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A Hilbert Calculus for Logic of Truth-Functional Contingency*

Fei Liang Zhongxu Tian[✉] Xinyu Yang

Abstract. A statement is *truth-functionally contingent*, if it is neither a tautology nor a contradiction in classical propositional logic. The logic of truth-functional contingency, is to capture all these contingent statements. In this paper, we introduce a sound and complete Hilbert calculus for the logic of truth-functional contingency, where every formula introduced in a deduction is a contingent formula, and it is introduced only if it is a contingent axiom, or it follows by one of the contingent rules of inference from contingent formulas introduced earlier in the deduction.

1 Introduction

The notion of *contingency* goes back to Aristotle, who develops a logic of statements about contingency. ([1]) From modal logic point of view, [5] firstly defines the contingent statement as *possibly true* and *possibly false*. The contingency logic, in which "contingency" is considered as a modal operator, has been well studied in [3, 4], etc. In this paper, we focus on *contingency* in classical propositional logic. Here, a statement φ is *contingent*, if there is an assignment v and an assignment u , s.t. $v(\varphi) = t$ and $u(\varphi) = f$. In other words, a statement is *contingent* if it is neither a tautology nor a contradiction. The logic of truth-functional contingency, is to capture all these contingent statements.

There are many works on the complementary sentential logics, such as [2, 7, 9], etc., all of which introduce different systems to capture non-tautologies in classical propositional logic. The first proper system for truth-functional contingency was introduced in [6]. Using this system, in order to obtain a contingent formula, we can translate a formula into its perfect disjunctive normal form by transformation rules, and then remove all redundant variables occurring in it. T. Tiomkin introduced

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Fei Liang School of Philosophy and Social Development, Shandong University
Institute of Concept and Reasoning, Shandong University
f.liang@sdu.edu.cn

Zhongxu Tian[✉] School of Philosophy and Social Development, Shandong University
201800011025@mail.sdu.edu.cn

Xinyu Yang School of Philosophy and Social Development, Shandong University
201900011027@mail.sdu.edu.cn

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a sound, complete and cut-free calculus for contingent sequent in classical propositional logic. ([8]) However, the cut rules and extension rules in [8] mix classical sequents with contingent sequents together. The standard Hilbert style calculus for this logic is still missing. In this paper, we are going to introduce a Hilbert calculus for the logic of truth-functional contingency, where every formula introduced in a deduction is a contingent formula, and it is introduced only if it is a contingent axiom, or it follows by one of the contingent rules of inference from contingent formulas introduced earlier in the deduction.

The paper is organized as follows. In the next section, we present syntax and semantics of truth-functional contingency. In Section 3, we introduce a Hilbert calculus for the logic of truth-functional contingency. In Section 4, we prove the soundness and completeness of the calculus. Finally, we present some directions for future research.

2 Preliminaries

In this section, we present syntax and semantics of truth-functional contingency which will be used in the following sections.

Fix a denumerable set $\mathcal{V}$ of atomic propositions. The language is defined recursively as follows:

$$\mathcal{F} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \supset \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi)$$

where $p \in \mathcal{V}$. In what follows, we will use $p, q, r, \ldots$ (with or without subscripts) to denote any atomic proposition, and $\varphi, \psi, \chi, \ldots$ (with or without subscripts) to denote any formula in $\mathcal{F}$. The outmost parenthesis of formulas will be omitted.

An assignment $v : \mathcal{V} \to \{t, f\}$ is defined as the same as in classical logic, and it can be uniquely extended to $\bar{v} : \mathcal{F} \to \{t, f\}$.

Definition 1 (Truth-functionally contingent formula) For any formula $\varphi \in \mathcal{F}$, we say φ is a *truth-functionally contingent formula*, abbreviated as contingent formula, if there is an assignment v and an assignment u, s.t. $\bar{v}(\varphi) = t$ and $\bar{u}(\varphi) = f$.

In what follows, we use $\models_c \varphi$ if φ is a contingent formula. The logic **LC** of truth-functional contingency is the set of all contingent formulas.

3 A Hilbert Calculus for LC

In this section, we will introduce a sound and complete Hilbert calculus HLC for **LC**, which is inspired by [2].

Before giving the calculus, we first introduce some notations which will be used in this section. For any formula φ, we let $\mathrm{Var}(\varphi) = \{p \mid p \text{ occurs in } \varphi\}$. Let the formula be the form of $\Delta = \Delta_1$ or $\Delta = \Delta_1 \supset (\Delta_2 \supset ...(\Delta_{n-1} \supset \Delta_n)...)$, where $\Delta_i = p_i$ or $\Delta_i = \neg p_i$, and $p_i \neq p_j$ for $i \neq j$.

Definition 2 (Hilbert calculus HLC) The Hilbert calculus HLC is defined as follows:

- **Axioms:**
  (A1) p (A2) $\neg p$

- **Rules:** let $p \notin \mathrm{Var}(\varphi)$, and $\mathrm{Var}(\varphi \supset \psi) \subseteq \mathrm{Var}(\Delta)$,

(R1a) $\dfrac{\varphi}{p \supset \varphi}$ (R1b) $\dfrac{\varphi}{\neg p \supset \varphi}$

(R2a) $\dfrac{\varphi}{\varphi \supset p}$ (R2b) $\dfrac{\varphi}{\varphi \supset \neg p}$

(R3) $\dfrac{\varphi \supset \psi}{\varphi \supset (\varphi \supset \psi)}$ (R4) $\dfrac{\varphi \supset \psi \qquad \psi}{(\chi \supset \varphi) \supset \psi}$

(R5) $\dfrac{\neg\varphi \supset \psi \qquad \psi}{(\varphi \supset \chi) \supset \psi}$ (R6) $\dfrac{\neg\varphi \supset \psi \qquad \varphi \supset \chi}{\varphi}$

