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A Logic of von Wright's Deontic Necessity*

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Abstract. In this paper, we build a bridge between G. von Wright's deontic logic and E. Bezerra and G. Venturi's $\boxplus$-logic, in the sense that on one hand, we give an interpretation of $\boxplus$-operator as von Wright's deontic necessity, and on the other hand, we give the exact semantics of von Wright's deontic modalities. Inspired by an almost definability schema, we explain why the canonical model of the minimal $\boxplus$-logic is defined in that way. We also present various axiomatizations of $\boxplus$-logic, among which the transitive system is also inspired by the schema in question. We explain why the two non-equivalent semantics for $\boxplus$ involved in the literature, one of which is standard and the other is non-standard, come to give the same logic. We conclude with some discussions about notions of deontic non-contingency and deontic contingency.

1 Introduction

In his seminal work [20], von Wright investigates some logical properties of deontic concepts such as obligation, permission, and forbiddance. As von Wright observes, these concepts resemble the alethic ones — necessity, possibility, and impossibility — in many respects.

However, there is a crucial difference between the deontic concepts and alethic ones. Although every tautology is necessary, and every contradiction is impossible, this cannot be extended to obligation and forbidden. Von Wright suggests the following Principle of Deontic Contingency: a tautologous act is not necessarily obligatory, and a contradictory act is not necessarily forbidden.[1] This means that, for instance, the semantics of obligation should be different from that of necessity. It is then natural to ask what the exact semantics is for these modalities. Unfortunately, to our knowledge, von Wright and his followers have not dealt with this issue yet. As we will argue below, among other similarities, the obligation operator $\Box$ have the D schema, but no the necessitation rule, thus von Wright's $\Box$ should be understood as our $\boxplus$ operator, rather than the classical obligation operator (which will have the necessitation rule).

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[1]This principle is formulated in terms of acts rather than propositions. However, a similar criterion can be defined for propositions, by simply replacing "act" with "proposition" in the formulation above. This criterion seems to be adopted by e.g. A. Prior ([16]) and E. Lemmon ([12]).

On the other hand, in a recent paper ([1]), Bezerra and Venturi introduce a modality $\boxplus$ and present a minimal system of $\boxplus$-logic. Semantically, $\boxplus\varphi$ is interpreted as "φ is both necessary and possible". According to the semantics, the $\boxplus$-operator has many common properties with the serial necessity (that is, necessity over serial frames);[2] for instance, **K**- and **D**-axioms are valid on the class of all frames, except for the failure of the necessitation rule.

However, the interpretation of $\boxplus$ is left open. This can be seen in the following paragraphs in [1, pp. 8–9]:

> As for the interpretation of the $\boxplus$-operator, however, the matter is a bit more complicated. … As regards the failure of the necessitation rule, even though Lemmon argues that it is a desirable aspect of this logical system, this seems to go against the current state of deontic logics, where it is usually accepted that every tautology should be obligatory.
>
> We leave the problem of the right interpretation of the $\boxplus$-operator open. We just remark that if any can be found, this should be in a context where axiom ($D_{\boxplus}$) plays a fundamental conceptual role and where the lack of necessitation does not cause harm to a faithful interpretation.

As we will demonstrate below, $\boxplus$-operator (and its dual) possesses all logical properties of the obligation modality (and its dual, permission) listed in [20]. This may give an interpretation of $\boxplus$-operator as the obligation in the previous sense, or as we will say, von Wright's deontic necessity, on one hand, and also give the exact semantics of the deontic modalities in question, on the other hand. In doing so, we build a bridge between von Wright's deontic logic and Bezerra and Venturi's $\boxplus$-logic.

Bezerra and Venturi's work is connected to the pioneering work of Lemmon ([12]) on regular logics. Inspired by neighborhood semantics of regular logics, Lemmon proposes a Kripke semantics for $\boxplus$ (Lemmon uses a different symbol though), which involves a set of "normal" worlds and the usual truth-condition for the necessity operator $\Box$, see [13] and [17] for a systematic discussion of these logics and Sec. 5 below. Rather than doing this, Bezerra and Venturi ([1]) introduce a non-standard semantics for $\boxplus$, which involves a class of standard frames with a non-standard truth-condition — conjoined universal and existential — for $\Box$. It turns out the two semantics give the same logic. Although Bezerra and Venturi mention this, they do not attempt to explain why it should be so. In this paper, we will also give an explanation of it: the latter semantics is equivalent to the special case of the former in which the normal worlds are precisely those with successors.

Coming back to the minimal system given in [1], in order to show the completeness, the authors there adopt a canonical model construction. The construction contains a complicated definition of canonical relation, which though is not given any intuitive explanation, except for the statement that "The proof of completeness is inspired by [18]" ([1], p. 6). In the present paper, as in [7, 8, 4], we introduce a special

[2]Maybe because of this fact, in an earlier version of [1], $\boxplus$ is called the operator of *serial necessity*.

almost definability schema, which says that $\boxplus\varphi$ and $\Box\varphi$ are equivalent given some proposition. As we will see, with help of the schema, we will give an explanation of why the canonical relation is defined in that way. The schema also helps us find a complete axiomatization of transitive $\boxplus$-logic.

The rest of the paper is organized as follows. Sec. 2 introduces the syntax and semantics of von Wright's deontic necessity, including the so-called deontic tautologies concerning obligation and permission operators in [20], and an almost definability schema of great importance. Sec. 3 proposes the minimal system of deontic logic in the sense of von Wright's deontic necessity and shows its completeness, where the canonical relation is inspired by the previous almost definability schema. It also includes an explication of why the canonical model in [1] is defined in that way. Sec. 4 explores various extensions, where the transitive axiom is also inspired by the almost definability schema. Sec. 5 briefly reviews the pioneering work of Lemmon and gives an explanation of why the two semantics give the same logic and an alternative proof of completeness of a system given in [17]. We finally conclude in Sec. 6 with some discussions on deontic non-contingency and deontic contingency.

2 Syntax and Semantics

Throughout the current paper, we fix **P** to be a nonempty set of propositional variables and $p \in \mathbf{P}$.

Definition 1. The language $\mathcal{L}(\Box, \boxplus)$ is inductively defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \boxplus\varphi$$

Without the construct $\boxplus\varphi$, we obtain the language $\mathcal{L}(\Box)$ of standard modal logic; without the construct $\Box\varphi$, we obtain the language $\mathcal{L}(\boxplus)$ of *the logic of von Wright's deontic necessity*.

