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Undecidability Results of Modal Definability in Extended Modal Languages*

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Abstract. In the present paper, we apply the methodology in Balbiani and Tinchev(2016) to show that for the modal language with universal modality $\mathcal{L}_U$, tense language $\mathcal{L}_T$, hybrid languages $\mathcal{L}_H$, $\mathcal{L}_{H(@)}$, Chagrova's theorem holds that the modal/tense/hybrid definability of first-order sentences with respect to certain classes of frames is undecidable, by using similar techniques as stable classes of Kripke frames.

1 Introduction

In the model theory of modal logic, the modal definability problem of first-order formulas can be stated as follows: given a first-order formula α, for the class of Kripke frames it defines, whether there is a modal formula φ that defines the same class of Kripke frames. The celebrated Goldblatt-Thomason Theorem ([16]) states that given an elementary class of Kripke frames, it is modally definable if and only if it is closed under taking disjoint unions, generated subframes and bounded morphic images and reflects ultrafilter extensions. However, this theorem does not provide an algorithm to check if a given first-order formula is modally definable. As is shown by Chagrova in [10], this problem is undecidable.[1]

The problem of modal definability is further studied in [1–4, 12–15]. In [2], it is shown that modal definability for the class of all partitions is PSPACE-complete. In [3], it is further shown that modal definability in the modal language extended with universal modality for the class of all partitions is PSPACE-complete. In [12, 14], it is shown that for the modal language and the modal language with universal modality, the modal definability problem for the class of KD45-frames is PSPACE-complete. In [4], by applying the stable class technique, it is shown that with respect to certain frame classes, the modal definability problem of first-order sentences is undecidable, which is also an alternative proof of Chagrova's result. In [1], by using

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[1]For further similar results, see [7, 9, 11] and [8, Chapter 17].

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similar techniques, it is shown that with respect to the class of all Euclidean frames, the modal definability problem is undecidable.

The basic idea of the stable class technique can be described as follows: Given a class $\mathfrak{C}$ of Kripke frames, by showing that the class $\mathfrak{C}$ is stable, the problem of checking whether a first-order sentence α is valid in $\mathfrak{C}$ ($\mathfrak{C}$-validity problem) can be reduced to the modal definability problem of α in $\mathfrak{C}$ ($\mathfrak{C}$-modal definability problem). Therefore, if the first-order theory of $\mathfrak{C}$ is undecidable, then the modal definability of first-order sentences in $\mathfrak{C}$ is undecidable.

In the present paper, what we are going to investigate is to what extent can we apply the same technique to get similar undecidability results, if we extend the modal language by adding converse modality, nominals, the @-operator, universal modality, etc. Indeed, in the proof that $\mathfrak{C}$-validity problem can be reduced to $\mathfrak{C}$-modal definability problem, the only parts that uses properties of the modal language are the following: (a). the modal language contains a formula like $\bot$ such that it is valid on no Kripke frames in $\mathfrak{C}$; (b). the relation $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$ that $\mathbb{F}'$ validates more (or the same) modal formulas than $\mathbb{F}$. If we revise the definition of the $\preceq_{\mathcal{L}_M}$ relation by replacing modal formulas by tense/hybrid/…formulas, then we can get similar notions of stability in the extended modal languages, without changing the proof of the reducibility mentioned above. Therefore, once we have revised the definitions of stability according to the extended modal language, we can use the same technique to construct the witnesses of stability. What one needs to take care is that by adding expressivity to the modal language, the validity of extended modal formulas are preserved under less kinds of frame constructions, e.g. for tense logic, the notion of p-morphic image should be revised accordingly, and for the language with universal modality, the validity is not preserved under taking disjoint union or generated subframe anymore. Therefore, we need to take care of choosing appropriate frame $\mathbb{F}'$ to make sure that $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ holds for the extended modal language $\mathcal{L}$ in consideration.

The structure of the paper is as follows: Section 2 presents preliminaries on the extended modal languages and first-order language concerned in the paper. Section 3 sketches the stable class methodology as well as giving new undecidability results for the class of serial frames in the basic modal language. Section 4 gives the proofs that $\mathcal{L}_U$-, $\mathcal{L}_T$-, $\mathcal{L}_H$- and $\mathcal{L}_{H(@)}$-definability for certain classes of Kripke frames are undecidable. Section 5 gives conclusions and further directions of research.