(R7) $\dfrac{\varphi \supset \psi}{\neg\neg\varphi \supset \psi}$ (R8) $\dfrac{\varphi \supset (\psi \supset \chi)}{\psi \supset (\varphi \supset \chi)}$

(R9) $\dfrac{\varphi \supset (\psi \supset \chi)}{(\varphi \wedge \psi) \supset \chi}$ (R10) $\dfrac{\neg\varphi \supset (\neg\psi \supset \chi)}{\neg(\varphi \vee \psi) \supset \chi}$

(R11a) $\dfrac{(\varphi \supset \neg\psi) \supset \chi}{\neg(\varphi \wedge \psi) \supset \chi}$ (R11b) $\dfrac{(\psi \supset \neg\varphi) \supset \chi}{\neg(\varphi \wedge \psi) \supset \chi}$

(R12a) $\dfrac{\varphi \supset \psi \qquad \psi}{(\chi \vee \varphi) \supset \psi}$ (R12b) $\dfrac{\varphi \supset \psi \qquad \psi}{(\chi \vee \varphi) \supset \psi}$

(R13) $\dfrac{\varphi \supset \Delta \qquad \neg\psi \supset \Delta}{\neg(\varphi \supset \psi) \supset \Delta}$

A *deduction* in HLC is a finite sequence $\varphi_1, \ldots, \varphi_n$, s.t. φ_i ($i \in \{1, \ldots, n\}$) is either an axiom or a formula which is derived by applying rule(s) to its previous formula(s) occuring in the sequence. A formula φ is a *theorem* in HLC if there is a dedution $\varphi_1, \ldots, \varphi_n$, s.t. $\varphi = \varphi_n$. In what follows, we use $\vdash_c \varphi$ to mean φ is a theorem in HLC.

We first show some theorems in HLC, which will be used in the subsequent sections.

Corollary 1 The following formulas are theorems in HLC. For any Δ defined as above:

(1) $\vdash_c p \supset \neg p$;
(2) $\vdash_c \neg p \supset p$;
(3) $\vdash_c \Delta$;
(4) $\vdash_c \Delta_i \supset \Delta$ for $i < n$;
(5) $\vdash_c \neg\Delta_n \supset \Delta$;
(6) $\vdash_c \neg\Delta \supset \Delta$.

Proof We use proof tree to show above formulas are theorems in HLC. (1) can be shown as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q}\,\text{(A1)}}{p \supset q}\,\text{(R1a)}}{p \supset (p \supset q)}\,\text{(R3)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{q}\,\text{(A1)}}{p \supset q}\,\text{(R1a)}}{p \supset (p \supset q)}\,\text{(R3)}}{\neg\neg p \supset (p \supset q)}\,\text{(R7)}}{\neg(p \supset \neg p) \supset (p \supset q)}\,\text{(R13)} \qquad \dfrac{\dfrac{q}{\neg p \supset q}\,\text{(R1b)}}{(p \supset \neg p) \supset q}\,\text{(R4)}}{p \supset \neg p}\,\text{(R6)}$$

where $\Delta = p \supset q$ when we apply (R13) above. We omit the proof of (2) since it is quite similar to the proof above. (3) can be proven by applying (R1a) or (R1b) to (A1) or (A2) many times.

(4) can be shown by the following proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta_n}\,\text{(A1) or (A2)}}{\Delta_i \supset \Delta_n}\,\text{(R1a) or (R1b)}}{\Delta_{n-1} \supset (\Delta_i \supset \Delta_n)}\,\text{(R1a) or (R1b)}}{\Delta_i \supset (\Delta_{n-1} \supset \Delta_n)}\,\text{(R8)}$$

$$\vdots \quad \text{(R8),(R1a) or (R1b)}$$

$$\dfrac{\dfrac{\dfrac{\Delta_i \supset (\Delta_{i+1} \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots)}{\Delta_i \supset (\Delta_i \supset (\Delta_{i+1} \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots))}\,\text{(R3)}}{\Delta_{i-1} \supset (\Delta_i \supset (\Delta_i \supset (\Delta_{j+1} \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots)))}\,\text{(R1a) or (R1b)}}{\Delta_i \supset (\Delta_{i-1} \supset (\Delta_i \supset (\Delta_{j+1} \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots)))}\,\text{(R8)}$$

$$\vdots \quad \text{(R8),(R1a) or (R1b)}$$

$$\Delta_i \supset (\Delta_1 \supset \ldots \supset (\Delta_{i-1} \supset (\Delta_i \supset (\Delta_{i+1} \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n))))\ldots)$$

where $\Delta = \Delta_1 \supset (\ldots \supset (\Delta_{i-1} \supset (\Delta_i \supset (\Delta_{i+1} \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)))\ldots))$ for $i > 1$, and $\Delta = \Delta_i \supset (\ldots \supset (\Delta_{n-1} \supset \Delta_n))\ldots)$ for $i = 1$.

(5) can be proven as follows: If $\Delta_n = p_n$, then $\Delta = \Delta_1 \supset (\ldots \supset (\Delta_{i-1} \supset (\Delta_i \supset (\Delta_{i+1} \supset \ldots \supset (\Delta_{n-1} \supset p_n)))\ldots))$.

$$\dfrac{\dfrac{\dfrac{}{\neg p_n \supset p_n}\,(2)}{\Delta_{n-1} \supset (\neg p_n \supset p_n)}\,\text{(R1a) or (R1b)}}{\neg p_n \supset (\Delta_{n-1} \supset p_n)}\,\text{(R8)}$$

$$\vdots \qquad \text{(R8),(R1a) or (R1b)}$$

$$\neg p_n \supset (\Delta_1 \supset \ldots \supset (\Delta_{n-1} \supset p_n)\ldots)$$