Intuitively, $\Box\varphi$ is read "it is (alethic) necessary that φ", $\boxplus\varphi$ is read "it is *obligatory* (in the sense of von Wright) that φ", or "it is *deontically necessary* that φ". Other connectives are defined as usual; in particular, $\Diamond$ and $\diamondplus$ are, respectively, the dual of $\Box$ and $\boxplus$. Formula $\diamondplus\varphi$ is read "it is *permitted* (in the sense of von Wright) that φ", or "it is *deontically possible* that φ". In the sequel, we will focus on $\mathcal{L}(\boxplus)$.

The above languages are interpreted over standard models. A standard model, or simply *model*, is a triple $\mathcal{M} = \langle S, R, V\rangle$, where S is a nonempty set of worlds, R a binary relation on S, and V a valuation function for propositional variables. A (*standard*) *frame* is a model without valuations. A *pointed model* is a pair of a model and a world in it.

Given a model $\mathcal{M} = \langle S, R, V\rangle$ and a world $s \in S$, the semantics of $\mathcal{L}(\Box, \boxplus)$ is defined as follows.

$\mathcal{M}, s \vDash p$	$\iff$	$s \in V(p)$
$\mathcal{M}, s \vDash \neg\varphi$	$\iff$	$\mathcal{M}, s \nvDash \varphi$
$\mathcal{M}, s \vDash \varphi \wedge \psi$	$\iff$	$\mathcal{M}, s \vDash \varphi$ and $\mathcal{M}, s \vDash \psi$
$\mathcal{M}, s \vDash \Box\varphi$	$\iff$	for any $t \in S$ such that $sRt, \mathcal{M}, t \vDash \varphi$
$\mathcal{M}, s \vDash \boxplus\varphi$	$\iff$	both for any $t \in S$ such that $sRt, \mathcal{M}, t \vDash \varphi,$ and there is an $u \in S$ such that sRu and $\mathcal{M}, u \vDash \varphi$.

We say that φ is *true* at the world s in the model $\mathcal{M}$, if $\mathcal{M}, s \vDash \varphi$; we say that φ is *valid on a model* $\mathcal{M}$, denoted $\mathcal{M} \vDash \varphi$, if φ is true at every world in $\mathcal{M}$; we say that φ is *valid in a frame* $\mathcal{F}$, denoted $\mathcal{F} \vDash \varphi$, if φ is valid on every model based on $\mathcal{F}$; we say that φ is *valid on a class* $\mathbb{C}$ *of frames*, denoted $\mathbb{C} \vDash \varphi$, if φ is valid on every frame in $\mathbb{C}$. We say that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are $\mathcal{L}(\boxplus)$-equivalent, denoted $(\mathcal{M}, s) \equiv (\mathcal{M}', s')$, if for all $\varphi \in \mathcal{L}(\boxplus)$, we have that $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$.

As shown in [1, Prop. 1.1], over serial models, $\Box\varphi$ and $\boxplus\varphi$ are equivalent to each other. Also, one may easily verify that $\boxplus$ and $\Box$ are interdefinable, since $\vDash \boxplus\varphi \leftrightarrow (\Box\varphi \wedge \Diamond\varphi)$ (also, $\vDash \boxplus\varphi \leftrightarrow (\Box\varphi \wedge \Diamond\top)$), and $\vDash \Box\varphi \leftrightarrow (\boxplus\top \to \boxplus\varphi)$.[3]

We have the following validities (that is, in von Wright's terminology, deontic tautologies). About the deontic intuition of these formulas, we refer to [20, pp. 13–14].[4]

The following two laws concerns about the relation between obligation and permission.

$$
\begin{array}{ll}
(i)a & \oplus\varphi \leftrightarrow \neg\boxplus\neg\varphi \\
(i)b & \boxplus\varphi \to \oplus\varphi
\end{array}
$$

The following four laws concerns about the "dissolution" of deontic operators.

$$
\begin{array}{ll}
(ii)a & \boxplus(\varphi \wedge \psi) \leftrightarrow (\boxplus\varphi \wedge \boxplus\psi) \\
(ii)b & \oplus(\varphi \vee \psi) \leftrightarrow (\oplus\varphi \vee \oplus\psi) \\
(ii)c & (\boxplus\varphi \vee \boxplus\psi) \to \boxplus(\varphi \vee \psi) \\
(ii)d & \oplus(\varphi \wedge \psi) \to (\oplus\varphi \wedge \oplus\psi)
\end{array}
$$

[3] This is suggested by Lloyd Humberstone in a private communication to Bezerra and Venturi, see [1, Footnote 1]. Despite this interdefinability, we will propose an almost definability schema below (see Prop. 2 and the remarks preceding it), which can help us find an explanation of why the canonical model in [1] is defined in that way, and also a desired axiom in axiomatizing $\mathcal{L}(\boxplus)$ over transitive frames. In contrast, the interdefinability in question cannot provide either of them, as one may check.

[4] Here we replace O and P in [20] with $\boxplus$ and $\oplus$, respectively, and A, B, C there with φ, ψ, χ, respectively. Note that to avoid some parentheses, von Wright rules that "a deontic operator before a molecular complex of names of acts refers to the whole complex and not to its first constituent only" ([20], p. 5), e.g., $P\ A \vee B$ means $P(A \vee B)$. This regulation is then cancelled in [19, Chap. 5].

Here are seven laws on "commitment".[5]

$$\begin{array}{ll}
(iii)a & \boxplus\varphi \wedge \boxplus(\varphi \to \psi) \to \boxplus\psi \\
(iii)b & \diamondplus\varphi \wedge \boxplus(\varphi \to \psi) \to \diamondplus\psi \\
(iii)c & \neg\diamondplus\psi \wedge \boxplus(\varphi \to \psi) \to \neg\diamondplus\varphi \\
(iii)d & \boxplus(\varphi \to \psi \vee \chi) \wedge \neg\diamondplus\psi \wedge \neg\diamondplus\chi \to \neg\diamondplus\varphi \\
(iii)e & \neg(\boxplus(\varphi \vee \psi) \wedge \neg\diamondplus\varphi \wedge \neg\diamondplus\psi) \\
(iii)f & \boxplus\varphi \wedge \boxplus(\varphi \wedge \psi \to \chi) \to \boxplus(\psi \to \chi) \\
(iii)g & \boxplus(\neg\varphi \to \varphi) \to \boxplus\varphi
\end{array}$$

As explained by von Wright, $(iii)a$ says that if doing what we ought to do commits us to doing something else, then this new act is also something which we ought to do, $(iii)b$ says that if doing what we are free to do commits us to doing something else, then this new act is also something which we are free to do (in other words, doing the permitted can never commit us to doing the forbidden), $(iii)c$ says that if doing something commits us to doing the forbidden, then we are forbidden to do the first thing, $(iii)d$ says that an act which commits us to a choice between forbidden alternatives is forbidden, $(iii)e$ says that it is logically impossible to be obliged to choose between forbidden alternatives, $(iii)f$ says that if doing two things, the first of which we ought to do, commits us to doing a third thing, then doing the second thing alone commits us to doing the third thing, "Our commitments are not affected by our (other) obligations", and $(iii)g$ says that if failure to perform an act commits us to performing it, then this act is obligatory.

