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Sahlqvist Correspondence for Instantial Neighbourhood Logic*

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Abstract. In the present paper, we investigate the Sahlqvist-type correspondence theory for instantial neighbourhood logic (INL), which can talk about existential information about the neighbourhoods of a given world and is a mixture between relational semantics and neighbourhood semantics. The increased expressivity and its ability to talk about certain relational patterns of the neighbourhood function makes it possible to ask what kind of properties can this language define on the frame level, whether the "Sahlqvist" fragment of instantial neighbourhood logic could be larger than the rather small KW-fragment. (H. Hansen, 2003) We have two proofs of the correspondence results, the first proof is obtained by using standard translation and minimal valuation techniques directly, the second proof follows M. Gehrke et al. (2005) and H. Hansen (2003), where we use bimodal translation method to reduce the correspondence problem in instantial neighbourhood logic to normal bimodal logics in classical Kripke semantics. We give some remarks and future directions at the end of the paper.

1 Introduction

Recently, a variant of neighbourhood semantics for modal logics was given, under the name of instantial neighbourhood logic (INL), where existential information about the neighbourhoods of a given world can be added. ([5, 13, 2, 3, 4, 14, 15]) This semantics is a mixture between relational semantics and neighbourhood semantics, and its expressive power is strictly stronger than neighbourhood semantics.

In this semantics, the $(n+1)$-ary modality $\Box(\psi_1, \ldots, \psi_n; \varphi)$ is true at a world w if and only if there exists a neighbourhood $S \in N(w)$ such that φ is true everywhere in S, and each ψ_i is true at $w_i \in S$ for some w_i. This language has a natural interpretation as a logic of computation in open systems: $\Box(\psi_1, \ldots, \psi_n; \varphi)$ can be interpreted as "in the system, the agent has an action to enforce the condition φ while simultaneously allowing possible outcomes satisfying each of the conditions $\psi_1, \ldots, \psi_n$"; this language can describe not only what properties can be guaranteed by an agent's action, but also exactly what possible outcomes may be achieved from this action (see [3]).

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Instantial neighbourhood logic is first introduced in [4], where the authors defines the notion of bisimulation for instantial neighbourhood logic, gives a complete axiomatic system, and determines its precise SAT complexity; in [13], the canonical rules are defined for instantial neighbourhood logic; in [2], the game-theoretic aspects of instantial neighbourhood logic is studied; in [3], a propositional dynamic logic IPDL is obtained by combining instantial neighbourhood logic with propositional dynamic logic (PDL), its sound and complete axiomatic system is given as well as its finite model property and decidability; in [5], the duality theory for instantial neighbourhood logic is developed via coalgebraic method; in [14], a tableau system for instantial neighbourhood logic is given which can be used for mechanical proof and countermodel search; in [15], a cut-free sequent calculus and a constructive proof of its Lyndon interpolation theorem is given. However, the Sahlqvist-type correspondence theory is still unexplored, which is the theme of this paper; in addition, the increased expressivity makes it possible to ask what kind of properties can this language define on the frame level, whether the "Sahlqvist" fragment of instantial neighbourhood logic could be larger than the rather small KW-fragment in [10] in monotone modal logic.

In this paper, we define the Sahlqvist formulas in the instantial neighbourhood modal language, and give two different proofs of correspondence results. The first proof is given by standard translation and minimal valuation techniques as in [6, Section 3.6], while the second proof uses bimodal translation method in monotone modal logic and neighbourhood semantics ([10, 11, 12, 1]) to show that every Sahlqvist formula in the instantial neighbourhood modal language can be translated into a bimodal Sahlqvist formula in Kripke semantics, and hence has a first-order correspondent. The first proof is standard and it reveals how the instantial neighbourhood semantics have the relational pattern, and the second proof is simpler and easier to understand.

The structure of the paper is as follows: in Section 2, we give a brief sketch on the preliminaries of instantial neighbourhood logic, including its syntax and neighbourhood semantics. In Section 3, we define the standard translation of instantial neighbourhood logic into a two-sorted first-order language. In Section 4, we define Sahlqvist formulas in instantial neighbourhood logic, and prove the Sahlqvist correspondence theorem via standard translation and minimal valuation. In Section 5, we discuss the translation of instantial neighbourhood logic into normal bimodal logic, and prove Sahlqvist correspondence theorem via this bimodal translation. In Section 6, we give some examples. We give some remarks and further directions in Section 7.

2 Preliminaries on Instantial Neighbourhood Logic

In this section, we collect some preliminaries on instantial neighbourhood logic, which can be found in [4].

Syntax. The formulas of instantial neighbourhood logic are defined as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$$

where $p \in \mathsf{Prop}$ is a propositional variable, $\Box_n$ is an $(n{+}1)$-ary modality for each $n \in \mathbb{N}$. $\rightarrow, \leftrightarrow$ can be defined in the standard way. An occurence of p is said to be *positive* (resp. *negative*) in φ if p is under the scope of an even (resp. odd) number of negations. A formula φ is positive (resp. negative) if all occurences of propositional variables in φ are positive (resp. negative).

Semantics. For the semantics of instantial neighbourhood logic, we use neighbourhood frames to interpret the instantial neighbourhood modality, one and the same neighbourhood function for all the $(n{+}1)$-ary modalities for all $n \in \mathbb{N}$.

Definition 1 (Neighbourhood frames and models) A *neighbourhood frame* is a pair $\mathbb{F} = (W, N)$ where $W \neq \varnothing$ is the set of worlds, $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ is a map called a *neighbourhood function* (notice that there is no restriction on what additional properties N should satisfy, e.g. $w \in X$ for all $X \in N(w)$, or upward-closedness: $X \in N(w)$ and $X \subseteq Y$ implies $Y \in N(w)$), where $\mathcal{P}(W)$ is the powerset of W. A *valuation* on W is a map $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$. A triple $\mathbb{M} = (W, N, V)$ is called a *neighbourhood model* or a neighbourhood model based on (W, N) if (W, N) is a neighbourhood frame and V is a valuation on it.

The semantic clauses for the Boolean part is standard. For the instantial neighbourhood modality $\Box$, the satisfaction relation is defined as follows:

$\mathbb{M}, w \Vdash \Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$ iff there is $S \in N(w)$ such that for all $s \in S$ we have $\mathbb{M}, s \Vdash \varphi$ and for all $i = 1, \ldots, n$ there is an $s_i \in S$ such that $\mathbb{M}, s_i \Vdash \varphi_i$.

Semantic properties of instantial neighbourhood modalities. It is easy to see the following lemma, which states that the $(n{+}1)$-ary instantial neighbourhood modality $\Box_n$ is monotone in every coordinate, and is completely additive (and hence monotone) in the first n coordinates (even if the neighbourhood function is not upward-closed). This observation is useful in the algebraic correspondence analysis in instantial neighbourhood logic.

Lemma 1

1. For any $\mathbb{F} = (W, N)$, any $w \in W$ and any valuations $V_1, V_2 : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ such that $V_1(p) \subseteq V_2(p)$, $V_1(p_i) \subseteq V_2(p_i)$ for all $i = 1, \ldots, n$,

   if $\mathbb{F}, V_1, w \Vdash \Box_n(p_1, \ldots, p_n; p)$, then $\mathbb{F}, V_2, w \Vdash \Box_n(p_1, \ldots, p_n; p)$;

2. For any $\mathbb{F} = (W, N)$, any $w \in W$ and any valuation $V : \mathsf{Prop} \to \mathcal{P}(W)$, fix an $i \in \{1, \dots, n\}$ and a $v \in W$, and define $V_{i,v} : \mathsf{Prop} \to \mathcal{P}(W)$ such that $V_{i,v}(p_j) = V(p_j)$ for $j \neq i$, and $V_{i,v}(p_i) = \{v\}$. Then the following holds:

$$\mathbb{F}, V, w \Vdash \Box_n(p_1, \dots, p_i, \dots, p_n; p) \text{ iff there exists a } v \in V(p_i) \text{ such that } \mathbb{F}, V_{i,v}, w \Vdash \Box_n(p_1, \dots, p_i, \dots, p_n; p).$$

Algebraically, if we view the $(n+1)$-ary modality $\Box_n$ as an $(n+1)$-ary function $\Box_n^{\mathbb{A}} : \mathbb{A}^{n+1} \to \mathbb{A}$, then $\Box_n^{\mathbb{A}}(a_1, \dots, a_n; a)$ is completely additive (i.e. preserve arbitrary joins) in the first n coordinate, and monotone in the last coordinate. This observation is useful in the algebraic correspondence analysis (see Section 7).