If $\Delta_n = \neg p_n$, then $\Delta = \Delta_1 \supset (\ldots \supset (\Delta_{i-1} \supset (\Delta_i \supset (\Delta_{i+1} \supset \ldots \supset (\Delta_{n-1} \supset \neg p_n)))\ldots))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p_n \supset \neg p_n}\,(1)}{\neg\neg p_n \supset \neg p_n}\,\text{(R7)}}{\Delta_{n-1} \supset (\neg\neg p_n \supset \neg p_n)}\,\text{(R1a) or (R1b)}}{\neg\neg p_n \supset (\Delta_{n-1} \supset \neg p_n)}\,\text{(R8)}$$

$$\vdots \qquad \text{(R8),(R1a) or (R1b)}$$

$$\neg\neg p_n \supset (\Delta_1 \supset \ldots \supset (\Delta_{n-1} \supset \neg p_n)\ldots)$$

The proof of (6) is presented as follows:

$$\dfrac{\dfrac{}{\Delta_{n-1} \supset \Delta}\,(4) \qquad \dfrac{}{\neg\Delta_n \supset \Delta}\,(5)}{\neg(\Delta_{n-1} \supset \Delta_n) \supset \Delta}\,\text{(R13)}$$

$$\vdots \qquad \text{(R13), (4)}$$

$$\dfrac{\dfrac{\neg(\Delta_3 \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots) \supset \Delta \qquad \dfrac{}{\Delta_2 \supset \Delta}\,(4)}{\neg(\Delta_2 \supset \ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots) \supset \Delta}\,\text{(R13)} \qquad \dfrac{}{\Delta_1 \supset \Delta}\,(4)}{\neg(\Delta_1 \supset (\ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots) \supset \Delta}\,\text{(R13)}$$

where $\Delta = \Delta_1 \supset (\ldots \supset (\Delta_{n-1} \supset \Delta_n)\ldots)$. □

4 The Soundness and Completeness of HLC

In this section, we are going to show that HLC is sound and complete, that is, $\vdash_c \varphi$ iff $\models_c \varphi$ for any $\varphi \in \mathcal{F}$. We call HLC is *sound*, if every axiom in HLC is a contingent formula, and every rule is contingency-preserving, that is, if the premise(s) of a rule is(are) (a) contingent formula(s) then the conclusion of the rule is also a contingent formula. We call HLC is *complete* if every contingent formula is deducible in HLC.

Lemma 1 $\models_c \Delta$.

Proof It suffices to show that there exist assignments v and u, s.t. $\bar{v}(\Delta) = t$ and $\bar{u}(\Delta) = f$.

If $n = 1$, then $\Delta = \Delta_1 = p$ or $\neg p$. It is obvious that there exist assignments v and u, s.t. $\bar{v}(\Delta) = t$ and $\bar{u}(\Delta) = f$. If $n > 1$, then $\Delta = \Delta_1 \supset (... \supset (\Delta_{n-1} \supset \Delta_n)...)$. For any $1 \leq i \leq n$, there always exist v_i and u_i s.t. $\bar{v}_i(\Delta_i) = t$ and $\bar{u}_i(\Delta_i) = f$ by the construction of Δ (Definition 2). Since $\mathrm{Var}(\Delta_i) \neq \mathrm{Var}(\Delta_j)$ when $i \neq j$, we can define $\bar{v}$ and $\bar{u}$ as follows:

$$\bar{v}(p) = \begin{cases} \bar{u}_1(p) & p \in \mathrm{Var}(\Delta_1), \\ \bar{v}_i(p) & \text{otherwise.} \end{cases} \quad \bar{u}(p) = \begin{cases} \bar{v}_i(p) & p \in \mathrm{Var}(\Delta_i), \text{ for } 1 \leq i < n, \\ \bar{u}_n(p) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v}(\Delta) = t$ and $\bar{u}(\Delta) = f$. □

Theorem 1 (Soundness)**.** HLC *is sound.*

Proof That is to show that every theorem in HLC is a contingent formula. It is clear that the two axioms are contingent formulas, we only need to prove that all rules are contingency-preserving.

As to (R1a), suppose that $\models_c \varphi$, then there exist v' and u', s.t. $\bar{v'}(\varphi) = t$ and $\bar{u'}(\varphi) = f$. It is clear that $\models_c p$, that is, there exist v'' and u'' s.t. $\bar{v''}(p) = t$ and $\bar{u''}(p) = f$. Since $p \notin \mathrm{Var}(\varphi)$, we can define $\bar{v}$ and $\bar{u}$ as follows:

$$\bar{v}(p_i) = \begin{cases} \bar{u'}(p_i) & p_i \in \mathrm{Var}(\varphi), \\ \bar{v''}(p_i) & \text{otherwise.} \end{cases} \quad \bar{u}(p_i) = \begin{cases} \bar{v'}(p_i) & p_i \in \mathrm{Var}(\varphi), \\ \bar{u''}(p_i) & \text{otherwise.} \end{cases}$$

Hence, $\bar{v}(p \supset \varphi) = t$ and $\bar{u}(p \supset \varphi) = f$, this means $\models_c p \supset \varphi$. We omit the proofs of (R1b), (R2a) and (R2b), since they are quite similar to the proof above.

As to (R3), it suffices to show that if $\models_c \varphi \supset \psi$, then there exist assignments v and u, s.t. $\bar{v}(\varphi \supset (\varphi \supset \psi)) = t$ and $\bar{u}(\varphi \supset (\varphi \supset \psi)) = f$. Hence $\models_c \varphi \supset \psi$, then there exist $\bar{v}$ and $\bar{u}$ s.t. $\bar{v}(\varphi \supset \psi) = t$ and $\bar{u}(\varphi \supset \psi) = f$. We can define $\bar{v'}$ and $\bar{u'}$ as follows:

$$\bar{v'}(p_i) = \begin{cases} \bar{v}(p_i) & p_i \in \mathrm{Var}(\varphi \supset \psi), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases} \quad \bar{u'}(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\varphi \supset \psi), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v'}(\varphi \supset (\varphi \supset \psi)) = t$ and $\bar{u'}(\varphi \supset (\varphi \supset \psi)) = f$.