We take the validity of $(iii)a$ as an example, which is shown to be a deontic tautology in [20, p. 12] by using the truth-value table method.

Proposition 1. *$\boxplus\varphi \wedge \boxplus(\varphi \to \psi) \to \boxplus\psi$ is valid.*

Proof. Suppose, for reductio, that for some model $\mathcal{M} = \langle S, R, V\rangle$ and some $s \in S$, we have $\mathcal{M}, s \vDash \boxplus\varphi \wedge \boxplus(\varphi \to \psi)$ and $\mathcal{M}, s \nvDash \boxplus\psi$. Then *either* there is a t such that sRt and $\mathcal{M}, t \nvDash \psi$, *or* for any u such that sRu, $\mathcal{M}, u \nvDash \psi$. If the first case is true, from $\mathcal{M}, s \vDash \boxplus\varphi \wedge \boxplus(\varphi \to \psi)$, we have $\mathcal{M}, t \vDash \varphi \wedge (\varphi \to \psi)$, which implies that $\mathcal{M}, t \vDash \psi$: a contradiction. If the second case is true, from $s \vDash \boxplus\varphi$, it follows that there is an x such that sRx and $\mathcal{M}, x \vDash \varphi$, and thus $\mathcal{M}, x \nvDash \psi$, which entails that $\mathcal{M}, x \nvDash \varphi \to \psi$, contrary to the fact that $\mathcal{M}, s \vDash \boxplus(\varphi \to \psi)$ and sRx. $\square$

One of our main interests is to understand the definition of the canonical model introduced in [1]. The crucial observation is the following schema, which says that given some deontically necessary proposition, $\Box$ is definable in terms of $\boxplus$:

$$\boxplus\psi \to (\Box\varphi \leftrightarrow \boxplus\varphi) \quad (\star)$$

[5] Von Wright mistakenly counts the number of laws as six, but it is actually seven. Many people would find von Wright's discussion of commitment to be quite implausible, and have developed dyadic deontic logics with an operator $O(\cdot/\cdot)$ ("conditional obligation") where the intended reading of $O(\varphi/\psi)$ is "it ought to be that φ, given that ψ". (See Example 4.4.4, p. 241, in [10].)

In this sense, we say that $\Box$ is *almost* definable with $\boxplus$. Unlike the almost definability schemas in [7, 8, 4], in the current almost definability schema, ψ only occurs in the antecedent $\boxplus\psi$ but not in the consequent $\Box\varphi \leftrightarrow \boxplus\varphi$. This may be explained by the fact that $\Box\varphi$ and $\boxplus\varphi$ are logically equivalent at *any given point* with a successor, which can be provided by any formula of the form $\boxplus\psi$. In fact, the new schematic letter ψ does not need to appear at all since we can replace it (thanks to the monotony of $\boxplus$) with $\top$ — but we still stick to use ψ rather than $\top$ because the former is more convenient in Remark 1. We leave the reader to check the following result.

Proposition 2. $\boxplus\psi \to (\Box\varphi \leftrightarrow \boxplus\varphi)$ *is a validity of* $\mathcal{L}(\Box, \boxplus)$.

3 Minimal Logic

3.1 Proof system

Definition 2. The minimal system $\mathbf{K}^{\boxplus}$ consists of the following axiom schemas and inference rules:

TAUT	all instances of propositional tautologies
$\boxplus$D	$\boxplus\varphi \to \Diamond\!\!\!+\varphi$
$\boxplus\wedge$	$\boxplus\varphi \wedge \boxplus\psi \to \boxplus(\varphi \wedge \psi)$
MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
RM$\boxplus$	$\dfrac{\varphi \to \psi}{\boxplus\varphi \to \boxplus\psi}$

Intuitively, the axiom $\boxplus$D says that obligatory things are permitted, the axiom $\boxplus\wedge$ says that the conjunction of two obligatory things is also obligatory, and the rule RM$\boxplus$ concerns about the monotony of the obligation operator.

Notice that our proof system $\mathbf{K}^{\boxplus}$ and the one in [1] only differs in that $\boxplus(\varphi \to \psi) \to (\boxplus\varphi \to \boxplus\psi)$ (denoted $\mathrm{K}_{\boxplus}$ there) is used there, whereas we use the axiom $\boxplus\wedge$.[6] The intention behind $\mathrm{K}_{\boxplus}$ is presumably to have the form resembling the $\Box$-based axiom K. In comparison, the axiom $\boxplus\wedge$ is found to satisfy the need of the truth lemma (and thus the completeness proof) below. At the end of this section, we will give a syntactic proof of $\mathrm{K}_{\boxplus}$ in our proof system.

The rule RM$\boxplus$ is called RK$\boxplus$ in [1]. This, for us, is a bit confusing, in that it concerns about the monotony, rather than the normality, of the $\boxplus$ operator, compared with their $\Box$-counterparts in normal modal logics, usually denoted RM and RK, respectively, refer to e.g. [2, 10].

Notions of derivability and theorems are defined as usual, and we use $\vdash \varphi$ to denote φ is derivable in $\mathbf{K}^{\boxplus}$.

It is easy to show the following result.

[6] The proof system in [1] is called **D2** in [12] and **CD** in [17].

Fact 1. The following rule, denoted RE$\boxplus$, is derivable in $\mathbf{K}^{\boxplus}$:

$$\frac{\varphi \leftrightarrow \psi}{\boxplus\varphi \leftrightarrow \boxplus\psi}.$$

Recall that von Wright writes:

> It should, however, be observed that if there really existed an act, say A, which is such that $P(A\& \sim A)$ expresses a true proposition, then every act would be permitted. ([19, p. 38])

This can be expressed in the language $\mathcal{L}(\boxplus)$ as $\oplus(\varphi \wedge \neg\varphi) \to \oplus\psi$,[7] which is derivable in $\mathbf{K}^{\boxplus}$ from TAUT and RM$\boxplus$ immediately.

3.2 Completeness

The proof of the completeness is based on a canonical model construction.

Definition 3. The *canonical model* of $\mathbf{K}^{\boxplus}$ is a triple $\mathcal{M}^c = \langle S^c, R^c, V^c\rangle$, where

- $S^c = \{s \mid s$ is a maximal consistent set for $\mathbf{K}^{\boxplus}\}$.
- sR^ct iff there exists ψ such that (a) $\boxplus\psi \in s$ and (b) for all φ, if $\boxplus\varphi \in s$, then $\varphi \in t$.
- $V^c(p) = \{s \in S \mid p \in s\}$.