Getting standard neighbourhood semantics and Kripke semantics from INL. As we have already seen in [4], instantial neighbourhood logic can express standard monotone neighbourhood modalities by just taking $n = 0$, i.e.,

$$\mathbb{M}, w \Vdash \Box_0\varphi \text{ iff there is } S \in N(w) \text{ such that for all } s \in S \text{ we have } \mathbb{M}, s \Vdash \varphi.$$

Indeed, from the definition of N we can define some induced $(n+1)$-ary relations, and instantial neighbourhood logic can reason about these relational structures. Here we take binary relation and the binary modality $\Box_1$ as an example:

We can define the following binary relation $R_{1,\top}$ based on the neighbourhood function N:

$$R_{1,\top}wv \quad \text{iff} \quad \text{there exists an } X \in N(w) \text{ such that } v \in X \quad \text{iff} \quad v \in \bigcup N(w).$$

Then it is easy to see that

$$\mathbb{M}, w \Vdash \Box_1(\varphi_1; \top) \text{ iff there exists a } v \text{ such that } R_{1,\top}wv \text{ and } \mathbb{M}, v \Vdash \varphi_1.$$

Therefore, instantial neighbourhood logic can talk about certain relational structures behind the neighbourhood function. Indeed, we will expand on this phenomenon later on (see Section 4.2) when we analyze when instantial neighbourhood logic become "normal".

3 Standard Translation of Instantial Neighbourhood Logic

3.1 Two-sorted first-order language $\mathcal{L}_1$ and standard translation

Given the INL language, we consider the corresponding two sorted first-order language $\mathcal{L}_1$, which is going to be interpreted in a two-sorted domain $W_w \times W_s$. For a more detailed treatment, see [10, 7]. This language is used in the treatment of the standard translation for neighbourhood semantics. The major pattern of this language is that we treat worlds and subsets of worlds as two different sorts, which makes it different from standard first-order language. In addition, allowing quantification over subsets of worlds makes the language have some flavor of second-order logic, but here we treat those subsets of worlds as first-order objects in the second domain W_s.

This language has the following ingredients:

1. world variables $x, y, z, \ldots$, to be interpreted as possible worlds in the world domain W_w;
2. subset variables $X, Y, Z, \ldots$, to be interpreted as objects in the subset domain $W_s = \{X \mid X \subseteq W_w\}$;[1]
3. a binary relation symbol $R_\ni$, to be interpreted as the reverse membership relation $R^\ni \subseteq W_s \times W_w$ such that $R^\ni Xx$ iff $x \in X$;
4. a binary relation symbol R_N, to be interpreted as the neighbourhood relation $R^N \subseteq W_w \times W_s$ such that $R^N xX$ iff $X \in N(x)$;
5. unary predicate symbols P_1, P_2,…, to be interpreted as subsets of the world domain W_w.

We also consider the following second-order language $\mathcal{L}_2$ which is obtained by adding second-order quantifiers $\forall P_1, \forall P_2$,…over the world domain W_w. Existential second-order quantifiers $\exists P_1, \exists P_2, \ldots$ are interpreted in the standard way. Notice that here the second-order variables P_1,…are different from the subset variables $X, Y, Z, \ldots$, since the former are interpreted as subsets of W_w, and the latter are interpreted as objects in W_s.

Now we define the standard translation $ST_w(\varphi)$ as follows:

Definition 2 (Standard translation) For any INL formula φ and any world symbol x, the standard translation $ST_x(\varphi)$ of φ at x is defined as follows:

- $ST_x(p) := Px$;
- $ST_x(\bot) := x \neq x$;
- $ST_x(\top) := x = x$;
- $ST_x(\neg\varphi) := \neg ST_x(\varphi)$;
- $ST_x(\varphi \wedge \psi) := ST_x(\varphi) \wedge ST_x(\psi)$;
- $ST_x(\varphi \vee \psi) := ST_x(\varphi) \vee ST_x(\psi)$;
- $ST_x(\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)) = \exists X(R_N xX \wedge \forall y(R_\ni Xy \to ST_y(\varphi)) \wedge$ $\exists y_1(R_\ni Xy_1 \wedge ST_{y_1}(\varphi_1)) \wedge \ldots \wedge \exists y_n(R_\ni Xy_n \wedge ST_{y_n}(\varphi_n)))$.

For any neighbourhood frame $\mathbb{F} = (W, N)$, it is natural to define the following corresponding two-sorted Kripke frame $\mathbb{F}^2 = (W, \mathcal{P}(W), R^\ni, R^N)$, where

[1]Notice that here the subset variables are treated as first-order variables in the subset domain W_s, rather than second-order variables in the world domain W_w. Indeed, when talking about standard translation in neighbourhood semantics, it is not possible to avoid talking about subsets of the domain, since the elements in $N(w)$ are subsets of W. Therefore, we follow the tradition in monotone modal logic [10, p.34] to call this two-sorted language first-order.

1. $R^{\ni} \subseteq \mathcal{P}(W) \times W$ such that for any $x \in W$ and $X \in \mathcal{P}(W)$, $R^{\ni}Xx$ iff $x \in X$;
2. $R^{N} \subseteq W \times \mathcal{P}(W)$ such that for any $x \in W$ and $X \in \mathcal{P}(W)$, $R^{N}xX$ iff $X \in N(x)$.

Given a two-sorted Kripke frame $\mathbb{F}^2 = (W, \mathcal{P}(W), R^{\ni}, R^{N})$, a valuation V is defined as a map $V : \mathsf{Prop} \to \mathcal{P}(W)$. Notice that here the $\mathcal{P}(W)$ in the definition of V is understood as the powerset of the first domain, rather than the second domain itself.

For this standard translation, it is easy to see the following correctness result:

Theorem 3. *For any neighbourhood frame $\mathbb{F} = (W, N)$, any valuation V on $\mathbb{F}$, any $w \in W$, any INL formula φ,*

$$(\mathbb{F}, V, w) \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^2, V \vDash ST_x(\varphi)[w].$$

4 Sahlqvist Correspondence Theorem in Instantial Neighbourhood Logic via Standard Translation

In this section, we will define the Sahlqvist formulas in instantial neighbourhood logic and prove the correspondence theorem via standard translation and minimal valuation method. First we recall the definition of Sahlqvist formulas in normal modal logic. Then we identify the special situations where the instantial neighbourhood modalities "behave well", i.e. have good quantifier patterns in the standard translation. Finally, we define INL-Sahlqvist formulas step by step in the style of [6, Section 3.6], and prove the correspondence theorem. The reason why we still have a proof by standard translation and minimal valuation method is that it helps to recognize the "relational" pattern in this neighbourhood-type semantics.

4.1 Sahlqvist formulas in normal modal logic

In this subsection we recall the syntactic definition of Sahlqvist formulas in normal modal logic (see [6, Section 3.6, Definition 3.51]).

Definition 4 (Sahlqvist formulas in normal modal logic) A *boxed atom* is a formula of the $\Box_{i_1} \ldots \Box_{i_n} p$, where $\Box_{i_1}, \ldots, \Box_{i_n}$ are (not necessarily distinct) boxes. In the case where $n = 0$, the boxed atom is just p.

A *Sahlqvist antecedent* φ is a formula built up from $\bot, \top$, boxed atoms, and negative formulas, using $\wedge, \vee$ and existential modal operators $\Diamond$ (unary diamond) and Δ (polyadic diamond). A *Sahlqvist implication* is an implication $\varphi \to \psi$ in which ψ is positive and φ is a Sahlqvist antecedent.