As to (R4), it suffices to show that if $\models_c \varphi \supset \psi$ and $\models_c \psi$, then there exist assignments v and u, s.t. $\bar{v}((\chi \supset \varphi) \supset \psi) = t$ and $\bar{u}((\chi \supset \varphi) \supset \psi) = f$. Hence $\models_c \varphi \supset \psi$, then there exist $\bar{v}$ and $\bar{u}$ s.t. $\bar{v}(\varphi \supset \psi) = t$ and $\bar{u}(\varphi \supset \psi) = f$, hence $\models_c \psi$, there exist assignments v and u, s.t. $\bar{v'}((\psi) = t$ and $\bar{u'}(\psi) = f$. We can define $\bar{v''}$ and $\bar{u''}$ as follows:

$$\bar{v''}(p_i) = \begin{cases} \bar{v'}(p_i) & p_i \in \mathrm{Var}(\psi), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases} \quad \bar{u''}(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\varphi \supset \psi), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v}''((\chi \supset \varphi) \supset \psi) = t$ and $\bar{u}''((\chi \supset \varphi) \supset \psi) = f$.

As to (R5), it suffices to show that if $\models_c \neg\varphi \supset \psi$ and $\models_c \psi$, then there exist assignments v and u, s.t. $\bar{v}((\varphi \supset \chi) \supset \psi) = t$ and $\bar{u}((\varphi \supset \varphi) \supset \psi) = f$. Hence $\models_c \neg\varphi \supset \psi$, then there exist $\bar{v}$ and $\bar{u}$ s.t. $\bar{v}(\neg\varphi \supset \psi) = t$ and $\bar{u}(\neg\varphi \supset \psi) = f$, hence $\models_c \psi$, there exist assignments v and u, s.t. $\bar{v}'(\psi) = t$ and $\bar{u}'(\psi) = f$. We can define $\bar{v}''$ and $\bar{u}''$ as follows:

$$\bar{v}''(p_i) = \begin{cases} \bar{v}'(p_i) & p_i \in \mathrm{Var}(\psi), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases} \quad \bar{u}''(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\neg\varphi \supset \psi), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v}''((\varphi \supset \chi) \supset \psi) = t$ and $\bar{u}''((\varphi \supset \chi) \supset \psi) = f$.

As to (R6), it suffices to show that if $\models_c \neg\varphi \supset \psi$ and $\models_c \varphi \supset \chi$, then there exist assignments v and u, s.t. $\bar{v}(\varphi) = t$ and $\bar{u}(\varphi) = f$. Hence $\models_c \neg\varphi \supset \psi$, then there exist $\bar{v}$ and $\bar{u}$, s.t. $\bar{v}(\neg\varphi \supset \psi) = t$ and $\bar{u}(\neg\varphi \supset \psi) = f$, hence $\models_c \varphi \supset \chi$, there exist assignments v and u, s.t. $\bar{v}'(\varphi \supset \chi) = t$ and $\bar{u}'(\varphi \supset \chi) = f$. We can define $\bar{v}''$ and $\bar{u}''$ as follows:

$$\bar{v}''(p_i) = \begin{cases} \bar{v}'(p_i) & p_i \in \mathrm{Var}(\varphi \supset \chi), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases} \quad \bar{u}''(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\neg\varphi \supset \psi), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v}''(\varphi) = t$ and $\bar{u}''(\varphi) = f$.

As to (R7), it suffices to show that if $\models_c \varphi \supset \psi$, then there exist assignments v and u, s.t. $\bar{v}(\neg\neg\varphi \supset \psi) = t$ and $\bar{u}(\neg\neg\varphi \supset \psi) = f$. Hence $\models_c \varphi \supset \psi$, then there exist $\bar{v}$ and $\bar{u}$ s.t. $\bar{v}(\varphi \supset \psi) = t$ and $\bar{u}(\varphi \supset \psi) = f$. We can define $\bar{v}'$ and $\bar{u}'$ as follows:

$$\bar{v}'(p_i) = \begin{cases} \bar{v}(p_i) & p_i \in \mathrm{Var}(\varphi \supset \psi), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases} \quad \bar{u}'(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\varphi \supset \psi), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v}'(\neg\neg\varphi \supset \psi) = t$ and $\bar{u}'(\neg\neg\varphi \supset \psi) = f$.

As to (R8), it suffices to show that if $\models_c \varphi \supset (\psi \supset \chi)$, then there exist assignments v and u, s.t. $\bar{v}(\psi \supset (\varphi \supset \chi)) = t$ and $\bar{u}(\psi \supset (\varphi \supset \chi)) = f$. Hence $\models_c \varphi \supset (\psi \supset \chi)$, then there exist $\bar{v}$ and $\bar{u}$ s.t. $\bar{v}(\varphi \supset (\psi \supset \chi)) = t$ and $\bar{u}(\varphi \supset (\psi \supset \chi)) = f$. We can define $\bar{v}'$ and $\bar{u}'$ as follows:

$$\bar{v}'(p_i) = \begin{cases} \bar{v}(p_i) & p_i \in \mathrm{Var}(\varphi \supset (\psi \supset \chi)), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases}$$
$$\bar{u}'(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\varphi \supset (\psi \supset \chi)), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v}'(\psi \supset (\varphi \supset \chi)) = t$ and $\bar{u}'(\varphi \supset (\varphi \supset \chi)) = f$.