2 Preliminaries

In this section, we collect preliminary definitions and propositions for modal logic, tense logic and hybrid logic. For more details, see [4–6].

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2.1 Extended modal languages

Syntax Given a set of propositional variables Prop, a set of nominals Nom, the syntax for modal logic $\mathcal{L}_M$, modal logic with universal modality $\mathcal{L}_U$, tense logic $\mathcal{L}_T$, hybrid logic $\mathcal{L}_H$, hybrid logic with @-operator $\mathcal{L}_{H(@)}$ (we call these languages (extended) modal languages) are defined as follows:

$$\mathcal{L}_M: \ \ \varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi$$

where $p \in \mathrm{Prop}$.

$\mathcal{L}_U$ is obtained by adding the clauses $U\varphi$ and $E\varphi$ to $\mathcal{L}_M$;

$\mathcal{L}_T$ is obtained by adding the clauses $\blacksquare\varphi$ and $\blacklozenge\varphi$ to $\mathcal{L}_M$;

$\mathcal{L}_H$ is obtained by adding the clause $\mathbf{i}$ to $\mathcal{L}_M$ where $\mathbf{i} \in \mathrm{Nom}$;

$\mathcal{L}_{H(@)}$ is obtained by adding the clause $@_{\mathbf{i}}\varphi$ to $\mathcal{L}_H$.

Semantics Given the (extended) modal languages, they are interpreted on Kripke frames $\mathbb{F} = (W, R)$ where $W \neq \varnothing$, $R \subseteq W \times W$. A Kripke model is a tuple $\mathbb{M} = (W, R, V)$ where (W, R) is a Kripke frame and $V : \mathrm{Prop} \cup \mathrm{Nom} \to P(W)$ is an assignment such that $V(\mathbf{i})$ is a singleton for all nominals $\mathbf{i} \in \mathrm{Nom}$.

Given a Kripke model $\mathbb{M} = (W, R, V)$, the satisfaction relation is defined as follows:

$\mathbb{M}, w \Vdash p$	iff	$w \in V(p)$;
$\mathbb{M}, w \Vdash \mathbf{i}$	iff	$V(\mathbf{i}) = \{w\}$;
$\mathbb{M}, w \Vdash \bot$	:	never;
$\mathbb{M}, w \Vdash \top$	:	always;
$\mathbb{M}, w \Vdash \neg\varphi$	iff	$\mathbb{M}, w \nVdash \varphi$;
$\mathbb{M}, w \Vdash \varphi \wedge \psi$	iff	$\mathbb{M}, w \Vdash \varphi$ and $\mathbb{M}, w \Vdash \psi$;
$\mathbb{M}, w \Vdash \varphi \vee \psi$	iff	$\mathbb{M}, w \Vdash \varphi$ or $\mathbb{M}, w \Vdash \psi$;
$\mathbb{M}, w \Vdash \varphi \to \psi$	iff	$\mathbb{M}, w \nVdash \varphi$ or $\mathbb{M}, w \Vdash \psi$;
$\mathbb{M}, w \Vdash \Box\varphi$	iff	for all $v \in W$, if Rwv then $\mathbb{M}, v \Vdash \varphi$;
$\mathbb{M}, w \Vdash \Diamond\varphi$	iff	there exists $v \in W$ such that Rwv and $\mathbb{M}, v \Vdash \varphi$;
$\mathbb{M}, w \Vdash U\varphi$	iff	for all $v \in W$, $\mathbb{M}, v \Vdash \varphi$;
$\mathbb{M}, w \Vdash E\varphi$	iff	there exists $v \in W$ such that $\mathbb{M}, v \Vdash \varphi$;
$\mathbb{M}, w \Vdash \blacksquare\varphi$	iff	for all $v \in W$, if Rvw then $\mathbb{M}, v \Vdash \varphi$;
$\mathbb{M}, w \Vdash \blacklozenge\varphi$	iff	there exists $v \in W$ such that Rvw and $\mathbb{M}, v \Vdash \varphi$;
$\mathbb{M}, w \Vdash @_{\mathbf{i}}\varphi$	iff	$\mathbb{M}, V(\mathbf{i}) \Vdash \varphi$.