A Sahlqvist formula is a formula that is built up from Sahlqvist implications by applying boxes and conjunctions, and by applying disjunctions only between formulas that do not share any proposition variables.

As we can see from the definition above, the Sahlqvist antecedents are built up by $\bot, \top, p, \Box_{i_1} \ldots \Box_{i_n} p$ and negative formulas using $\wedge, \vee, \Diamond, \Delta$. If we consider the standard translations of Sahlqvist antecedents, the inner part is translated into universal quantifiers, and the outer part are translated into existential quantifiers.

4.2 Special cases where the instantial neighbourhood modalities become "normal"

As is mentioned in [4, Section 7] and as we can see in the definition of the standard translation, the quantifier pattern of $\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$ is similar to the case of monotone modal logic ([10]) which has an $\exists\forall$ pattern. As a result, even with two layers of INL modalities the complexity goes beyond the Sahlqvist fragment. However, we can still consider some special situations where we can reduce the modality to an n-ary normal diamond or a unary normal box.

n-ary normal diamond. We first consider the case $\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$ where φ is a *pure formula* without any propositional variables, i.e., all propositional variables are substituted by $\bot$ or $\top$. In this case $ST_x(\varphi)$ is a first-order formula $\alpha_\varphi(x)$ without any unary predicate symbols $P_1, P_2 \cdots$. Therefore, in the shape of the standard translation of $\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$, the universal quantifier $\forall y$ is not touched during the computation of minimal valuation, since there is no unary predicate symbol there. Indeed, we can consider the following equivalent form of $ST_x(\Box_n(\varphi_1, \ldots, \varphi_n; \varphi))$:

$$\begin{aligned} ST_x(\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)) =& \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge \\ & \forall y(R_\ni X y \to \alpha_\varphi(y)) \wedge (ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))) \end{aligned}$$

Now $ST_x(\Box_n(\varphi_1, \ldots, \varphi_n; \varphi))$ is essentially in a form similar to $ST_x(\Diamond\psi)$ in the normal modal logic case; indeed, when we compute the minimal valuation here, $R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge \forall y(R_\ni X y \to \alpha_\varphi(y))$ can be recognized as an entirety and stay untouched during the process. Indeed, here we can use the formula $\exists X(R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge \forall y(R_\ni X y \to \alpha_\varphi(y)))$ to define an $(n+1)$-ary relation symbol $R_{n,\varphi} x y_1 \ldots y_n$, and we denote the semantic counterpart of this relation also by $R_{n,\varphi}$, then it is easy to see that

$$\mathbb{M}, w \Vdash \Box_n(\varphi_1, \ldots, \varphi_n; \varphi) \text{ iff there exist } v_1, \ldots, v_n \text{ such that } R_{n,\varphi} w v_1 \ldots v_n \text{ and } \mathbb{M}, v_i \Vdash \varphi_i \text{ for } 1 \le i \le n.$$

This is exactly how the n-ary Δ modality is defined in standard modal logic settings. From now onwards we can denote $\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$ by $\Delta_{n,\varphi}(\varphi_1, \ldots, \varphi_n)$ where φ is pure.

Unary Normal Box. As we can see from above, in $\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$, we can replace propositional variables in φ by $\bot$ and $\top$ to obtain n-ary normal diamond modalities. By using the composition with negations, we can get the unary normal box modality, i.e. we can have a modality

$$\nabla_{1,\varphi}(\varphi_1) = \neg\Delta_{1,\varphi}(\neg\varphi_1) = \neg\Box_1(\neg\varphi_1; \varphi).$$

Now we can consider the standard translation of $\nabla_{1,\varphi}(\varphi_1)$:

$$\begin{array}{rcl}
ST_x(\nabla_{1,\varphi}(\varphi_1)) & \leftrightarrow & \neg ST_x(\Box_1(\neg\varphi_1; \varphi)) \\
& \leftrightarrow & \neg\exists X \exists y_1 (R_N x X \wedge R_\ni X y_1 \wedge ST_{y_1}(\neg\varphi_1) \\
& & \wedge \forall y (R_\ni X y \rightarrow \alpha_\varphi(y))) \\
& \leftrightarrow & \forall X \forall y_1 \neg (R_N x X \wedge R_\ni X y_1 \wedge ST_{y_1}(\neg\varphi_1) \\
& & \wedge \forall y (R_\ni X y \rightarrow \alpha_\varphi(y))) \\
& \leftrightarrow & \forall X \forall y_1 (R_N x X \wedge R_\ni X y_1 \wedge \forall y (R_\ni X y \\
& & \rightarrow \alpha_\varphi(y)) \rightarrow ST_{y_1}(\varphi_1)),
\end{array}$$

where $\forall y(R_\ni X y \rightarrow \alpha_\varphi(y))$ does not contain unary predicate symbols $P_1, P_2, \cdots$. Now we can see that $ST_x(\nabla_{1,\varphi}(\varphi_1))$ has a form similar to $ST_x(\Box\psi)$ where $\Box$ is a normal unary box, by taking $R_N x X \wedge R_\ni X y_1 \wedge \forall y(R_\ni X y \rightarrow \alpha_\varphi(y))$ as an entirety.

4.3 The definition of INL-Sahlqvist formulas in instantial neighbourhood logic

Now we can define the INL-Sahlqvist formulas in instantial neighbourhood logic step by step in the style of [6, Section 3.6]. The basic proof structure is similar to the basic modal logic setting, namely we first rewrite the standard translation of the modal formula into a specific shape, and then read off the minimal valuation directly from the shape, while here the manipulation of quantifiers is more complicated and needs to take more care.

4.3.1 Very simple INL-Sahlqvist implications

Definition 5 (Very simple INL-Sahlqvist implications) A *very simple INL-Sahlqvist antecedent* φ is defined as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \Delta_{n,\theta}(\varphi_1, \ldots, \varphi_n) \mid \Box_n(\varphi_1, \ldots, \varphi_n; p)$$

where $p \in \mathsf{Prop}$ is a propositional variable, θ is a pure formula without propositional variables. A *very simple INL-Sahlqvist implication* is an implication $\varphi \rightarrow \psi$ where ψ is positive , and φ is a very simple INL-Sahlqvist antecedent.

For very simple INL-Sahlqvist implications, we allow n-ary normal diamonds $\Delta_{n,\theta}$ in the construction of φ, while for the $(n+1)$-ary modality $\Box_n$, we only allow propositional variables to occur in the $(n+1)$-th coordinate.

We can show that very simple INL-Sahlqvist implications have first-order correspondents:

Theorem 6. *For any given very simple INL-Sahlqvist implication $\varphi \to \psi$, there is a two-sorted first-order local correspondent $\alpha(x)$ such that for any neighbourhood frame $\mathbb{F} = (W, N)$, any $w \in W$,*

$$\mathbb{F}, w \Vdash \varphi \to \psi \quad \textit{iff} \quad \mathbb{F}^2 \vDash \alpha(x)[w].$$

Proof The proof strategy is similar to [6, Theorem 3.42, Theorem 3.49], with some differences in treating $\Box_n(\varphi_1, \ldots, \varphi_n; p)$.

We first start with the two-sorted second-order translation of $\varphi \to \psi$, namely $\forall P_1 \ldots \forall P_n \forall x(ST_x(\varphi) \to ST_x(\psi))$, where $ST_x(\varphi), ST_x(\psi)$ are the two-sorted first-order standard translations of φ, ψ.