As to (R9), it suffices to show that if $\models_c \varphi \supset (\psi \supset \chi)$, then there exist assignments v and u, s.t. $\bar{v}((\varphi \wedge \psi) \supset \chi) = t$ and $\bar{u}((\varphi \wedge \psi) \supset \chi) = f$. Hence

$\models_c \varphi \supset (\psi \supset \chi)$, then there exist $\bar{v}$ and $\bar{u}$ s.t. $\bar{v}(\varphi \supset (\psi \supset \chi)) = t$ and $\bar{u}(\varphi \supset (\psi \supset \chi)) = f$. We can define $\bar{v'}$ and $\bar{u'}$ as follows:

$$\bar{v'}(p_i) = \begin{cases} \bar{v}(p_i) & p_i \in \mathrm{Var}(\varphi \supset (\psi \supset \chi)), \\ \bar{u}(p_i) & \text{otherwise.} \end{cases}$$
$$\bar{u'}(p_i) = \begin{cases} \bar{u}(p_i) & p_i \in \mathrm{Var}(\varphi \supset (\psi \supset \chi)), \\ \bar{v}(p_i) & \text{otherwise.} \end{cases}$$

It is clear that $\bar{v'}((\varphi \wedge \psi) \supset \chi) = t$ and $\bar{u'}((\varphi \wedge \psi) \supset \chi) = f$.

As to (R10), if $\models_c \neg\varphi \supset (\neg\psi \supset \chi)$, then there exists $\bar{v}$ such that $\bar{v}(\neg\varphi \supset (\neg\psi \supset \chi)) = f$, hence, $\bar{v}(\varphi) = f$ and $\bar{v}(\psi) = \bar{v}(\chi) = f$, therefore $\bar{v}(\neg(\varphi \vee \psi) \supset \chi) = f$; and there exists $\bar{v'}$ such that $\bar{v'}(\neg\varphi \supset (\neg\psi \supset \chi)) = t$, hence $\bar{v'}(\varphi) = t$ or $\bar{v'}(\psi) = t$ or $\bar{v'}(\chi) = t$, all of these three cases ensure that $\bar{v'}(\neg(\varphi \vee \psi) \supset \chi) = t$, therefore we have $\models_c \neg(\varphi \vee \psi) \supset \chi$.

As to (R11a), if $\models_c (\varphi \supset \neg\psi) \supset \chi$, then there exists $\bar{v}$ such that $\bar{v}((\varphi \supset \neg\psi) \supset \chi) = f$, hence $\bar{v}(\chi) = f$, and $\bar{v}(\varphi) = f$ or $\bar{v}(\psi) = f$, therefore $\bar{v}(\neg(\varphi \wedge \psi) \supset \chi) = f$; and there exists $\bar{v'}$ such that $\bar{v'}((\varphi \supset \neg\psi) \supset \chi) = t$, hence $\bar{v'}(\chi) = t$ or $\bar{v'}(\varphi) = \bar{v'}(\psi) = t$, both of these two cases ensure that $\bar{v'}(\neg(\varphi \wedge \psi) \supset \chi) = t$, therefore we have $\models_c \neg(\varphi \wedge \psi) \supset \chi$.

As to (R11b), if $\models_c (\psi \supset \neg\varphi) \supset \chi$, then there exists $\bar{v}$ such that $\bar{v}((\psi \supset \neg\varphi) \supset \chi) = f$, hence $\bar{v}(\chi) = f$, and $\bar{v}(\varphi) = f$ or $\bar{v}(\psi) = f$, therefore $\bar{v}(\neg(\varphi \wedge \psi) \supset \chi) = f$; and there exists $\bar{v'}$ such that $\bar{v'}((\psi \supset \neg\varphi) \supset \chi) = t$, hence $\bar{v'}(\chi) = t$ or $\bar{v'}(\psi) = \bar{v'}(\varphi) = t$, both of these two cases ensure that $\bar{v'}(\neg(\varphi \wedge \psi) \supset \chi) = t$, therefore we have $\models_c \neg(\varphi \wedge \psi) \supset \chi$.

As to (R12a), if $\models_c \varphi \supset \psi$ and $\models_c \psi$, then there exists $\bar{v}$ such that $\bar{v}(\varphi \supset \psi) = t$, which means $\bar{v}(\varphi) = \bar{v}(\varphi \vee \chi) = t$, and $\bar{v}(\psi) = f$, therefore $\bar{v}((\varphi \vee \chi) \supset \psi) = f$; and there exists $\bar{v'}$ such that $\bar{v'}(\psi) = t$, which ensures $\bar{v'}((\varphi \vee \chi) \supset \psi) = t$, and finally we have $\models_c (\varphi \vee \chi) \supset \psi$. The proof of (R12b) is quite similar to the proof above and we omit it.

(R13) is a little more complex. Let Δ be as explained in the rule. If $\models_c \neg\psi \supset \Delta$, then there exists $\bar{v}$ such that $\bar{v}(\neg\psi \supset \Delta) = t$, which means $\bar{v}(\psi) = t$ or $\bar{v}(\Delta) = t$, both of these two cases ensure that $\bar{v}(\neg(\varphi \supset \psi) \supset \Delta) = t$. If $\models_c \varphi \supset \Delta$, then there exists $\bar{v'}$ such that $\bar{v'}(\varphi \supset \Delta) = f$, therefore $\bar{v'}(\Delta) = f$ and $\bar{v'}(\varphi) = t$. Because of the construction of Δ, there exists and only exists an assignment which makes Δ false, that is $\bar{v'}(\Delta_1) = ... = \bar{v'}(\Delta_{n-1}) = t$, $\bar{v'}(\Delta_n) = f$; additionally, for $\models_c \neg\psi \supset \Delta$, there exists a $\bar{u}$ such that $\bar{u}(\neg\psi \supset \Delta) = f$, therefore $\bar{u}(\Delta) = \bar{u}(\neg\psi) = f$. Since there exists and only exists an assignment that makes Δ false, we conclude that $\bar{v'} = \bar{u}$. In this case, $\bar{v'}(\neg(\varphi \supset \psi) \supset \Delta) = f$. Finally, we have $\models_c \neg(\varphi \supset \psi) \supset \Delta$. In summary, the soundness holds. □

Lemma 2 Let $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Delta)$ and $\Delta = \Delta_1 \supset (\Delta_2 \supset ... \supset (\Delta_{n-1} \supset \Delta_n)...)$ with $p_i \neq p_j$ for $i \neq j$, and $\Delta_i = p_i$ or $\Delta_i = \neg p_i$. Then, $\models_c \varphi \supset \Delta$ implies $\vdash_c \varphi \supset \Delta$.