The above definition of R^c is inspired by the schema $(\star)$, namely $\boxplus\psi \to (\Box\varphi \leftrightarrow \boxplus\varphi)$. Recall that in the construction of canonical model for the minimal normal modal logic, the canonical relation R^c is defined such that sR^ct iff for all φ, if $\Box\varphi \in s$, then $\varphi \in t$. Now given the schema $(\star)$, $\Box\varphi \in s$ can be replaced with $\boxplus\varphi \in s$ provided that $\boxplus\psi \in s$ for some ψ. Intuitively, if $\neg\boxplus\chi$ holds on a world for all χ, then so does $\neg\boxplus\top$, which implies that s is a dead point.

Note that due to the absence of ψ in the condition (b), the definition of R^c is equivalent to the following:

sR^ct iff (a) there exists ψ such that $\boxplus\psi \in s$ and (b) for all φ, if $\boxplus\varphi \in s$, then $\varphi \in t$.

Remark 1. One may easily verify that our definition of R^c is equivalent to (but simpler than) the definition of $R^{\boxplus}$ given in [1], where the canoical model $\mathcal{M}^{\boxplus} = \langle W^{\boxplus}, R^{\boxplus}, V^{\boxplus}\rangle$ is defined as follows:

1. $W^{\boxplus} = \{w \mid w$ is a maximal consistent set for $\mathbf{K}^{\boxplus}\}$ such that $W^{\boxplus} = W^s \cup W^{\neg s}$ where:
    - $w \in W^s$ iff for some φ we have $\boxplus\varphi \in w$;

[7] We thank an anonymous referee for this observation.

- $w \in W^{\neg s}$ iff there is no φ such that $\boxplus\varphi \in w$.

2. $R^{\boxplus} \subseteq W^{\boxplus} \times W^{\boxplus}$ is defined as folows: for all $w, y \in W^{\boxplus}$,
   - if $w \in W^s$, then $wR^{\boxplus}y$ iff $\lambda(w) \subseteq y$, where $\lambda(w) = \{\varphi \mid \boxplus\varphi \in w\}$.
   - if $w \in W^{\neg s}$, then there is no $y \in W^{\boxplus}$ such that $wR^{\boxplus}y$.
3. $V^{\boxplus}(p) = \{w \in W^{\boxplus} \mid p \in w\}$.

In this way, with the help of the schema $(\star)$, we have given an explanation of why the canonical model in [1] is defined as above.

Define $\eta(s) = \{\chi \mid \boxplus\chi \in s\}$ for $s \in S^c$. The following result lists some properties of the function η, which say that $\eta(s)$ is closed under conjunction and logical deduction. This will be used in the proof of the truth lemma.

Proposition 3.

1. *If* $\varphi, \psi \in \eta(s)$*, then* $\varphi \wedge \psi \in \eta(s)$.
2. *If* $\varphi \in \eta(s)$ *and* $\vdash \varphi \to \psi$*, then* $\psi \in \eta(s)$.

Proof. For 1, suppose that $\varphi, \psi \in \eta(s)$, then $\boxplus\varphi, \boxplus\psi \in s$. By axiom $\boxplus\wedge$, we infer that $\boxplus(\varphi \wedge \psi) \in s$, and therefore $\varphi \wedge \psi \in \eta(s)$.

For 2, assume that $\varphi \in \eta(s)$ and $\vdash \varphi \to \psi$, then $\boxplus\varphi \in s$. Applying the rule RM$\boxplus$, we derive that $\vdash \boxplus\varphi \to \boxplus\psi$, and therefore $\boxplus\psi \in s$, that is, $\psi \in \eta(s)$. $\square$

Now we are close to the demonstration of the truth lemma.

Lemma 1 (Truth Lemma)**.** *For all* $s \in S^c$*, for all* $\varphi \in \mathcal{L}(\boxplus)$*, we have*

$$\mathcal{M}^c, s \vDash \varphi \iff \varphi \in s.$$

Proof. By induction on φ. The only non-trivial case is $\boxplus\varphi$.

Suppose that $\boxplus\varphi \in s$, to show that $\mathcal{M}^c, s \vDash \boxplus\varphi$. By induction hypothesis, it suffices to prove that (1) for all $t \in S^c$, if sR^ct, then $\varphi \in t$, and (2) there exists $u \in S^c$ such that sR^cu and $\varphi \in u$. (1) is immediate by the supposition and the definition of R^c. For (2), by Lindenbaum's Lemma, it remains to demonstrate that $\eta(s) \cup \{\varphi\}$ is consistent.

If $\eta(s) \cup \{\varphi\}$ is not consistent, we consider two cases.

- $\eta(s) = \emptyset$. Then $\vdash \neg\varphi$, and thus $\vdash \varphi \to \neg\varphi$. Moreover, by supposition, we infer that $\varphi \in \eta(s)$. Then by the item 2 of Prop. 3, we derive that $\boxplus\neg\varphi \in s$.
- $\eta(s) \neq \emptyset$. Then there are $\chi_1, \cdots, \chi_n \in \eta(s)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \neg\varphi$. By the item 1 of Prop. 3, we have $\chi_1 \wedge \cdots \wedge \chi_n \in \eta(s)$. Then by the item 2 of Prop. 3, we infer that $\neg\varphi \in \eta(s)$, namely $\boxplus\neg\varphi \in s$.

In either case, we have $\boxplus\neg\varphi \in s$. However, by supposition and axiom $\boxplus$D, we also have $\Diamond\!\!\!+\varphi \in s$, that is, $\neg\boxplus\neg\varphi \in s$: a contradiction.

We have now shown that $\eta(s) \cup \{\varphi\}$ is consistent, as desired.

Conversely, assume that $\boxplus\varphi \notin s$, to prove that $\mathcal{M}^c, s \nvDash \boxplus\varphi$. For this, suppose, to exploit the induction hypothesis, that there exists $u \in S^c$ such that sR^cu and $\varphi \in u$, to find a $t \in S^c$ such that sR^ct and $\varphi \notin t$. For this, by Lindenbaum's Lemma, we only need to show that $\eta(s) \cup \{\neg\varphi\}$ is consistent.

By supposition, it follows that there exists ψ such that $\boxplus\psi \in s$. This provides the non-emptiness of $\eta(s)$. If $\eta(s) \cup \{\neg\varphi\}$ is not consistent, then as in the previous proof of the consistency of $\eta(s) \cup \{\varphi\}$, we can obtain that $\boxplus\varphi \in s$, which is contrary to the assumption.