For any very simple INL-Sahlqvist antecedent φ, we consider the shape of $\beta = ST_x(\varphi)$ defined inductively,

$$\beta ::= Px \mid x \neq x \mid x = x \mid \beta \wedge \beta \mid \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge$$

$$\forall y(R_\ni Xy \to \alpha_\theta(y)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1$$

$$\wedge \ldots \wedge R_\ni X y_n \wedge \forall y(R_\ni Xy \to Py) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))$$

Now we can denote $R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n$ as $R_n X x y_1 \ldots y_n$ and $R_{-1,\theta} X$ for $\forall y(R_\ni Xy \to \alpha_\theta(y))$, and thus get

$$\beta ::= Px \mid x \neq x \mid x = x \mid \beta \wedge \beta \mid$$

$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1 \ldots y_n \wedge R_{-1,\theta} X \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid$$

$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1 \ldots y_n \wedge \forall y(R_\ni Xy \to Py) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))$$

By using the equivalences

$$\exists y \delta(y) \wedge \gamma \leftrightarrow \exists y(\delta(y) \wedge \gamma) \text{ (where } y \text{ does not occur in } \gamma)$$

and

$$\exists X \delta(X) \wedge \gamma \leftrightarrow \exists X(\delta(X) \wedge \gamma) \text{ (where } X \text{ does not occur in } \gamma),$$

It is easy to see that the two-sorted first-order formula $\beta = ST_x(\varphi)$ is equivalent to a formula of the form $\exists \overline{X} \exists \overline{y}(\mathrm{REL}^{\overline{\theta}, \overline{X}, x, \overline{y}} \wedge \mathrm{ATProp})$, where:

- $\mathrm{REL}^{\overline{\theta}, \overline{X}, x, \overline{y}}$ is a (possibly empty) conjunction of formulas of the form $R_n X x y_1 \ldots y_n$ or $R_{-1,\theta} X$;
- ATProp is a conjunction of formulas of the form $\forall y(R_\ni Xy \to Py)$ or Pw or $w = w$ or $w \neq w$.

Therefore, by using the equivalences

$$(\exists y\delta(y) \rightarrow \gamma) \leftrightarrow \forall y(\delta(y) \rightarrow \gamma) \text{ (where } y \text{ does not occur in } \gamma)$$

and

$$(\exists X\delta(X) \rightarrow \gamma) \leftrightarrow \forall X(\delta(X) \rightarrow \gamma) \text{ (where } X \text{ does not occur in } \gamma),$$

it is immediate that $\forall P_1 \ldots \forall P_n \forall x(ST_x(\varphi) \rightarrow ST_x(\psi))$ is equivalent to

$$\forall P_1 \ldots \forall P_n \forall \overline{X} \forall x \forall \overline{y}(\mathrm{REL}^{\overline{\theta},\overline{X},x,\overline{y}} \wedge \mathrm{ATProp} \rightarrow \mathrm{POS}),^{2}$$

where $\mathrm{REL}^{\overline{\theta},\overline{X},x,\overline{y}}$ and ATProp are given as above, and POS is the standard translation $ST_x(\psi)$.

Now we can use similar strategy as in [6, Theorem 3.42, Theorem 3.49]. To make it easier for later parts in the paper, we still mention how the minimal valuation and the resulting first-order correspondent formula look like. Without loss of generality we suppose that for any unary predicate P that occurs in the POS also occurs in AT; otherwise we can substitute P by $\lambda u.u \neq u$ for P to eliminate P (see [6, Theorem 3.42]).

Now consider a unary predicate symbol P occuring in ATProp, and $Px_1, \ldots, Px_n, \forall y(R_{\ni}X_1y \rightarrow Py), \ldots, \forall y(R_{\ni}X_my \rightarrow Py)$ are all occurences of P in ATProp. By taking $\sigma(P)$ to be

$$\lambda u.u = x_1 \vee \ldots \vee u = x_n \vee R_{\ni}X_1u \vee \ldots \vee R_{\ni}X_mu,$$

we get the minimal valuation. The resulting first-order correspondent formula is

$$\forall \overline{X} \forall x \forall \overline{y}(\mathrm{REL}^{\overline{\theta},\overline{X},x,\overline{y}} \rightarrow [\sigma(P_1)/P_1, \ldots, \sigma(P_k)/P_k]\mathrm{POS}). \qquad \square$$

From the proof above, we can see that the part corresponding to $\Delta_{n,\theta}(\varphi_1, \ldots, \varphi_n)$ is essentially treated in the same way as an n-ary diamond in the normal modal logic setting, and $\Box_n(\varphi_1, \ldots, \varphi_n; p)$ is treated as $\Delta(\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n \wedge \Box p)$ where Δ is an $(n{+}1)$-ary normal diamond, $\Diamond$ is a unary normal diamond and $\Box$ is a unary normal box, therefore we can guarantee the compositional structure of quantifiers in the antecedent to be $\exists\forall$ as a whole.

4.3.2 Simple INL-Sahlqvist implications

Similar to simple Sahlqvist implications in basic modal logic, here we can define simple INL-Sahlqvist implications:

[2]Notice that the quantifiers $\forall P_1 \ldots \forall P_n$ are second-order quantifiers over the world domain, and $\forall \overline{X}$ are first-order quantifiers over the subset domain.

Definition 7 (Simple INL-Sahlqvist implications) A *pseudo-boxed atom* ζ is defined as follows:

$$\zeta ::= p \mid \bot \mid \top \mid \zeta \wedge \zeta \mid \nabla_{1,\theta}(\zeta)$$

where θ is a pure formula without propositional variables. Based on this, a *simple INL-Sahlqvist antecedent* φ is defined as follows:

$$\varphi ::= \zeta \mid \varphi \wedge \varphi \mid \Delta_{n,\theta}(\varphi_1, \ldots, \varphi_n) \mid \Box_n(\varphi_1, \ldots, \varphi_n; \zeta)$$

where θ is a pure formula without propositional variables and ζ is a pseudo-boxed atom. A *simple INL-Sahlqvist implication* is an implication $\varphi \to \psi$ where ψ is positive, and φ is a simple INL-Sahlqvist antecedent.

Theorem 8. *For any given simple INL-Sahlqvist implication $\varphi \to \psi$, there is a two-sorted first-order local correspondent $\alpha(x)$ such that for any neighbourhood frame $\mathbb{F} = (W, N)$, any $w \in W$,*

$$\mathbb{F}, w \Vdash \varphi \to \psi \quad \textit{iff} \quad \mathbb{F}^2 \vDash \alpha(x)[w].$$

Proof We use similar proof strategy as [6, Theorem 3.49]. The part that we need to take care of is the way to compute the minimal valuation. Now without loss of generality (by renaming quantified variables) we have the following shape of $\beta = ST_x(\zeta)$ defined inductively for any pseudo-boxed atom ζ:

$$\beta ::= Px \mid x \neq x \mid x = x \mid \beta \wedge \beta \mid$$

$$\forall X \forall y_1 (R_N x X \wedge R_\ni X y_1 \wedge \forall y (R_\ni X y \to \alpha_\theta(y)) \to ST_{y_1}(\zeta)).$$

The shape of $\beta = ST_x(\varphi)$ is defined inductively for any simple Sahlqvist antecedent φ:

$$\beta ::= ST_x(\zeta) \mid \beta \wedge \beta \mid \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge$$

$$\forall y (R_\ni X y \to \alpha_\theta(y)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge$$

$$R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge \forall y (R_\ni X y \to ST_y(\zeta)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))$$

Now we use the abbreviation $R_n X x y_1 \ldots y_n$ for $R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n$ and $R_{-1,\theta} X$ for $\forall y (R_\ni X y \to \alpha_\theta(y))$ (note that the only possible free variable in $\alpha_\theta(y)$ is y), then by the equivalence $(\exists X \alpha \to \beta) \leftrightarrow \forall X (\alpha \to \beta)$, the shape of $\beta = ST_x(\zeta)$ can be given as follows:

$$\beta ::= Px \mid x \neq x \mid x = x \mid \beta \wedge \beta \mid \forall y_1 (\exists X (R_1 X x y_1 \wedge R_{-1,\theta} X) \to ST_{y_1}(\zeta))$$

The shape of $\beta = ST_x(\varphi)$ can be given as follows:

$$\beta ::= ST_x(\zeta) \mid \beta \wedge \beta \mid$$
$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1 \ldots y_n \wedge R_{-1,\theta} X \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid$$
$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1 \ldots y_n \wedge \forall y (R_\ni X y \to ST_y(\zeta)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)).$$