Proof By induction on the complexity of the formula φ. If $\varphi = p_j$, there are two subcases.

Subcase 1: when $j < n$. If $\Delta_j = \neg p_j$, for any assignment v, either p_j or $\neg p_j$ is false, so that $p_j \supset \Delta$ is a tautology. Hence, the assumption implies that Δ_j must be p_j. Then the proof is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta_n}\,\text{(A1) or (A2)}}{p_j \supset \Delta_n}\,\text{(R1a)}}{\Delta_{n-1} \supset (p_j \supset \Delta_n)}\,\text{(R1a) or (R1b)}}{p_j \supset (\Delta_{n-1} \supset \Delta_n)}\,\text{(R8)}$$

$$\vdots \qquad \text{(R8), (R1a) or (R1b)}$$

$$\dfrac{\dfrac{\dfrac{p_j \supset (\Delta_{j+1} \supset ... \supset (\Delta_{n-1} \supset \Delta_n)...)}{p_j \supset (p_j \supset (\Delta_{j+1} \supset ... \supset (\Delta_{n-1} \supset \Delta_n)...))}\,\text{(R3)}}{\Delta_{j-1} \supset (p_j \supset (p_j \supset (\Delta_{j+1} \supset ... \supset (\Delta_{n-1} \supset \Delta_n)...)))}\,\text{(R1a) or (R1b)}}{p_j \supset (\Delta_{j-1} \supset (p_j \supset (\Delta_{j+1} \supset ... \supset (\Delta_{n-1} \supset \Delta_n)...)))}\,\text{(R8)}$$

$$\vdots \qquad \text{(R8), (R1a) or (R1b)}$$

$$p_j \supset (\Delta_1 \supset ... \supset (\Delta_{j-1} \supset (p_j \supset (\Delta_{j+1} \supset ... \supset (\Delta_{n-1} \supset \Delta_n))))...)$$

Subcase 2: when $j = n$. If $\Delta_n = p_n$, then $p_n \supset \Delta$ is a tautology, so that Δ_n must be $\neg p_n$, by Corollary 1(1), we have the proof as follows:

$$\dfrac{\dfrac{\dfrac{}{p_n \supset \neg p_n}\,\text{Corollary 1(1)}}{\Delta_{n-1} \supset (p_n \supset \neg p_n)}\,\text{(R1a) or (R1b)}}{p_n \supset (\Delta_{n-1} \supset \neg p_n)}\,\text{(R8)}$$

$$\vdots \qquad \text{(R8), (R1a) or (R1b)}$$

$$p_n \supset (\Delta_1 \supset ... \supset (\Delta_{n-1} \supset \Delta_n)...)$$

If $\varphi = \neg\psi$. By subinduction on the complexity of ψ:

If $\psi = p_j$, the proof of it is omitted since it is quite analogous to the proof above.

If $\psi = \neg\chi$, i.e. $\models_c \neg\neg\chi \supset \Delta$, it is clear that $\models_c \chi \supset \Delta$, then by induction hypothesis, $\vdash_c \chi \supset \Delta$, then by (R6), we have $\vdash_c \neg\neg\chi \supset \Delta$.

If $\psi = \chi \wedge \delta$, i.e. $\models_c \neg(\chi \wedge \delta) \supset \Delta$, then by Lemma 1, there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\neg\chi \supset \Delta) = \bar{v}(\neg\delta \supset \Delta) = t$. And there exists u such that $\bar{u}(\neg(\chi \wedge \delta) \supset \Delta) = f$, hence $\bar{u}(\Delta) = f$, and $\bar{u}(\chi)$ or $\bar{u}(\delta)$ is false, which makes $\bar{u}(\neg\chi \supset \Delta) = f$ or $\bar{u}(\neg\delta \supset \Delta) = f$. Now it is clear in this case we have

$\models_c \neg\chi \supset \Delta$ or $\models_c \neg\delta \supset \Delta$. By induction hypothesis, $\vdash_c \neg\chi \supset \Delta$ or $\vdash_c \neg\delta \supset \Delta$. Corollary 1(3) and $\vdash_c \neg\chi \supset \Delta$ imply that $\vdash_c \neg(\chi \wedge \delta) \supset \Delta$ by (R5) and (R11a), Corollary 1(3) and $\vdash_c \neg\delta \supset \Delta$ imply that $\vdash_c \neg(\chi \wedge \delta) \supset \Delta$ by (R4) and (R11a).

If $\psi = \chi \vee \delta$, i.e. $\models_c \neg(\chi \vee \delta) \supset \Delta$, then by Lemma 1, there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\neg\chi \supset \Delta) = \bar{v}(\neg\delta \supset \Delta) = t$. And there exists u such that $\bar{u}(\neg(\chi \vee \delta) \supset \Delta) = f$, hence, $\bar{u}(\Delta) = f$, $\bar{u}(\chi) = f$ and $\bar{u}(\delta) = f$, which makes $\bar{u}(\neg\chi \supset \Delta) = f$ and $\bar{u}(\neg\delta \supset \Delta) = f$. Now it is clear that we have $\models_c \neg\chi \supset \Delta$ and $\models_c \neg\delta \supset \Delta$. By induction hypothesis, $\vdash_c \neg\chi \supset \Delta$ and $\vdash_c \neg\delta \supset \Delta$. The proof of $\vdash_c \neg(\chi \vee \delta) \supset \Delta$ is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\delta \supset \Delta \qquad \dfrac{}{\neg\Delta \supset \Delta}\,\text{Corollary1(6)}}{\neg(\neg\delta \supset \Delta) \supset \Delta}\,\text{(R13)} \qquad \neg\chi \supset \Delta}{\neg(\neg\chi \supset (\neg\delta \supset \Delta)) \supset \Delta}\,\text{(R13)} \qquad \dfrac{\dfrac{\neg\delta \supset \Delta}{(\neg\delta \supset \Delta) \supset q (q \notin \mathrm{Var}(\Delta))}\,\text{(R2a)} \qquad \dfrac{}{q}\,\text{(A1)}}{(\neg\chi \supset (\neg\delta \supset \Delta)) \supset q}\,\text{(R4)}}{\neg\chi \supset (\neg\delta \supset \Delta)}\,\text{(R6)}}{\neg(\chi \vee \delta) \supset \Delta}\,\text{(R10)}$$