We have thus shown that $\eta(s) \cup \{\neg\varphi\}$ is consistent, as desired. □

As a corollary, we can obtain the relationship between s having successors and s containing some $\boxplus\psi$.

Corollary 1. *The following conditions are equivalent:*

(1) *s has an R^c-successor;*

(2) *s contains some $\boxplus\psi$.*

Proof. The direction from (1) to (2) is immediate from the definition of R^c. The other direction follows from the proof of Lem. 1. □

We claim that every $\mathbf{K}^{\boxplus}$-consistent set of formulas can be extended to a maximal $\mathbf{K}^{\boxplus}$-consistent set in the standard way (Lindenbaum's Lemma).

With Lem. 1 and Lindenbaum's Lemma in hand, it is now a routine exercise to show the completeness of $\mathbf{K}^{\boxplus}$.

Theorem 1. $\mathbf{K}^{\boxplus}$ *is sound and strongly complete with respect to the class of all frames.*

We conclude this section with a syntactic proof of $\mathrm{K}_{\boxplus}$ in our system.

Proposition 4. $\vdash \boxplus(\varphi \to \psi) \to (\boxplus\varphi \to \boxplus\psi)$.

Proof. We have the following proof sequences in $\mathbf{K}^{\boxplus}$:

$$\begin{array}{lll}
(i) & \boxplus(\varphi \to \psi) \wedge \boxplus\varphi \to \boxplus((\varphi \to \psi) \wedge \varphi) & \boxplus\wedge \\
(ii) & (\varphi \to \psi) \wedge \varphi \to \psi & \mathrm{TAUT} \\
(iii) & \boxplus((\varphi \to \psi) \wedge \varphi) \to \boxplus\psi & (ii), \mathrm{RM}\boxplus \\
(iv) & \boxplus(\varphi \to \psi) \wedge \boxplus\varphi \to \boxplus\psi & (i), (iii) \\
(v) & \boxplus(\varphi \to \psi) \to (\boxplus\varphi \to \boxplus\psi) & (iv)
\end{array}$$

□

4 Extensions

This part explores some extensions of $\mathbf{K}^{\boxplus}$ over special frames and shows their completeness. First, define $\mathbf{D}^{\boxplus} = \mathbf{K}^{\boxplus} + \boxplus\top$.

Theorem 2. $\mathbf{D}^{\boxplus}$ *is sound and strongly complete with respect to the class of serial frames.*

Proof. The soundness follows from that of $\mathbf{K}^{\boxplus}$ and the validity of $\boxplus\top$, where the latter is straightforward.

For completeness, by Thm. 1, it suffices to show that R^c is serial. By axiom $\boxplus\top$, it remains to prove that $\eta(s)$ is consistent.

If not, then there are $\chi_1, \cdots, \chi_m \in \eta(s)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_m \to \bot$. Then by Prop. 3, we can obtain that $\bot \in \eta(s)$, that is, $\boxplus\bot \in s$. However, by axiom $\boxplus\top$ and axiom $\boxplus$D, $\diamondplus\top \in s$, that is, $\neg\boxplus\bot \in s$: a contradiction. $\square$

Define $\mathbf{T}^{\boxplus} = \mathbf{D}^{\boxplus} + \boxplus\mathrm{T}$, where $\boxplus\mathrm{T}$ denotes $\boxplus\varphi \to \varphi$.[8] Note that $\boxplus\top$ is indispensable in $\mathbf{T}^{\boxplus}$, which is different from the case in normal modal logics **T** and **D** (recall that the seriality axiom $\mathrm{D} = \Box\varphi \to \Diamond\varphi$ is derivable in $\mathbf{T} = \mathbf{K} + \mathrm{T}$, where **K** is the minimal normal modal logic, and $\mathrm{T} = \Box\varphi \to \varphi$). To see the "indispensability" part, define an auxiliary semantics which interprets all formulas of the form $\boxplus\varphi$ as $\bot$. One may easily verify that $\mathbf{T}^{\boxplus} - \boxplus\top$ is sound with respect to the auxiliary semantics, but $\boxplus\top$ is not.

Theorem 3. $\mathbf{T}^{\boxplus}$ *is sound and strongly complete with respect to the class of reflexive frames.*

Proof. The soundness is straightforward. For completeness, it suffices to prove that R^c is reflexive. That is, for all $s \in S^c$, sR^cs.

By axiom $\boxplus\top$, it suffices to prove that for all φ, if $\boxplus\varphi \in s$, then $\varphi \in s$. This follows immediately from axiom $\boxplus\mathrm{T}$. $\square$

Now define $\mathbf{K4}^{\boxplus} = \mathbf{K}^{\boxplus} + \boxplus 4$, where $\boxplus 4$ denotes

$$\boxplus\psi \wedge \boxplus\varphi \to \boxplus(\boxplus\chi \to \boxplus\varphi).^{9}$$

Note that $\boxplus 4$ is also inspired by the schema $(\star)$. Recall that $(\star)$ says that given some proposition $\boxplus\psi$, all $\Box\varphi$ can be replaced with $\boxplus\varphi$. In this way, we obtain the

[8] It may be worth mentioning the difference between the axiom (schema) $\boxplus\mathrm{T}$ and the axiom $\boxplus\top$.

[9] On [17, p. 220], a simpler axiom, called 4_0, is given as follows: $\boxplus\varphi \to \boxplus(\boxplus\top \to \boxplus\varphi)$. However, as will be explained below, our axiom $\boxplus 4$ is obtained from axiom 4 in normal modal logic via the schema $(\star)$. It is easy to see that 4_0 is derivable from $\boxplus 4$ by letting ψ and χ be, respectively, φ and $\top$.

axiom $\boxplus 4$ from axiom 4 ($\Box\varphi \to \Box\Box\varphi$). In detail,

$$\begin{array}{lll} & \boxplus\psi \to (\Box\varphi \to \Box(\boxplus\chi \to \Box\varphi) & (1) \\ \Leftrightarrow & \boxplus\psi \to (\boxplus\varphi \to \boxplus(\boxplus\chi \to \Box\varphi)) & (2) \\ \Leftrightarrow & \boxplus\psi \to (\boxplus\varphi \to \boxplus(\boxplus\chi \to \boxplus\varphi)) & (3) \\ \Leftrightarrow & \boxplus\psi \wedge \boxplus\varphi \to \boxplus(\boxplus\chi \to \boxplus\varphi) & (4) \end{array}$$

Instead of writing $\Box\varphi \to \Box\Box\varphi$, we write $\boxplus\psi \to (\Box\varphi \to \Box(\boxplus\chi \to \Box\varphi)$, because under the condition that $\boxplus\psi$ for some ψ, $\Box$ is definable (in the current setting, "definable" means "replacable") with $\boxplus$. The above transitions between (1) and (2) and between (2) and (3) follow from Prop. 2, and the transition from (3) to (4) is obtained via propositional reasoning.