Now we denote $\exists X(R_1 X x y_1 \wedge R_{-1,\theta} X)$ as $R_{-2,\theta} x y_1$, and we get the shape of pseudo-boxed atom $\beta = ST_x(\zeta)$ as follows:

$$\beta ::= Px \mid x \neq x \mid x = x \mid \beta \wedge \beta \mid \forall y_1 (R_{-2,\theta} x y_1 \to ST_{y_1}(\zeta)),$$

Now using the following equivalences:

- $(\varphi \to \forall z(\psi(z) \to \gamma)) \leftrightarrow \forall z(\varphi \wedge \psi(z) \to \gamma)$ (where z does not occur in φ);
- $(\varphi \to (\psi \to \gamma)) \leftrightarrow (\varphi \wedge \psi \to \gamma)$;
- $(\varphi \to (\psi \wedge \gamma)) \leftrightarrow ((\varphi \to \psi) \wedge (\varphi \to \gamma))$;
- $\forall z(\psi(z) \wedge \gamma(z)) \leftrightarrow (\forall z \psi(z) \wedge \forall z \gamma(z))$;

For any pseudo-boxed atom ζ, the first-order formula $ST_x(\zeta)$ is equivalent to a conjunction of two-sorted first-order formulas of the form $\forall \overline{y}(\mathrm{REL}^{\overline{\theta},x,\overline{y}} \to \mathrm{AT})$ or Px or $x \neq x$ or $x = x$, where:

- $\mathrm{REL}^{\overline{\theta},x,\overline{y}}$ is a (possibly empty) conjunction of formulas of the form $R_{-2,\theta} yz$;
- AT is a formula of the form Pw or $w = w$ or $w \neq w$ where w is bound by $\forall \overline{y}$ (here we do not need to take the conjunction because of $\forall z(\psi(z) \wedge \gamma(z)) \leftrightarrow (\forall z \psi(z) \wedge \forall z \gamma(z))$).

It is easy to see that $\mathrm{REL}^{\overline{\theta},x,\overline{y}}$ does not contain any unary predicate symbol P_i. By the equivalence $(\exists x \varphi(x) \to \psi) \leftrightarrow \forall x(\varphi(x) \to \psi)$ where ψ does not contain x, we can transform $\forall \overline{y}(\mathrm{REL}^{\overline{\theta},x,\overline{y}} \to \mathrm{AT})$ into $\forall y(\exists \overline{y}' \mathrm{REL}^{\overline{\theta},x,\overline{y}} \to \mathrm{AT}(y))$, where $\mathrm{AT}(y)$ is Py or $y = y$ or $y \neq y$.

We can introduce a new binary relation symbol $R_{\overline{\theta}} xy$ which is $\exists \overline{y}' \mathrm{REL}^{\overline{\theta},x,\overline{y}}$. Then $\beta = ST_x(\zeta)$ is a conjunction of formulas of the form $\forall y(R_{\overline{\theta}} xy \to \mathrm{AT}(y))$ or Px or $x \neq x$ or $x = x$.

Now we somehow come back to the situation of the basic normal modal logic case, where $R_{\overline{\theta}}$ is a real relation symbol. The shape of $\beta = ST_x(\varphi)$ for simple INL-Sahlqvist antecedent φ can be recursively defined as follows:

$$\beta ::= \forall y(R_{\overline{\theta}} xy \to \mathrm{AT}(y)) \mid Px \mid x \neq x \mid x = x \mid \beta \wedge \beta \mid$$
$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1 \ldots y_n \wedge R_{-1,\theta} X \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid$$
$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1 \ldots y_n \wedge \forall y (R_\ni X y \to ST_y(\zeta)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))$$

Since $ST_y(\zeta)$ is a conjunction of formulas of the form $\forall z(R_{\overline{\theta}} yz \to \mathrm{AT}(z))$ or $y = y$ or $y \neq y$ or Py, we have

$$
\begin{array}{ll}
 & \forall y(R_{\ni}Xy \to \bigwedge_i \forall z_i(R_{\overline{\theta}_i}yz_i \to \mathrm{AT}(z_i))) \\
\leftrightarrow & \bigwedge_i \forall y(R_{\ni}Xy \to \forall z_i(R_{\overline{\theta}_i}yz_i \to \mathrm{AT}(z_i))) \\
\leftrightarrow & \bigwedge_i \forall y \forall z_i(R_{\ni}Xy \to (R_{\overline{\theta}_i}yz_i \to \mathrm{AT}(z_i))) \\
\leftrightarrow & \bigwedge_i \forall z_i(\exists y(R_{\ni}Xy \wedge R_{\overline{\theta}_i}yz_i) \to \mathrm{AT}(z_i))).
\end{array}
$$

So $\forall y(R_{\ni}Xy \to ST_y(\zeta))$ is equivalent to a conjunction of formulas of the form $\forall z_i(\exists y(R_{\ni}Xy \wedge R_{\overline{\theta}_i}yz_i) \to \mathrm{AT}(z_i)))$ or $\forall y(R_{\ni}Xy \to y = y)$ (i.e. $\top$) or $\forall y(R_{\ni}Xy \to y \neq y)$ or $\forall y(R_{\ni}Xy \to Py)$.

Now the situation is similar to the very simple INL-Sahlqvist implication case. We can see how the minimal valuation is computed:

- for the $\forall y(R_{\overline{\theta}}xy \to \mathrm{AT}(y))$ part, when $\mathrm{AT}(y)$ is Py, its corresponding minimal valuation is $\lambda u.R_{\overline{\theta}}xu$; when $\mathrm{AT}(y)$ is $y = y$ or $y \neq y$, we can replace $\mathrm{AT}(y)$ by $\top$ or $\bot$, respectively;
- for the $x \neq x$ part, it is equivalent to $\bot$;
- for the $x = x$ part, it is equivalent to $\top$;
- for the Px part, its corresponding minimal valuation is $\lambda u.x = u$;
- for the $\forall z_i(\exists y(R_{\ni}Xy \wedge R_{\overline{\theta}_i}yz_i) \to \mathrm{AT}(z_i))$ part, when $\mathrm{AT}(z_i)$ is Pz_i, its corresponding minimal valuation is $\lambda u.\exists y(R_{\ni}Xy \wedge R_{\overline{\theta}_i}yu)$; when $\mathrm{AT}(z_i)$ is $z_i = z_i$ or $z_i \neq z_i$, we can replace $\mathrm{AT}(y)$ by $\top$ or $\bot$, respectively;
- for the $\forall y(R_{\ni}Xy \to Py)$ part, its corresponding minimal valuation is $\lambda u.R_{\ni}Xu$.

Now for each propositional variable p_i, we take the minimal valuation to be the union of all the corresponding minimal valuations where there an occurence of P_i. By essentially the same argument as in [6, Theorem 3.49], we get the first-order correspondent of $\varphi \to \psi$. □

4.3.3 INL-Sahlqvist implications

In the present section, we add negated formulas and disjunctions in the antecedent part, which is analogous to the first half of [6, Definition 3.51].

Definition 9 (INL-Sahlqvist implications) An *INL-Sahlqvist antecedent* φ is defined as follows:

$$\varphi ::= \zeta \mid \gamma \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Delta_{n,\theta}(\varphi_1, \ldots, \varphi_n) \mid \Box_n(\varphi_1, \ldots, \varphi_n; \zeta) \mid \Box_n(\varphi_1, \ldots, \varphi_n; \gamma)$$

where θ is a pure formula without propositional variables, ζ is a pseudo-boxed atom and γ is a negative formula. An *INL-Sahlqvist implication* is an implication $\varphi \to \psi$ where ψ is positive, and φ is an INL-Sahlqvist antecedent.