If $\psi = \chi \supset \delta$, i.e. $\models_c \neg(\chi \supset \delta) \supset \Delta$, then by Lemma 1, there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\chi \supset \Delta) = \bar{v}(\neg\delta \supset \Delta) = t$. And there exists u such that $\bar{u}(\neg(\chi \supset \delta) \supset \Delta) = f$, hence $\bar{u}(\Delta) = f$, $\bar{u}(\chi) = t$ and $\bar{u}(\delta) = f$, which make $\bar{u}(\chi \supset \Delta) = f$ and $\bar{u}(\neg\delta \supset \Delta) = f$. Now it is clear that we have $\models_c \chi \supset \Delta$ and $\models_c \neg\delta \supset \Delta$. By induction hypothesis, $\vdash_c \chi \supset \Delta$ and $\vdash_c \neg\delta \supset \Delta$. Then by (R13), we have $\vdash_c \neg(\chi \supset \delta) \supset \Delta$.

If $\varphi = \psi \wedge \chi$, i.e. $\models_c (\psi \wedge \chi) \supset \Delta$, then by Lemma 1, there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\psi \supset \Delta) = \bar{u}(\chi \supset \Delta) = t$, and there exists u such that $\bar{u}((\psi \wedge \chi) \supset \Delta) = f$, hence $\bar{u}(\Delta) = f$ and $\bar{u}(\psi) = \bar{u}(\chi) = t$. Therefore, $\bar{u}(\psi \supset \Delta) = f$ and $\bar{u}(\chi \supset \Delta) = f$. Now it is clear that we have $\models_c \psi \supset \Delta$ and $\models_c \chi \supset \Delta$. By induction hypothesis, $\vdash_c \psi \supset \Delta$ and $\vdash_c \chi \supset \Delta$. The proof of $\vdash_c (\psi \supset \chi) \supset \Delta$ is as follows:

$$\dfrac{\dfrac{\dfrac{\psi \supset \Delta \qquad \dfrac{\chi \supset \Delta \qquad \dfrac{}{\neg\Delta \supset \Delta}\,\text{Corollary1(6)}}{\neg(\chi \supset \Delta) \supset \Delta}\,\text{(R13)}}{\neg(\psi \supset (\chi \supset \Delta)) \supset \Delta}\,\text{(R13)} \qquad \dfrac{\dfrac{\chi \supset \Delta}{(\chi \supset \Delta) \supset q \quad (q \notin \mathrm{Var}(\Delta))}\,\text{(R2a)} \qquad \dfrac{}{q}\,\text{(A1)}}{(\psi \supset (\chi \supset \Delta)) \supset q}\,\text{(R4)}}{\psi \supset (\chi \supset \Delta)}\,\text{(R6)}}{(\psi \wedge \chi) \supset \Delta}\,\text{(R9)}$$

If $\varphi = \psi \vee \chi$, i.e. $\models_c (\psi \vee \chi) \supset \Delta$, then by Lemma 1, there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\psi \supset \Delta) = \bar{v}(\chi \supset \Delta) = t$. And there exists u such that $\bar{u}((\psi \vee \chi) \supset \Delta) = f$, hence $\bar{u}(\Delta) = f$, and $\bar{u}(\psi) = t$ or $\bar{u}(\chi) = t$. Therefore, $\bar{u}(\psi \supset \Delta) = f$ or $\bar{u}(\chi \supset \Delta) = f$. Now it is clear that we have $\models_c \psi \supset \Delta$ or $\models_c \chi \supset \Delta$. By induction hypothesis, $\vdash_c \psi \supset \Delta$ or $\vdash_c \chi \supset \Delta$. Corollary 1(3) and $\vdash_c \psi \supset \Delta$ imply that $\vdash_c (\psi \vee \chi) \supset \Delta$ by (R12a), Corollary 1(3) and $\vdash_c \chi \supset \Delta$ imply that $\vdash_c (\psi \vee \chi) \supset \Delta$ by (R12b).

If $\varphi = \psi \supset \chi$, i.e. $\models_c (\psi \supset \chi) \supset \Delta$, then by Lemma 1, there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\neg\psi \supset \Delta) = \bar{v}(\chi \supset \Delta) = t$. And there exists u such that $\bar{u}((\psi \supset \chi) \supset \Delta) = f$, hence $\bar{u}(\Delta) = f$, and $\bar{u}(\psi) = f$ or $\bar{u}(\chi) = t$. Therefore,

$\bar{u}(\neg\psi \supset \Delta) = f$ or $\bar{u}(\chi \supset \Delta) = f$. Now it is clear that we have $\models_c \neg\psi \supset \Delta$ or $\models_c \chi \supset \Delta$. By induction hypothesis, $\vdash_c \neg\psi \supset \Delta$ or $\vdash_c \chi \supset \Delta$. Corollary1(3) and $\vdash_c \neg\psi \supset \Delta$ imply that $\vdash_c (\psi \supset \chi) \supset \Delta$ by (R5), Corollary1(3) and $\vdash_c \chi \supset \Delta$ imply that $\vdash_c (\psi \supset \chi) \supset \Delta$ by (R4). □