Theorem 4. $\mathbf{K4}^{\boxplus}$ *is sound and strongly complete with respect to the class of transitive frames.*

Proof. For soundness, it suffices to show the validity of $\boxplus 4$. For this, suppose, for a contradiction, that there is a model $\mathcal{M} = \langle S, R, V\rangle$ and $s \in S$ such that $\mathcal{M}, s \vDash \boxplus\psi$ and $\mathcal{M}, s \vDash \boxplus\varphi$, but $\mathcal{M}, s \nvDash \boxplus(\boxplus\chi \to \boxplus\varphi)$. By $\mathcal{M}, s \nvDash \boxplus(\boxplus\chi \to \boxplus\varphi)$, we have *either* (i) there exists u such that sRu and $\mathcal{M}, u \nvDash \boxplus\chi \to \boxplus\varphi$, *or* (ii) for all x, if sRx, then $\mathcal{M}, x \nvDash \boxplus\chi \to \boxplus\varphi$. We consider the two cases:

- Case (i). By $\mathcal{M}, u \nvDash \boxplus\chi \to \boxplus\varphi$, we have $\mathcal{M}, u \vDash \boxplus\chi$ and $\mathcal{M}, u \nvDash \boxplus\varphi$. From the latter we have two subcases: *either* (i1) there exists v such that uRv and $\mathcal{M}, v \nvDash \varphi$, *or* (i2) for all y, if uRy, then $\mathcal{M}, y \nvDash \varphi$. If (i1) is the case, then by sRu, uRv and the transitivity of R, we derive that sRv, and thus $\mathcal{M}, v \vDash \varphi$ due to the supposition that $\mathcal{M}, s \vDash \boxplus\varphi$: a contradiction. If (i2) is the case, then from $\mathcal{M}, u \vDash \boxplus\chi$, it follows that u has a successor u', and thus $\mathcal{M}, u' \nvDash \varphi$. Similar to the first subcase, we can also arrive at a contradiction.
- Case (ii). Since $\mathcal{M}, s \vDash \boxplus\psi$, it follows that there exists t such that sRt, and thus $\mathcal{M}, t \nvDash \boxplus\chi \to \boxplus\varphi$. Then similar to case (i), we can reach a contradiction, as desired.

For completeness, suppose that $s, t, u \in S^c$ such that $sR^c t$ and $tR^c u$. By $sR^c t$, it follows that there exists ψ such that (a) $\boxplus\psi \in s$ and (b) for all φ, if $\boxplus\varphi \in s$, then $\varphi \in t$. By $tR^c u$, it follows that there exists χ such that (a') $\boxplus\chi \in t$, and (b') for all φ', if $\boxplus\varphi' \in t$, then $\varphi' \in u$. To prove that $sR^c u$, assume that $\boxplus\varphi \in s$, it suffices to show that $\varphi \in u$.

From (a) and the assumption and axiom $\boxplus 4$, we obtain that $\boxplus(\boxplus\chi \to \boxplus\varphi) \in s$. Then using (b), we derive that $\boxplus\chi \to \boxplus\varphi \in t$. This together with (a') implies that $\boxplus\varphi \in t$. Now using (b'), we conclude that $\varphi \in u$, as desired. $\Box$

5 Some Thoughts on Related Work

Inspired by neighborhood semantics of regular logics, in [17], a relational semantics of $\mathcal{L}(\boxplus)$ is proposed as follows.[10] A tuple $\mathcal{M} = \langle S, N, R, V\rangle$ is said to be a *non-standard model*, if $\langle S, R, V\rangle$ is a (standard) model defined as before, and $N \subseteq S$ is a set of *normal worlds*.[11] $\mathcal{L}(\boxplus)$ is interpreted on non-standard models where the cases for Boolean formulas are as usual and

$$\mathcal{M}, s \Vdash \boxplus\varphi \iff \text{both } s \in N \text{ and for each } t \in S \text{ such that } sRt, \mathcal{M}, t \Vdash \varphi.$$

It is shown in [17, p. 221] (and also [13, p. 62, Corollary]) that $\mathbf{K}^{\boxplus}$ (called **D2** or **CD** there) is determined by the class of all non-standard frames satisfying the condition that every normal world is serial; in symbol, if $s \in N$, then sRt for some t. Lemmon [13, p. 58] calls such frames "deontic model structures". To be consistent with our context, we will use the term "deontic non-standard frames" instead, and call those models underlying such frames "deontic non-standard models".

Now that we have two different semantics for $\boxplus$, one equipped with a set N of normal worlds (namely $\Vdash$), and the other not using such worlds but changing the truth condition so that $\boxplus$ has the conjunctive requirement "all accessible worlds verify φ and some accessible world verifies φ" for the truth of $\boxplus\varphi$ at a world (namely $\vDash$), which comes to give the same logic. Although Bezerra and Venturi mention this in [1], they do not attempt to explain why it should be so. In what follows, we give an explanation of it: the latter semantics is equivalent to the special case of the former in which the normal elements are precisely those with successors.[12]

Proposition 5. *For each* (*standard*) *model $\mathcal{M}$ and each world s in $\mathcal{M}$, there exists a denotic non-standard model $\mathcal{M}'$ and a world s' in $\mathcal{M}'$ such that for all $\varphi \in \mathcal{L}(\boxplus)$, we have*

$$\mathcal{M}, s \vDash \varphi \textit{ iff } \mathcal{M}', s' \Vdash \varphi.$$

Proof. Let $\mathcal{M} = \langle S, R, V\rangle$ be a (standard) model. Define $\mathcal{M}' = \langle S, N, R, V\rangle$ such that $N = \{s \in S \mid sRt \text{ for some } t \in S\}$, and let $s' = s$. It is straightforward

[10] For the sake of comparsion, we here use $\boxplus$ to replace the symbol $\Box$ adopted in [17]. The change of symbol is inessential to the results below.

[11] In his definition of *relational frames*, rather than using N, Segerberg uses Q to denote the set of singular elements, and the elements not in Q (that is, the elements in $S\backslash Q$, where S is the domain of the relational frame) are called *normal*, see [17, p. 23]. It should be easily seen from this and (iv$'$) therein that $S\backslash Q$ is just our N. It is also worth noting that the notion of "relational frames" is called "model structures" by Lemmon on page 56 in [13].