Theorem 10. *For any given INL-Sahlqvist implication $\varphi \to \psi$, there is a two-sorted first-order local correspondent $\alpha(x)$ such that for any neighbourhood frame $\mathbb{F} = (W, N)$, any $w \in W$,*

$$\mathbb{F}, w \Vdash \varphi \to \psi \quad \textit{iff} \quad \mathbb{F}^2 \vDash \alpha(x)[w].$$

Proof We use similar proof strategy as [6, Theorem 3.54]. The part that we need to take care of is the way to compute the minimal valuation. Now for each INL-Sahlqvist antecedent φ, we consider the shape of $\beta = ST_x(\varphi)$:

$$\beta ::= ST_x(\zeta) \mid ST_x(\gamma) \mid \beta \wedge \beta \mid \beta \vee \beta \mid \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1 \wedge$$

$$\ldots \wedge R_\ni X y_n \wedge \forall y (R_\ni X y \to \alpha_\theta(y)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid$$

$$\exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n \wedge \forall y (R_\ni X y \to ST_y(\zeta))$$

$$\wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid \exists X \exists y_1 \ldots \exists y_n (R_N x X \wedge R_\ni X y_1 \wedge$$

$$\ldots \wedge R_\ni X y_n \wedge \forall y (R_\ni X y \to ST_y(\gamma)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))$$

where θ is a pure formula without propositional variables, ζ is a pseudo-boxed atom and γ is a negative formula.

We use the abbreviation $R_n X x y_1, \ldots y_n$ for $R_N x X \wedge R_\ni X y_1 \wedge \ldots \wedge R_\ni X y_n$ and $R_{-1,\theta} X$ for $\forall y (R_\ni X y \to \alpha_\theta(y))$, we can rewrite the shape of $\beta = ST_x(\varphi)$ as follows:

$$\beta ::= ST_x(\zeta) \mid ST_x(\gamma) \mid \beta \wedge \beta \mid \beta \vee \beta \mid$$

$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1, \ldots y_n \wedge R_{-1,\theta} X \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid$$

$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1, \ldots y_n \wedge \forall y (R_\ni X y \to ST_y(\zeta)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n)) \mid$$

$$\exists X \exists y_1 \ldots \exists y_n (R_n X x y_1, \ldots y_n \wedge \forall y (R_\ni X y \to ST_y(\gamma)) \wedge ST_{y_1}(\varphi_1) \wedge \ldots \wedge ST_{y_n}(\varphi_n))$$

where θ is a pure formula without propositional variables, ζ is a pseudo-boxed atom and γ is a negative formula.

Using the equivalence $\exists y \delta(y) \wedge \gamma \leftrightarrow \exists y (\delta(y) \wedge \gamma)$ (where y does not occur in γ), $\exists y (\alpha \vee \beta) \leftrightarrow \exists y \alpha \vee \exists y \beta$, $(\alpha \vee \beta) \wedge \gamma \leftrightarrow (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$, it is easy to see that the first-order formula $\beta = ST^E_x(\varphi)$ is equivalent to a formula of the form $\bigvee_i \exists \overline{X}_i \exists \overline{y}_i (\mathrm{REL}_i^{\overline{X}_i, x, \overline{y}_i} \wedge \text{PS-BOXED-AT}_i \wedge \mathrm{NEG}_i)$, where:

- $\mathrm{REL}_i^{\overline{X}_i, x, \overline{y}_i}$ is a (possibly empty) conjunction of formulas of the form $R_n X x y_1 \ldots y_n$ and $R_{-1,\theta} X$;
- PS-BOXED-AT$_i$ is a conjunction of formulas of the form $ST_y(\zeta)$ and $\forall y (R_\ni X y \to ST_y(\zeta))$ where ζ is a pseudo-boxed atom;

- NEG_i is a conjunction of formulas of the form $ST_y(\gamma)$ and $\forall y(R_{\ni}Xy \to ST_y(\gamma))$ where γ is a negative formula.

Now let us consider the standard translation of INL-Sahlqvist implication $\varphi \to \psi$ where φ is an INL-Sahlqvist antecedent and ψ is a positive formula. For $\beta = ST_x^E(\varphi \to \psi)$, we have the following equivalence:

$$\begin{aligned}
& \bigvee_i \exists \overline{X}_i \exists \overline{y}_i (\mathrm{REL}_i^{\overline{X}_i, x, \overline{y}_i} \wedge \text{PS-BOXED-AT}_i \wedge \mathrm{NEG}_i) \to ST_x(\psi) \\
\Leftrightarrow \ & \bigwedge_i (\exists \overline{X}_i \exists \overline{y}_i (\mathrm{REL}_i^{\overline{X}_i, x, \overline{y}_i} \wedge \text{PS-BOXED-AT}_i \wedge \mathrm{NEG}_i) \to ST_x(\psi)) \\
\Leftrightarrow \ & \bigwedge_i \forall \overline{X}_i \forall \overline{y}_i (\mathrm{REL}_i^{\overline{X}_i, x, \overline{y}_i} \wedge \text{PS-BOXED-AT}_i \wedge \mathrm{NEG}_i \to ST_x(\psi)) \\
\Leftrightarrow \ & \bigwedge_i \forall \overline{X}_i \forall \overline{y}_i (\mathrm{REL}_i^{\overline{X}_i, x, \overline{y}_i} \wedge \text{PS-BOXED-AT}_i \to \neg \mathrm{NEG}_i \vee ST_x(\psi))
\end{aligned}$$

Now it is easy to see that $\neg\mathrm{NEG}_i \vee ST_x(\psi)$ is equivalent to a first-order formula which is positive in all unary predicates. We can now use essentially the same proof strategy as Theorem 8. □

As we can see from the proofs above, the key point is the quantifier pattern of the two-sorted standard translation of the modalities, i.e. the outer part of the structure of an INL-Sahlqvist antecedent are translated into existential quantifiers, and the inner part is translated into universal quantifiers.

4.3.4 INL-Sahlqvist formulas

In the present section, we build Sahlqvist formulas from Sahlqvist implications by applying $\nabla_{1,\theta}(\cdot)$ (where θ is pure), $\wedge$ and $\vee$, which is analogous to the second half of [6, Definition 3.51].

Definition 11 (INL-Sahlqvist formulas) An *INL-Sahlqvist formula* φ is defined as follows:

$$\varphi ::= \varphi_0 \mid \nabla_{1,\theta}(\varphi) \mid \varphi \wedge \varphi \mid \varphi \overline{\vee} \varphi$$

where φ_0 is an INL-Sahlqvist implication, θ is a pure formula without propositional variables, $\varphi \overline{\vee} \varphi$ is a disjunction such that the two φs share no propositional variable.

Theorem 12. *For any given INL-Sahlqvist formula φ, there is a two-sorted first-order local correspondent $\alpha(x)$ such that for any neighbourhood frame $\mathbb{F} = (W, N)$, any $w \in W$,*

$$\mathbb{F}, w \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^2 \vDash \alpha(x)[w].$$

Proof Similar to [6, Lemma 3.53]. □

5 Bimodal Translation of Instantial Neighbourhood Logic

In the present section we give the second proof of Sahlqvist correspondence theorem, by using a bimodal translation into a normal bimodal language. The methodology is similar to [10, 7], but with slight differences.

5.1 Normal bimodal language and two-sorted Kripke frame

In this subsection, we introduce the normal bimodal language and two-sorted Kripke frame. For a more detailed treatment, see [10, 7].

As we can see in Section 3, for any given neighbourhood frame $\mathbb{F} = (W, N)$, there is an associated two-sorted Kripke frame $\mathbb{F}^2 = (W, \mathcal{P}(W), R^{\ni}, R^N)$, where

1. $R^{\ni} \subseteq \mathcal{P}(W) \times W$ such that for any $x \in W$ and $X \in \mathcal{P}(W)$, $R^{\ni}Xx$ iff $x \in X$;
2. $R^N \subseteq W \times \mathcal{P}(W)$ such that for any $x \in W$ and $X \in \mathcal{P}(W)$, $R^N xX$ iff $X \in N(x)$.

In this kind of semantic structures, we can define the following two-sorted normal bimodal language:

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond_N \theta$$

$$\theta ::= \Diamond_{\ni} \varphi \mid \neg\theta \mid \theta \wedge \theta \mid \theta \vee \theta$$

where φ is a formula of the *world type* and will be interpreted in the first domain, and θ is a formula of the *subset type* and will be interpreted in the second domain. We can also define $\Box_{\ni}$ and $\Box_N$ in the standard way.