Theorem 2 (Completeness)**.** HLC *is complete.*

Proof That is to show that every contingent formula is a theorem in HLC. The proof strategy is as follows: we first show that $\models_c \varphi \supset \Delta$ and $\models_c \neg\varphi \supset \Delta'$ by the assumption $\models_c \varphi$, and then use Lemma 2 to show that $\vdash_c \varphi$. Since $\models_c \varphi$, there exist u and w, such that $\bar{u}(\varphi) = t$ and $\bar{w}(\varphi) = f$. Assume that $\mathrm{Var}(\varphi) = \{p_1, ..., p_n\}$, define q_i:

$$q_i = \begin{cases} p_i & \bar{u}(p_i) = t, \\ \neg p_i & \bar{u}(p_i) = f. \end{cases}$$

Then $\bar{u}(q_i) = t$ for $i \leq n$. Let $\Delta = q_1 \supset (q_2 \supset ... \supset (q_{n-1} \supset \neg q_n)...)$, so that $\bar{u}(\Delta) = f$ and hence $\bar{u}(\varphi \supset \Delta) = f$.

Define r_i:

$$r_i = \begin{cases} p_i & \bar{w}(p_i) = t, \\ \neg p_i & \bar{w}(p_i) = f. \end{cases}$$

Then $\bar{w}(q_i) = t$ for $i \leq n$. Let $\Delta' = r_1 \supset (r_2 \supset ... \supset (r_{n-1} \supset \neg r_n)...)$, so that $\bar{u}(\Delta') = f$ and hence $\bar{w}(\neg\varphi \supset \Delta') = f$. By Lemma 1 and the construction of Δ and Δ', there exists v such that $\bar{v}(\Delta) = t$, which makes $\bar{v}(\varphi \supset \Delta) = t$, and v' such that $\bar{v'}(\Delta') = t$, which makes $\bar{v}(\neg\varphi \supset \Delta') = t$. Therefore, $\models_c \varphi \supset \Delta$ and $\models_c \neg\varphi \supset \Delta'$, by Lemma 2, we have $\vdash_c \varphi \supset \Delta$ and $\vdash_c \neg\varphi \supset \Delta'$. Then by (R6), it is obtained that $\vdash \varphi$. □

Corollary 2 $\vdash_c \varphi$ iff $\vdash_c \neg\varphi$.

Proof From left to right direction, assume that $\vdash_c \varphi$, then by Theorem 1, we have $\models_c \varphi$, which means there exists v such that $\bar{v}(\varphi) = t$, i.e. $\bar{v}(\neg\varphi) = f$, and there exists u such that $\bar{u}(\varphi) = f$, i.e. $\bar{v}(\neg\varphi) = t$, therefore, $\models_c \neg\varphi$. By Theorem 2, we have $\vdash_c \neg\varphi$. If $\vdash_c \neg\varphi$, by Theorem 1, we have $\models_c \neg\varphi$, which means there exists v such that $\bar{v}(\neg\varphi) = t$, i.e. $\bar{v}(\varphi) = f$ and there exists u such that $\bar{u}(\neg\varphi) = f$, i.e. $\bar{v}(\varphi) = t$, therefore $\models_c \varphi$. By Theorem 2, we have $\vdash_c \varphi$. □

5 Future work

In this paper, we introduced a complete and sound Hilbert calculus for the logic of truth-functional contingency, which captures all contingent formulas in classical logic. The quantificational extension of the logic **LC** will be our next work. Inspired by [8], to develop a pure sequent calculus for contingent sequent without using classical sequent will be a challenge work in future. Another interesting topic is to study the relation between contingency logic ([4]), as we introduced in Section 1, and **LC** in this paper.

References

[1] A. P. Brogan, 1967, "Aristotle's logic of statements about contingency", *Mind*, **76(301)**: 49–61.

[2] X. Caicedo, 1978, "A formal system for the non-theorems of the propositional calculus", *Notre Dame Journal of Formal Logic*, **19(1)**: 147–151.

[3] J. Fan and H. van Ditmarsch, 2019, "Neighborhood contingency logic", *Notre Dame Journal of Formal Logic*, **60(4)**: 683–699.

[4] J. Fan, Y. Wang and H. van Ditmarsch, 2014, "Almost necessary", *Advances in Model Logic*, **Vol. 10**, pp. 178–196.

[5] H. Montgomery and R. Routley, 1966, "Contingency and non-contingency bases for normal modal logics", *Logique et Analyse*, **9(35)**: 318–328.

[6] C. Morgan, A. Hertel and P. Hertel, 2007, "A sound and complete proof theory for propositional logical contingencies", *Notre Dame Journal of Formal Logic*, **48(4)**: 521–530.

[7] C. G. Morgan, 1973, "Sentential calculus for logical falsehoods", *Notre Dame Journal of Formal Logic*, **14(3)**: 347–353.

[8] M. Tiomkin, 2013, "A sequent calculus for a logic of contingencies." *Journal of Applied Logic*, **11(4)**: 530–535.

[9] A. C. Varzi, 1990, "Complementary sentential logics", *Bulletin of the Section of Logic*, **19(4)**: 112–116.

一个真值函项偶然逻辑的希尔伯特演算系统

梁 飞 田中旭[✉] 杨新宇

摘 要

如果一个命题在经典命题逻辑中既不是一个重言式也不是一个矛盾式，则称它是真值函项偶然的。真值函项偶然逻辑即是为了刻画所有真值函项偶然的命题。本文将给出一个关于真值函项偶然逻辑的可靠且完全的希尔伯特演算。在此演算中，通过演绎所得到的公式要么是偶然公理，要么是由偶然规则推出的。

梁 飞 山东大学哲学与社会发展学院
山东大学概念与推理研究所
f.liang@sdu.edu.cn
田中旭[✉] 山东大学哲学与社会发展学院
201800011025@mail.sdu.edu.cn
杨新宇 山东大学哲学与社会发展学院
201900011027@mail.sdu.edu.cn