hold:*

(1) if $\langle a \rangle \top \notin L$ *for every* $a \in Q$*, then* $L \subseteq Th(\bullet)$.
(2) if $\langle a \rangle \top \in L$ *for some* $a \in Q$*, then* $L \subseteq Th(\circ_b)$ *for some* $b \in Q$.

Proof. (1) Assume $\langle a \rangle \top \notin L$ for every $a \in Q$. Let $\mathbb{M}^L = (W^L, \sigma^L, V^L)$ be the canonical model for L. Let $\Gamma = \{[a]\bot : a \in Q\}$. Now we show that Γ is L-consistent. Suppose not. Then $\bigvee_{a \in X} \langle a \rangle \top \in L$ for some finite $\varnothing \neq X \subseteq Q$. Let $\bigwedge X = b$. Clearly $\bigvee_{a \in X} \langle a \rangle \top \to \langle b \rangle \top \in L$. Then $\langle b \rangle \top \in L$ which contradicts the assumption. Hence Γ is L-consistent. Let $\Sigma \in W^L$ and $\Gamma \subseteq \Sigma$. Clearly there is no $\Delta \in W^L$ with $\sigma^L(\Sigma, \Delta)!$. Let $\mathbb{M} = (W, \sigma, V)$ be the model generated from $\mathbb{M}^L$ by Σ. Due to $[a]\bot \in \Sigma$ for all $a \in Q$, one obtains that $\mathfrak{F} = (W, \sigma)$ consists of a single state Σ with $\sigma(\Sigma, \Sigma)$ undefined. Obviously $\mathbb{M}, \Sigma \models L$. Note that L is closed under substitution. By Lemma 10, one obtains $\bullet \in \mathrm{Fr}_Q(L)$, i.e., $L \subseteq Th(\bullet)$.

(2) Assume $\langle a\rangle\top \in L$ for some $a \in Q$. Let $b = \bigvee\{c \in Q : \langle c\rangle\top \in L\}$. Then $\langle b\rangle\top \in L$. Since L is consistent, let $\mathbb{F} = (W, \sigma)$ be a Q-frame and $\mathbb{F} \models L$. Then $\mathbb{F} \models \langle b\rangle\top$. Hence $R^\sigma_b(w) \neq \varnothing$ for all $w \in W$. Let η be the function which maps each $w \in W$ to $\circ$. Then η is a surjective bounded morphism from $\mathbb{F}$ to $\circ_b$. By Proposition 3, $\circ_b \models L$, i.e., $L \subseteq Th(\circ_b)$. □

The notion of total Q-frame is given in Definition 2. Here we consider a particular class of total Q-frames. Let Q_0 be an *i.w.o* set with bottom element 0. Obviously every finite chain is certainly such an *i.w.o* set. The class of all total Q_0-frames is denoted by $\mathcal{F}_{Q_0}$. Note that, although $\mathcal{F}_{Q_0}$ is not closed under taking disjoint unions, it is closed under taking generated subframes, bounded morphic images, and ultrafilter extensions. Moreover, Propositions 2, 3 and 4 hold for total Q_0-frames. Now we introduce normal total Q_0-modal logics.

Definition 15. A *normal total Q_0-modal logic* is a set of formulas $L \subseteq Fm(Q_0)$ which contains (Tau), (K_a), (C_{Q_0}), and the following formulas:

$$(T_0)\ [0]p \to p,\ (4_0)\ [0]p \to [0][0]p,\ (B_0)\ p \to [0]\langle 0\rangle p.$$

and is closed under (MP), (Gen) and (Sub). The minimal normal total Q_0-modal logic is denoted by $\mathrm{K}^\flat_{Q_0}$. Let $\mathrm{NExt}^\flat(L)$ be the set of all normal total Q_0-modal logics which contain L.

Theorem 7. $\mathrm{K}^\flat_{Q_0} = Th(\mathcal{F}_{Q_0})$.