A formula φ is true in a model $\mathbb{M}$ (notation: $\mathbb{M} \Vdash \varphi$), if $\mathbb{M} \Vdash \varphi$ for all $v \in W$. φ is valid in a frame $\mathbb{F}$ (notation: $\mathbb{F} \Vdash \varphi$), if φ is true in all models based on $\mathbb{F}$. φ is valid in a frame class $\mathfrak{C}$ (notation: $\mathfrak{C} \Vdash \varphi$), if φ is valid in all frames in $\mathfrak{C}$. A frame

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$\mathbb{F}$ is $\mathcal{L}$-weaker than a frame $\mathbb{F}'$ (notation $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$), if for all $\mathcal{L}$-formulas φ, if $\mathbb{F} \Vdash \varphi$ then $\mathbb{F}' \Vdash \varphi$.

2.2 Frame constructions and validity preservation

In the stable class techniques, the $\preceq_{\mathcal{L}}$ relation is an important technical tool, and it can be shown by proving that certain frame constructions preserve $\mathcal{L}$-validity. Notice that here we only talk about frame constructions rather than model constructions, we do not need to revise the definitions of the frame constructions for hybrid logic.

Definition 1 (Generated subframe). Given two Kripke frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$, we say that $\mathbb{F}'$ is a generated subframe of $\mathbb{F}$ if $W' \subseteq W$, $R' = R \cap (W' \times W')$, and for all $w \in W'$ and $v \in W$ such that Rwv, we have $v \in W'$.

Definition 2 (Disjoint union). Given Kripke frames $\mathbb{F}_i = (W_i, R_i)$ $(i \in I)$ with disjoint domains, their disjoint union $\biguplus_i \mathbb{F}_i = (W, R)$ is defined as $W := \bigcup_{i \in I} W_i$, $R := \bigcup_{i \in I} R_i$.

Definition 3 (Bounded morphic image). Given two Kripke frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$, we say that $\mathbb{F}'$ is a bounded morphic image of $\mathbb{F}$ if there is a surjective map $f : W \to W'$ such that the following conditions hold:

- for all $w, v \in W$, if Rwv then $R'f(w)f(v)$;
- for all $w \in W$, $v' \in W'$, if $R'f(w)v'$ then there exists a $v \in W$ such that Rwv and $f(v) = v'$.

Definition 4 (Tense generated subframe). Given two Kripke frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$, we say that $\mathbb{F}'$ is a tense generated subframe of $\mathbb{F}$ if

- $W' \subseteq W$;
- $R' = R \cap (W' \times W')$;
- for all $w \in W'$ and $v \in W$ such that Rwv, we have $v \in W'$;
- for all $w \in W'$ and $v \in W$ such that Rvw, we have $v \in W'$.

Definition 5 (Tense bounded morphic image). Given two Kripke frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$, we say that $\mathbb{F}'$ is a tense bounded morphic image of $\mathbb{F}$ if there is a surjective map $f : W \to W'$ such that the following conditions hold:

- for all $w, v \in W$, if Rwv then $R'f(w)f(v)$;
- for all $w \in W$, $v' \in W'$, if $R'f(w)v'$ then there exists a $v \in W$ such that Rwv and $f(v) = v'$;
- for all $w \in W$, $v' \in W'$, if $R'v'f(w)$ then there exists a $v \in W$ such that Rvw and $f(v) = v'$.

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It is easy to see that the modal language $\mathcal{L}_M$ is preserved under taking the first three kinds of frame constructions defined above:

Proposition 1.