Given a two-sorted Kripke frame $\mathbb{F}^2 = (W, \mathcal{P}(W), R^{\ni}, R^N)$, a valuation V is defined as a map $V : \mathsf{Prop} \to \mathcal{P}(W)$, where propositional variables are interpreted as subsets of the first domain. The satisfaction relation $\Vdash$ is defined as follows, for any $w \in W$ and any X in $\mathcal{P}(W)$ (here we omit the Boolean connectives):

- $\mathbb{F}^2, V, w \Vdash p$ iff $w \in V(p)$;
- $\mathbb{F}^2, V, w \Vdash \Diamond_N \theta$ iff there is an $X \in \mathcal{P}(W)$ such that $R^N wX$ and $\mathbb{F}^2, V, X \Vdash \theta$;
- $\mathbb{F}^2, V, X \Vdash \Diamond_{\ni} \varphi$ iff there is a $w \in W$ such that $R^{\ni} Xw$ and $\mathbb{F}^2, V, w \Vdash \varphi$.

5.2 Bimodal translation

Now we are ready to define the translation τ from the INL language to the two-sorted normal bimodal language:

Definition 13 (Bimodal translation) Given any INL formula φ, the bimodal translation $\tau(\varphi)$ is defined as follows:

- $\tau(p) = p$;

- $\tau(\bot) = \bot$;
- $\tau(\top) = \top$;
- $\tau(\neg\varphi) = \neg\tau(\varphi)$;
- $\tau(\varphi_1 \wedge \varphi_2) = \tau(\varphi_1) \wedge \tau(\varphi_2)$;
- $\tau(\varphi_1 \vee \varphi_2) = \tau(\varphi_1) \vee \tau(\varphi_2)$;
- $\tau(\varphi_1 \to \varphi_2) = \tau(\varphi_1) \to \tau(\varphi_2)$;
- $\tau(\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)) = \Diamond_N(\Diamond_\ni\tau(\varphi_1) \wedge \ldots \wedge \Diamond_\ni\tau(\varphi_n) \wedge \Box_\ni\tau(\varphi))$.

It is easy to see the following correctness result:

Theorem 14. *For any neighbourhood frame $\mathbb{F} = (W, N)$, any valuation V on $\mathbb{F}$, any $w \in W$, any INL formula φ,*

$$(\mathbb{F}, V, w) \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^2, V, w \Vdash \tau(\varphi).$$

5.3 Sahlqvist correspondence theorem via bimodal translation

To discuss the Sahlqvist correspondence theorem via bimodal translation, we first discuss how the Sahlqvist fragment in normal bimodal logic looks like.

First of all, we have the following observation that for $\nabla_{1,\theta}(\zeta)$ where θ is pure, its bimodal translation is $\Box_N(\Box_\ni\tau(\zeta) \vee \neg\Box_\ni\tau(\theta))$, i.e. $\Box_N(\Box_\ni\tau(\theta) \to \Box_\ni\tau(\zeta))$. This formula is not a box itself applied to $\tau(\zeta)$, but its standard translation into first-order logic is

$$\forall X \forall y_1(R_N xX \wedge R_\ni Xy_1 \wedge \forall y(R_\ni Xy \to \alpha_\theta(y)) \to ST_{y_1}(\zeta)),$$

which means that we can treat $R_N xX \wedge R_\ni Xy_1 \wedge \forall y(R_\ni Xy \to \alpha_\theta(y))$ as an entirety and therefore we can treat $\Box_N(\Box_\ni\tau(\theta) \to \Box_\ni\tau(\zeta))$ like a boxed formula. From here onwards we will also call formulas of the shape $\Box_N(\Box_\ni\tau(\theta) \to \Box_\ni\tau(\zeta))$ boxed atoms if $\tau(\zeta)$ is a boxed atom.

Now, similar to the normal modal logic case, we can define the bimodal Sahlqvist antecedents in the normal bimodal logic built up by boxed atoms and negative formulas in the inner part generated by $\wedge$, $\vee$, $\Diamond_\ni$, $\Diamond_N$, where the formulas are of the right type, and therefore bimodal Sahlqvist implications are defined in the standard way. A bimodal Sahlqvist formula is built up from bimodal Sahlqvist implications by applying boxes, $\Box_N(\Box_\ni\tau(\theta) \to \Box_\ni(\cdot))$, $\wedge$ and $\vee$ where θ is pure and $\vee$ is only applied to formulas which share no propositional variable.

Theorem 15. *For any bimodal Sahlqvist formula φ, there is a two-sorted first-order local correspondent $\alpha(x)$ such that for any neighbourhood frame $\mathbb{F} = (W, N)$, any $w \in W$,*

$$\mathbb{F}, w \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^2 \vDash \alpha(x)[w].$$

Proof By adaptation of the proofs of Theorem 3.42, 3.49, 3.54 and Lemma 3.53 in [6] to the bimodal setting. □

Now we can prove Sahlqvist correspondence theorem by using bimodal translation:

Theorem 16. *For any INL-Sahlqvist implication $\varphi \to \psi$, $\tau(\varphi \to \psi)$ is a Sahlqvist implication in the normal bimodal language.*

Proof As we know, the shape of an INL-Sahlqvist antecedent is given as follows:

$$\zeta ::= p \mid \bot \mid \top \mid \zeta \wedge \zeta \mid \nabla_{1,\theta}(\zeta)$$

$$\varphi ::= \zeta \mid \gamma \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Delta_{n,\theta}(\varphi_1, \ldots, \varphi_n) \mid \Box_n(\varphi_1, \ldots, \varphi_n; \zeta) \mid \Box_n(\varphi_1, \ldots, \varphi_n; \gamma),$$

where θ is a pure INL formula without propositional variables, ζ is a pseudo-boxed atom, and γ is a negative formula. Therefore, the bimodal translations of $\tau(\zeta)$ and $\tau(\varphi)$ have the following shape:

$$\tau(\zeta) ::= p \mid \bot \mid \top \mid \tau(\zeta) \wedge \tau(\zeta) \mid \neg\Diamond_N(\Diamond_\ni \neg\tau(\zeta) \wedge \Box_\ni \tau(\theta))$$

$$\tau(\varphi) ::= \tau(\zeta) \mid \tau(\gamma) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid$$

$$\Diamond_N(\Diamond_\ni \tau(\varphi_1) \wedge \ldots \wedge \Diamond_\ni \tau(\varphi_n) \wedge \Box_\ni \tau(\theta)) \mid$$

$$\Diamond_N(\Diamond_\ni \tau(\varphi_1) \wedge \ldots \wedge \Diamond_\ni \tau(\varphi_n) \wedge \Box_\ni \tau(\zeta)) \mid$$

$$\Diamond_N(\Diamond_\ni \tau(\varphi_1) \wedge \ldots \wedge \Diamond_\ni \tau(\varphi_n) \wedge \Box_\ni \tau(\gamma))$$

Now we analyze the shape above. For the bimodal translation of a pseudo-boxed atom ζ in the INL language, $\neg\Diamond_N(\Diamond_\ni \neg\tau(\zeta) \wedge \Box_\ni \tau(\theta))$ is equivalent to $\Box_N(\Box_\ni \tau(\zeta) \vee \neg\Box_\ni \tau(\theta))$. since θ is a pure formula without propositional variables, $\tau(\zeta)$ can be treated as a conjunction of boxed atoms in the bimodal language.

Now we examine $\tau(\varphi)$. It is built up by $\tau(\zeta)$ (a conjunction of boxed atoms) and $\tau(\gamma)$ (a negative formula), generated by $\wedge, \vee$ and the three special shapes of $\tau(\Box_n(\varphi_1, \ldots, \varphi_n; \varphi))$ where φ are pure formulas without propositional variables (the θ case), pseudo-boxed atoms (the ζ case) or negative formulas (the γ case). It is easy to see that $\tau(\varphi)$ is built up by pure formulas[3], boxed atoms and negative formulas in the bimodal language, generated by $\Diamond_\ni, \Diamond_N, \wedge, \vee$, thus of the shape of Sahlqvist antecedent in the bimodal language. Therefore, $\tau(\varphi \to \psi)$ is a Sahlqvist implication in the normal bimodal language. □

[3]Indeed, pure formulas are both negative and positive formulas in every propositional variable p, since their values are constants and p does not occur in them.