Proof. Let $\mathbb{F} = (W, \sigma)$ be a total Q_0-frame. Then $u \in R^\sigma_0(w)$ for all $w, u \in W$. It follows that (T_0), (4_0) and (B_0) are valid in $\mathcal{F}_{Q_0}$. Then $\mathrm{K}^\flat_{Q_0} \subseteq Th(\mathcal{F}_{Q_0})$. For the completeness, let $\mathbb{M} = (W, \sigma, V)$ be the canonical model for $\mathrm{K}^\flat_{Q_0}$. Note that $0 \in X_{Q_0}(\Sigma, \Theta)$ and so $\sigma(\Sigma, \Theta) = \bigvee X_{Q_0}(\Sigma, \Theta)$ which is a total function. If $\varphi \notin \mathrm{K}^\flat_{Q_0}$, then $\mathbb{M} \not\models \varphi$. Hence $Th(\mathcal{F}_{Q_0}) \subseteq \mathrm{K}^\flat_{Q_0}$. □

Proposition 7. *For every consistent normal total Q_0-modal logic L, there exists $a \in Q_0$ with $L \subseteq Th(\circ_a)$.*

Proof. By $\langle 0\rangle\top \in L$ and Theorem 6, $L \subseteq Th(\circ_a)$ for some $a \in Q_0$. □

Finally, we shall consider the embedding from a normal Q-modal logic to a normal Q'-modal logic where Q is embedded into Q' in some way. In what follows, Q and Q' are supposed to be *i.w.o* sets.

Definition 16. Let $(Q, \leq_Q, 1)$ and $(Q', \leq_{Q'}, 1')$ be *i.w.o* sets. A function $f : Q \to Q'$ is an *embedding*, notation $f : Q \hookrightarrow Q'$, if $f(1) = 1'$ and for all $a, b \in Q$, $a \leq_Q b$ if and only if $fa \leq_{Q'} fb$. An embedding $f : Q \hookrightarrow Q'$ is *upward*, if $f(Q)$ is an upset in Q'. For every $f : Q \hookrightarrow Q'$, the function $\varepsilon_f : Fm(Q) \to Fm(Q')$ is defined inductively as follows:

$$\varepsilon_f(\varphi) = \varphi, \text{ if } \varphi \in \mathbb{P} \cup \{\bot\}.$$
$$\varepsilon_f(\varphi \to \psi) = \varepsilon_f(\varphi) \to \varepsilon_f(\psi).$$
$$\varepsilon_f([a]\varphi) = [fa]\varepsilon_f(\varphi).$$

For every set of formulas $\Sigma \subseteq Fm(Q)$, let $\varepsilon_f(\Sigma) = \{\varepsilon_f(\varphi) : \varphi \in \Sigma\}$.

Lemma 11. *Let $f : Q \hookrightarrow Q'$. For every set of formulas $\Sigma \cup \{\varphi\} \subseteq Fm(Q)$, if $\varphi \in \mathrm{K}_Q \oplus \Sigma$, then $\varepsilon_f(\varphi) \in \mathrm{K}_{Q'} \oplus \varepsilon_f(\Sigma)$.*

Proof. Assume $\varphi \in \mathrm{K}_Q \oplus \Sigma$. The case $\varphi \in \Sigma$ is obvious. If φ is an instance of (Tau), by the definition of ε_f, so is $\varepsilon_f(\varphi)$. If φ be (K_a), then $\varepsilon_f(\varphi)$ is (K_{fa}). Suppose that φ is (C_Q), i.e., $\varphi = [a]p \to [b]p$ with $a \leq_Q b$. Then $fa \leq_{Q'} fb$. Hence $\varepsilon_f(\varphi) = [fa]p \to [fb]p$. If φ is obtained by a rule (R) in $\mathrm{K}_Q \oplus \Sigma$, then $\varepsilon_f(\varphi)$ is also obtained by (R) in $\mathrm{K}_{Q'} \oplus \varepsilon_f(\Sigma)$. □

Let $f : Q \hookrightarrow Q'$. For every Q-frame $\mathbb{F} = (W, \sigma)$, one defines the Q'-frame $\mathbb{F}^f = (W, \sigma^f)$ by setting for all $w, u \in W$:

$$\sigma^f(w,u) = \begin{cases} f(\sigma(w,u)) & \text{if } \sigma(w,u)!. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For every Q-model $\mathbb{M} = (\mathbb{F}, V)$, let $\mathbb{M}^f = (\mathbb{F}^f, V)$.