- *Given two Kripke frames $\mathbb{F}$ and $\mathbb{F}'$, if $\mathbb{F}'$ is a generated subframe of $\mathbb{F}$, then for any $\mathcal{L}_M$-formula φ, if $\mathbb{F} \Vdash \varphi$, then $\mathbb{F}' \Vdash \varphi$ (i.e., $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$);*
- *Given a class of frames $\{\mathbb{F}_i \mid i \in I\}$, for any $\mathcal{L}_M$-formula φ, if $\mathbb{F}_i \Vdash \varphi$ for all $i \in I$, then $\biguplus_i \mathbb{F}_i \Vdash \varphi$;*
- *Given two Kripke frames $\mathbb{F}$ and $\mathbb{F}'$, if $\mathbb{F}'$ is a bounded morphic image of $\mathbb{F}$, then for any $\mathcal{L}_M$-formula φ, if $\mathbb{F} \Vdash \varphi$, then $\mathbb{F}' \Vdash \varphi$ (i.e., $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$).*

We can obtain similar results for extended modal languages:

Proposition 2.

- *For the modal language with universal modality $\mathcal{L}_U$, its validity is preserved under taking bounded morphic images;*
- *For the tense logic language $\mathcal{L}_T$, its validity is preserved under taking tense generated subframes, disjoint unions and tense bounded morphic images;*
- *For the hybrid logic languages $\mathcal{L}_H$ and $\mathcal{L}_{H(@)}$, their validities are preserved under taking generated subframes.*

2.3 First-order language

In this subsection we give the necessary notations and definitions in first-order logic and relativization. We follow the presentations in [4].

Syntax Given a set of individual variables Var, the first-order language $\mathcal{L}^1$ is defined as follows:

$$\alpha ::= Rxy \mid x = y \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \to \beta \mid \forall x\alpha \mid \exists x\alpha$$

We use $\overline{x}$ to denote a list of individual variables $x_1, \ldots, x_n$, and use $\alpha(\overline{x})$ to indicate that all free individual variables are among $\overline{x}$. When a first-order formula α does not contain free variables, we call it a sentence.

Truth Given a frame $\mathbb{F} = (W, R)$, the satisfaction relation between first-order formula $\alpha(\overline{x})$ and $\mathbb{F}$ with respect to a list $\overline{s}$ of worlds in $\mathbb{F}$ (notation: $\mathbb{F} \vDash \alpha(\overline{x})[\overline{s}]$) is defined as follows:

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$$
\begin{array}{lll}
\mathbb{F} \vDash Rx_ix_j[\overline{s}] & \text{iff} & Rs_is_j; \\
\mathbb{F} \vDash x_i = x_j[\overline{s}] & \text{iff} & s_i = s_j; \\
\mathbb{F} \vDash (\neg\alpha)[\overline{s}] & \text{iff} & \mathbb{F} \nvDash \alpha[\overline{s}]; \\
\mathbb{F} \vDash (\alpha \wedge \beta)[\overline{s}] & \text{iff} & \mathbb{F} \vDash \alpha[\overline{s}] \text{ and } \mathbb{F} \vDash \beta[\overline{s}]; \\
\mathbb{F} \vDash (\alpha \vee \beta)[\overline{s}] & \text{iff} & \mathbb{F} \vDash \alpha[\overline{s}] \text{ or } \mathbb{F} \vDash \beta[\overline{s}]; \\
\mathbb{F} \vDash (\alpha \to \beta)[\overline{s}] & \text{iff} & \mathbb{F} \vDash \alpha[\overline{s}] \text{ implies } \mathbb{F} \vDash \beta[\overline{s}]; \\
\mathbb{F} \vDash \forall x\alpha(\overline{x}, x)[\overline{s}] & \text{iff} & \text{for all } s \in W, \mathbb{F} \vDash \alpha(\overline{x}, x)[\overline{s}, s]; \\
\mathbb{F} \vDash \exists x\alpha(\overline{x}, x)[\overline{s}] & \text{iff} & \text{there exists } s \in W \text{ such that } \mathbb{F} \vDash \alpha(\overline{x}, x)[\overline{s}, s].
\end{array}
$$

A first-order formula $\alpha(\overline{x})$ is valid in $\mathbb{F}$ (notation: $\mathbb{F} \vDash \alpha(\overline{x})$), if $\alpha(\overline{x})$ is satisfied in $\mathbb{F}$ with respect to all $\overline{s}$ in $\mathbb{F}$. A first-order formula α is said to be valid in a class $\mathfrak{C}$ of frames (notation $\mathfrak{C} \vDash \alpha$), if α is valid in all frames in $\mathfrak{C}$. The first-order theory of the frame class $\mathfrak{C}$ is $Th(\mathfrak{C}) := \{\varphi \mid \varphi \text{ is a first-order sentence and } \mathfrak{C} \vDash \varphi\}$.