[12] That *only* normal points have successors is a condition that can be imposed on the current models without affecting the logic. (Those meeting this condition are the refined models of Lemmon ([14]).) The work done in validating $\boxplus$D is done by the converse: that *all* normal points have successors.

to see that $\mathcal{M}'$ is a deontic non-standard model. We proceed by induction on φ. The nontrivial case is $\boxplus\varphi$. We have the following equivalences:

$$\begin{array}{ll} & \mathcal{M}, s \vDash \boxplus\varphi \\ \iff & \text{both for each } t \in S \text{ such that } sRt, \mathcal{M}, t \vDash \varphi, \text{ and} \\ & \text{there is an } u \in S \text{ such that } sRu \text{ and } \mathcal{M}, u \vDash \varphi \\ \iff & \text{both for each } t \in S \text{ such that } sRt, \mathcal{M}, t \vDash \varphi, \text{ and} \\ & \text{there is an } u \in S \text{ such that } sRu \\ \iff & \text{both } s \in N \text{ and for each } t \in S \text{ such that } sRt, \mathcal{M}, t \Vdash \varphi \\ \iff & \mathcal{M}, s \Vdash \boxplus\varphi \end{array}$$

where the penultimate equivalence is due to the definition of N and induction hypothesis. □

With Prop. 5 in hand, we can give an alternative proof of $\mathbf{K}^{\boxplus}$ with respect to the class of deontic non-standard frames.

Theorem 5. $\mathbf{K}^{\boxplus}$ *is sound and strongly complete with respect to the class of deontic non-standard frames.*

Proof. The soundness is straightforward. For the completeness, by Thm. 1, every $\mathbf{K}^{\boxplus}$-consistent set is satisfiable in a (standard) model. Then by Prop. 5, every $\mathbf{K}^{\boxplus}$-consistent set is satisfiable in a deontic non-standard model. □

6 Discussion and Conclusion

In this paper, we built a bridge between von Wright's deontic logic ([20]) and Bezerra and Venturi's $\boxplus$-logic ([1]). On one hand, we provided the exact semantics for von Wright's deontic modalities; on the other hand, we provided a suitable interpretation of $\boxplus$-operator. Moreover, with the help of a schema, we gave an explanation of why the canonical relation in [1] is defined in that way. We also presented various axiomatizations of $\boxplus$-logic, among which the transitive system is also inspired by the schema in question. Last but not least, we explained why the two different semantics for $\boxplus$ involved in the literature give the same logic and gave an alternative proof of completeness of a system given in [17].

For the future work, one may study a notion of *deontic non-contingency* (or its dual, deontic contingency) and its logical properties. As known, (modal) non-contingency and contingency, usually denoted by Δ and ∇ respectively, are defined in terms of (modal) necessity and possibility, as $\Delta\varphi =: \Box\varphi \vee \Box\neg\varphi$ and $\nabla\varphi =: \Diamond\varphi \wedge \Diamond\neg\varphi$. (See e.g., [15, 3, 9, 11, 21, 7, 8, 6, 5]) Also, we have now a notion of deontic necessity. Then it is natural to introduce notions of deontic non-contingency and deontic contingency, as the following definitions show:

$$\begin{array}{ll} \mathbin{\triangle\!\!\!\!\!+}\,\varphi =: \boxplus\varphi \vee \boxplus\neg\varphi & (\text{Def. } \mathbin{\triangle\!\!\!\!\!+}) \\ \mathbin{\nabla\!\!\!\!\!+}\,\varphi =: \neg\mathbin{\triangle\!\!\!\!\!+}\,\varphi & (\text{Def. } \mathbin{\nabla\!\!\!\!\!+}) \end{array}$$

where $\triangle\!\!\!\!+\varphi$ and $\triangledown\!\!\!\!+\varphi$ are read "it is deontically non-contingent that φ" and "it is deontically contingent that φ", respectively.[13]

According to this definition and the semantics of $\boxplus$, we can define the semantics of $\triangle\!\!\!\!+$ in the following: given a model $\mathcal{M} = \langle W, R, V\rangle$,

$$\begin{array}{lcl} \mathcal{M}, w \vDash \triangle\!\!\!\!+\varphi & \Longleftrightarrow & \textit{either} \text{ (for any } u \in W \text{ such that } wRu, \mathcal{M}, u \vDash \varphi, \text{ and} \\ & & \text{there is an } x \text{ such that } wRx \text{ and } \mathcal{M}, x \vDash \varphi), \\ & & \textit{or} \text{ (for any } v \in W \text{ such that } wRv, \mathcal{M}, v \nvDash \varphi, \text{ and} \\ & & \text{there is a } y \text{ such that } wRy \text{ and } \mathcal{M}, y \nvDash \varphi). \end{array}$$

One may easily verify that $\vDash \triangle\!\!\!\!+\varphi \leftrightarrow (\Delta\varphi \wedge \Diamond\top)$. That is,

$$\begin{array}{lcl} \mathcal{M}, w \vDash \triangle\!\!\!\!+\varphi & \Longleftrightarrow & \textit{both} \text{ either for any } u \in W \text{ such that } wRu, \mathcal{M}, u \vDash \varphi \text{ or} \\ & & \text{for any } v \in W \text{ such that } wRv, \mathcal{M}, v \nvDash \varphi, \textit{ and} \\ & & \text{there is a } z \in W \text{ such that } wRz. \end{array}$$

Thus on the class of serial models, $\triangle\!\!\!\!+\varphi$ and $\Delta\varphi$ are equivalent, thus $\triangle\!\!\!\!+$ and Δ are interdefinable. This is similar to the case for $\boxplus$ and $\Box$: firstly, $\vDash \boxplus\varphi \leftrightarrow (\Box\varphi \wedge \Diamond\top)$; secondly, on the class of serial models, $\boxplus\varphi$ and $\Box\varphi$ are equivalent.

Different from the case for $\boxplus$ and $\Box$, $\triangle\!\!\!\!+$ and Δ are *not* interdefinable over the class of all models. For instance, consider the following two simple models:

$$\mathcal{M} \qquad s : p \qquad\qquad\qquad \mathcal{M}' \qquad s' : p$$

It should be straightforward to show that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by any $\mathcal{L}(\Delta)$-formula. However, $\mathcal{M}, s \nvDash \triangle\!\!\!\!+\varphi$ and $\mathcal{M}', s' \vDash \triangle\!\!\!\!+\varphi$ for all $\varphi \in \mathcal{L}(\triangle\!\!\!\!+)$.