Theorem 17. *For any INL-Sahlqvist formula φ, its bimodal translation $\tau(\varphi)$ is a bimodal Sahlqvist formula.*

Proof We prove by induction. For the basic case and the $\wedge$ and $\vee$ case, it is easy. For the $\nabla_{1,\theta}(\zeta)$ case where θ is pure and ζ is an INL-Sahlqvist formula, by induction hypothesis, $\tau(\zeta)$ is a bimodal Sahlqvist formula. By our definition, $\Box_N(\Box_\ni\tau(\theta) \to \Box_\ni\tau(\zeta))$ is also a bimodal Sahlqvist formula. $\Box$

Theorem 18. *For any INL-Sahlqvist formula φ, there is a two-sorted first-order local correspondent $\alpha(x)$ such that for any neighbourhood frame $\mathbb{F} = (W, N)$, any $w \in W$,*

$$\mathbb{F}, w \Vdash \varphi \quad \textit{iff} \quad \mathbb{F}^2 \vDash \alpha(x)[w].$$

Proof By Theorem 15 and Theorem 17. $\Box$

6 Examples

In this section, we give some examples of INL-Sahlqvist implications.

Example 19 Consider the formula $\Box_1(p; \top) \to \neg\Box_1(\neg p; \top)$, its standard translation is

$$\begin{array}{rl}
& ST_x(\Box_1(p;\top) \to \neg\Box_1(\neg p;\top)) \\
= & ST_x(\Box_1(p;\top)) \to ST_x(\neg\Box_1(\neg p;\top)) \\
= & \exists X(R_N xX \wedge \exists y_1(R_\ni X y_1 \wedge ST_{y_1}(p))) \to \neg\exists X(R_N xX \\
& \wedge\exists y_2(R_\ni X y_1 \wedge \neg ST_{y_2}(p))) \\
= & \exists y_1(R_{1,\top} x y_1 \wedge P y_1) \to \forall y_2(R_{1,\top} x y_2 \to P y_2) \\
= & \forall y_1(R_{1,\top} x y_1 \wedge P y_1 \to \forall y_2(R_{1,\top} x y_2 \to P y_2))
\end{array}$$

the minimal valuation for P is $\lambda z.z = y_1$, therefore the local first-order correspondent of $\Box_1(p; \top) \to \neg\Box_1(\neg p; \top)$ is

$$\forall y_1(R_{1,\top} x y_1 \to \forall y_2(R_{1,\top} x y_2 \to y_2 = y_1)),$$

i.e.,

$$\exists^{\leq 1} y_1 R_{1,\top} x y_1,$$

i.e.,

$$|\bigcup N(x)| \leq 1,$$

i.e., $N(x)$ is of one of the following form:

$$\varnothing, \{\varnothing\}, \{\{y\}\}, \{\varnothing, \{y\}\}.$$

Example 20 Consider the formula $\Box_1(\Box_1(p; \top); \top) \to \Box_1(p; \top)$, its standard translation is

$$\begin{array}{rl} & ST_x(\Box_1(\Box_1(p;\top);\top) \to \Box_1(p;\top)) \\ = & \exists y_1(R_{1,\top}xy_1 \wedge \exists y_2(R_{1,\top}y_1y_2 \wedge Py_2) \to \exists y_3(R_{1,\top}xy_3 \wedge Py_3) \\ = & \forall y_1\forall y_2(R_{1,\top}xy_1 \wedge R_{1,\top}y_1y_2 \wedge Py_2 \to \exists y_3(R_{1,\top}xy_3 \wedge Py_3)) \end{array}$$

the minimal valuation is $\lambda z.z = y_2$, therefore the local first-order correspondent of $\Box_1(\Box_1(p;\top);\top) \to \Box_1(p;\top)$ is

$$\forall y_1\forall y_2(R_{1,\top}xy_1 \wedge R_{1,\top}y_1y_2 \to \exists y_3(R_{1,\top}xy_3 \wedge y_3 = y_2)),$$

i.e.,

$$\forall y_1\forall y_2(R_{1,\top}xy_1 \wedge R_{1,\top}y_1y_2 \to R_{1,\top}xy_2),$$

i.e.,

$$\forall y_1 \in \bigcup N(x), \bigcup N(y_1) \subseteq \bigcup N(x).$$

As we can see from the examples, instantial neighbourhood logic can talk about the "relational part" of the neighbourhood function, this is one of the reason to investigate the correspondence theory of instantial neighbourhood logic.

7 Discussions and Further Directions

In this paper, we give two different proofs of the Sahlqvist correspondence theorem for instantial neighbourhood logic, the first one by standard translation and minimal valuation, and the second one by reduction using the bimodal translation into a normal bimodal language. We give some remarks and further directions here.

Algebraic correspondence method using the algorithm ALBA. In [8], Sahlqvist and inductive formulas (an extension of Sahlqvist formulas, see [9] for further details) are defined based on duality-theoretic and order-algebraic insights. The Ackermann lemma based algorithm ALBA is given, which effectively computes first-order correspondents of input formulas/inequalities, and succeed on the Sahlqvist and inductive formulas/inequalities. In this approach, Sahlqvist and inductive formulas are defined in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives. Indeed, in the dual complex algebra $\mathbb{A}$ of Kripke frame, the good properties of the connectives are the following:

- Unary $\Diamond$ is interpreted as a map $\Diamond^{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$, which preserves arbitrary joins, i.e. $\Diamond^{\mathbb{A}}(\bigvee a) = \bigvee \Diamond^{\mathbb{A}} a$ and $\Diamond^{\mathbb{A}}\bot = \bot$. Similarly, n-ary diamonds are interpreted as maps which preserve arbitrary joins in every coordinate.
- Unary $\Box$ is interpreted as a map $\Box^{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$, which preserves arbitrary meets, i.e. $\Box^{\mathbb{A}}(\bigwedge a) = \bigwedge \Box^{\mathbb{A}} a$ and $\Box^{\mathbb{A}}\top = \top$. Preserving arbitrary meets guarantees the map $\Box^{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ to have a left adjoint $\blacklozenge^{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ such that $\blacklozenge^{\mathbb{A}} a \leq b$ iff $a \leq \Box^{\mathbb{A}} b$.

As we have seen from Section 2, the algebraic interpretation of $\Box_n(\varphi_1, \ldots, \varphi_n; \varphi)$ preserves arbitrary joins in the first n coordinates, and is monotone in the last coordinate. Therefore, we can adapt the ALBA method to the instantial neighbourhood logic case. In addition to this, we can also define INL-inductive formulas based on the algebraic properties of the instantial neighbourhood connectives, to extend INL-Sahlqvist formulas to INL-inductive formulas as well as to the language of instantial neighbourhood logic with fixpoint operators.

Completeness and canonicity. Other issues that we do not study in the present paper include completeness of logics axiomatized by INL-Sahlqvist formulas and canonicity. For the proof of completeness, we need to establish the validity of INL-Sahlqvist formulas on their corresponding canonical frames, where canonicity and persistence might play a role (see [6, Chapter 5]).

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含例邻域逻辑的萨奎斯特对应理论

赵之光

摘 要

本文给出含例邻域逻辑的萨奎斯特对应理论。这种逻辑可以讨论一个可能世界的邻域的存在性信息，是关系语义和邻域语义的一种混合。增加的表达力和描述邻域函数的关系特征使我们可以在框架层面讨论这种语言可以定义什么性质，是否可以超越邻域语义的 KW 片段。我们给出对应定理的两个证明。第一个证明直接使用标准翻译和极小赋值的技术，第二个证明通过双模态翻译将含例邻域逻辑的对应问题转化为经典克里普克语义的双模态逻辑。

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