Lemma 12. *Let $f : Q \hookrightarrow Q'$ and $\mathbb{M} = (\mathbb{F}, V)$ be a Q-model with $\mathbb{F} = (W, \sigma)$. For every $w \in W$ and $\varphi \in Fm(Q)$, (i) $\mathbb{M}, w \models \varphi$ if and only if $\mathbb{M}^f, w \models \varepsilon_f(\varphi)$; and (ii) $\mathbb{F} \models \varphi$ if and only if $\mathbb{F}^f \models \varepsilon_f(\varphi)$.*

Proof. Clearly (ii) follows from (i). For (i), the proof proceeds by induction on the complexity $\delta(\varphi)$. Atomic and Boolean cases are obvious. Let $\varphi = [a]\psi$ and $\varepsilon_f(\varphi) = [fa]\varepsilon_f(\psi)$. Assume $\mathbb{M}^f, w \not\models \varepsilon_f(\varphi)$. Then $\sigma^f(w,u)! \geq fa$ and $\mathbb{M}^f, u \not\models \varepsilon_f(\psi)$ for some $u \in W$. Clearly $\sigma^f(w,u) = f(\sigma(w,u)) \geq fa$. Then $\sigma(w,u)! \geq a$. By induction hypothesis, $\mathbb{M}, u \not\models \psi$. Hence $\mathbb{M}, w \not\models [a]\psi$. The other direction is shown similarly. □

Theorem 8. *Let $f : Q \hookrightarrow Q'$ be upward and $\Sigma \subseteq Fm(Q)$. Suppose that $L = \mathrm{K}_Q \oplus \Sigma$ and $L' = \mathrm{K}_{Q'} \oplus \varepsilon_f(\Sigma)$ are complete. For every $\varphi \in Fm(Q)$, $\varphi \in L$ if and only if $\varepsilon_f(\varphi) \in L'$.*

Proof. By Lemma 11, if $\varphi \in L$, then $\varepsilon_f(\varphi) \in L'$. Assume $\varphi \notin L$. Since L is complete, there exists $\mathbb{F} \in \mathrm{Fr}_Q(L)$ with $\mathbb{F} \not\models \varphi$. By Lemma 12, $\mathbb{F}^f \in \mathrm{Fr}_{Q'}(L')$ and $\mathbb{F}^f \not\models \varepsilon_f(\varphi)$. Hence $\varepsilon_f(\varphi) \notin L'$. □

Corollary 1. *Let $f : Q \hookrightarrow Q'$ be upward. For every $\varphi \in Fm(Q)$, $\varphi \in \mathrm{K}_Q$ if and only if $\varepsilon_f(\varphi) \in \mathrm{K}_{Q'}$.*

7 Concluding Remarks

In the present work we contribute new multimodal logics over multivalued frames. Multivalued frames over a set of values Q are taken as the semantic ontology to interpret the corresponding multimodal language. We obtain Goldblatt-Thomason theorems for certain classes of Q-frames. In the study of normal Q-modal logics, we adjust the canonical model method and obtain some completeness results. The logics of singleton frames and related Makinson's classification theorem are established. There are many problems that are interesting for further exploration. Here we mention two of them: (i) throughout this paper, we considered only *i.w.o* sets as organizations of agents. One could change the set of values Q into a (possibly distributive or Boolean) lattice in general, and study related problems; and (ii) try to give a Goldblatt-Thomason theorem which characterizes the modal Q-definability of certain classes of total Q-frames.

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逆良序集上多值框架的模态逻辑

何凡

摘　要

通过引入取值集合，将 Kripke 框架推广为多值框架。本文假设取值集 Q 为逆良序的集合，与之对应的模态语言在 Q 上的多值框架中得到解释。本文证明了某些 Q-框架类的 Goldblatt-Thomason 定理。本文还引入了正规 Q-模态逻辑，并证明了其完全性与 Makinson 定理。

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