One may easily verify that the following formulas are valid. This is similar to the case in non-contingency except for the inference rule wGEN$\triangle\!\!\!\!+$, refer to e.g. [11].

$$\begin{array}{ll} \triangle\!\!\!\!+\text{Equ} & \triangle\!\!\!\!+\varphi \leftrightarrow \triangle\!\!\!\!+\neg\varphi \\ \triangle\!\!\!\!+\text{Con} & (\triangle\!\!\!\!+\varphi \wedge \triangle\!\!\!\!+\psi) \to \triangle\!\!\!\!+(\varphi \wedge \psi) \\ \triangle\!\!\!\!+\text{Dis} & \triangle\!\!\!\!+\varphi \to \triangle\!\!\!\!+(\varphi \vee \psi) \vee \triangle\!\!\!\!+(\neg\varphi \vee \chi) \\ \text{wGEN}\triangle\!\!\!\!+ & \dfrac{\varphi}{\triangle\!\!\!\!+\psi \to \triangle\!\!\!\!+\varphi} \\ \text{RE}\triangle\!\!\!\!+ & \dfrac{\varphi \leftrightarrow \psi}{\triangle\!\!\!\!+\varphi \leftrightarrow \triangle\!\!\!\!+\psi} \end{array}$$

[13]Von Wright himself, in the original paper ([20, p. 4]) writes "The difference between the permitted and the indifferent among the deontic modes is analogous to the difference between the possible and the contingent among the alethic modes." So he thinks of the deontic analogue of contingency ($\nabla\varphi$) as simply the following: it is permissible that φ and it is permissible that not-φ, making the analogue of noncontingency be simply: it is obligatory that φ or it is obligatory that not-φ, without any further conjunct. However, as we argue before, von Wright's obligation operator should be understood as our $\boxplus$, thus correspondingly, his deontic analogue of contingency should be understood as our notion of deontic contingency.

Note that we have no $\triangle\top$ as a validity, or equivalently, the rule $\frac{\varphi}{\triangle\varphi}$ is not validity-preserving. This is an important distinction between $\triangle$ and Δ. Also, note that the rule wGEN$\triangle$ is equivalent to a formula $\triangle\psi \to \triangle\top$.[14]

We have the following almost-definability schemas.

Proposition 6. $\triangledown\psi \to (\boxplus\varphi \leftrightarrow (\triangle\varphi \wedge \triangle(\psi \to \varphi)))$ *is valid.*

Proposition 7. $\triangle\chi \wedge \triangledown\psi \to (\Box\varphi \leftrightarrow (\triangle\varphi \wedge \triangle(\psi \to \varphi)))$ *is valid.*

In the current stage, we do not know whether the above validities (plus TAUT and MP) completely axiomatize the $\triangle$-logic over the class of all frames. The difficulty lies in the requirement that the evaluated point should be serial in the semantics. We will leave it for future work.

References

[1] E. Bezerra and G. Venturi, 2021, "A non-standard Kripke semantics for the minimal deontic logic", *Logic and Logical Philosophy*, **30(1)**: 125–135.

[2] B. F. Chellas, 1980, *Modal Logic: An Introduction*, Cambridge: Cambridge University Press.

[3] M. Cresswell, 1988, "Necessity and contingency", *Studia Logica*, **47(2)**: 145–149.

[4] J. Fan, 2019, "Bimodal logics with contingency and accident", *Journal of Philosophical Logic*, **48(2)**: 425–445.

[5] J. Fan, 2019, "Symmetric contingency logic with unlimitedly many modalities", *Journal of Philosophical Logic*, **48(5)**: 851–866.

[6] J. Fan and H. van Ditmarsch, 2015, "Neighborhood contingency logic", in M. Banerjee and S. Krishna (eds.), *Logic and Its Application*, **Vol. 8923**, pp. 88–99, Berlin& Heidelberg: Springer.

[7] J. Fan, Y. Wang and H. van Ditmarsch, 2014, "Almost necessary", *Advances in Modal Logic*, **Vol. 10**, pp. 178–196, Norcross, GA: College Publications.

[8] J. Fan, Y. Wang and H. van Ditmarsch, 2015, "Contingency and knowing whether", *The Review of Symbolic Logic*, **8(1)**: 75–107.

[9] L. Humberstone, 1995, "The logic of non-contingency", *Notre Dame Journal of Formal Logic*, **36(2)**: 214–229.

[10] L. Humberstone, 2015, *Philosophical Applications of Modal Logic*, Norcross, GA: College Publications.

[11] S. Kuhn, 1995, "Minimal non-contingency logic", *Notre Dame Journal of Formal Logic*, **36(2)**: 230–234.

[12] E. J. Lemmon, 1957, "New foundations for Lewis modal systems", *Journal of Symbolic Logic*, **22(2)**: 176–186.

[14] It may be interesting to see that unlike in the case of the almost definability schema $(\star)$, the formula ψ in wGEN$\triangle$ cannot be replaced with $\top$.

[13] E. J. Lemmon, 1966, "Algebraic semantics for modal logics I", *The Journal of symbolic Logic*, **31(1)**: 46–65.

[14] E. J. Lemmon, 1966, "Algebraic semantics for modal logics II", *The Journal of symbolic Logic*, **31(2)**: 191–218.

[15] H. Montgomery and R. Routley, 1966, "Contingency and non-contingency bases for normal modal logics", *Logique et Analyse*, **9(35)**: 318–328.

[16] A. N. Prior, 1955, *Formal Logic*, Oxford: Clarendon Press.

[17] K. Segerberg, 1971, *An Essay in Classical Modal Logic*, Uppsala: Uppsala University.

[18] C. Steinsvold, 2011, "Being wrong: Logics for false belief", *Notre Dame Journal of Formal Logic*, **52(3)**: 245–253.

[19] G. H. von Wright, 1951, *An Essay in Modal Logic*, Amsterdam: North Holland.

[20] G. H. von Wright, 1951, "Deontic logic", *Mind*, **60(237)**: 1–15.

[21] E. Zolin, 1999, "Completeness and definability in the logic of noncontingency", *Notre Dame Journal of Formal Logic*, **40(4)**: 533–547.

冯 • 赖特道义必然的逻辑

范杰

摘 要

在本文中，我们在冯 • 赖特（G. von Wright）的道义逻辑和贝泽拉（E. Bezerra）与文丘里（G. Venturi）的 田 逻辑之间架起一座桥梁：一方面，我们将 田 算子解释成冯 • 赖特的道义必然；另一方面，我们给出冯 • 赖特道义模态词的确切语义。受启发于一个几乎可定义模式，我们解释为什么极小 田 逻辑的典范模型以那种方式被定义。我们也提出 田 逻辑的各种公理化，其中传递系统也是受到上述模式的启发。我们解释为什么文献中关于 田 的两种不等价语义，其中一个是标准的，另一个是非标准的，能给出相同的逻辑。在结尾部分，我们将讨论道义非偶然和道义偶然的概念。

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