Relativization The relativization γ^α_x of a first-order formula γ with respect to another first-order formula α and an individual variable x is defined as follows:

$$
\begin{array}{lll}
(Rst)^\alpha_x & := & Rst; \\
(s = t)^\alpha_x & := & s = t; \\
(\neg\gamma)^\alpha_x & := & \neg\gamma; \\
(\gamma \wedge \delta)^\alpha_x & := & (\gamma)^\alpha_x \wedge (\delta)^\alpha_x; \\
(\gamma \vee \delta)^\alpha_x & := & (\gamma)^\alpha_x \vee (\delta)^\alpha_x; \\
(\gamma \to \delta)^\alpha_x & := & (\gamma)^\alpha_x \to (\delta)^\alpha_x; \\
(\forall y\gamma)^\alpha_x & := & \forall y(\alpha[x/y] \to (\gamma)^\alpha_x); \\
(\exists y\gamma)^\alpha_x & := & \exists y(\alpha[x/y] \wedge (\gamma)^\alpha_x).
\end{array}
$$

where $\alpha[x/y]$ is obtained by replacing all free occurrence of x in α by y. When writing γ^α_x, we assume that individual variables occurring in α and γ are disjoint.

Definition 6. Given two frames $\mathbb{F} = (W, R)$ and $\mathbb{F}' = (W', R')$, if there is a first-order formula $\alpha(\overline{x}, x)$ and a list $\overline{s} \in W$ such that $W' = \{s \in W \mid \mathbb{F} \vDash \alpha(\overline{x}, x)[\overline{s}, s]\}$ and $R' = R \cap (W' \times W')$, then we say that $\mathbb{F}'$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$.

For relativization and relativized reducts, we have the following theorem:

Theorem 3 (Relativization theorem)**.** *Given two frames $\mathbb{F}$, $\mathbb{F}'$, a first-order formula $\alpha(\overline{x}, x)$, a list of worlds $\overline{s}$ (corresponding to $\overline{x}$). If $\mathbb{F}'$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$, then for all first-order formula $\gamma(\overline{y})$ and list of worlds $\overline{t}$ (corresponding to $\overline{y}$) in $\mathbb{F}'$,*

$$\mathbb{F} \vDash (\gamma(\overline{y}))^{\alpha(\overline{x},x)}_x[\overline{s}, \overline{t}] \textit{ iff } \mathbb{F}' \vDash \gamma(\overline{y})[\overline{t}].$$

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3 Undecidability of Modal Definability: The Stable Class Technique

In this section, we recall the technique used in [4] to show the undecidability of modal definability of first-order sentences. The basic idea is to use the so-called stable class of frames. Balbiani and Tinchev ([4]) showed that if a class of frames $\mathfrak{C}$ is stable, then the validity problem of first-order sentences in $\mathfrak{C}$ is reducible to the modal definability problem with respect to $\mathfrak{C}$. Once the validity problem of first-order sentences in $\mathfrak{C}$ is undecidable, the modal definability problem with respect to $\mathfrak{C}$ is undecidable.

3.1 The stable class technique

Now we give the relevant definitions in [4].

Definition 7 (Modal definability)**.** Given a class of frames $\mathfrak{C}$, a first-order sentence α is modally definable with respect to $\mathfrak{C}$ if there is a modal formula φ such that for all frames $\mathbb{F} \in \mathfrak{C}$, $\mathbb{F} \vDash \alpha$ iff $\mathbb{F} \Vdash \varphi$.

For other extended modal languages, the definition is similar.

Definition 8 (Stable class)**.** A class of frames $\mathfrak{C}$ is stable if there is a first-order formula $\alpha(\overline{x}, x)$ and a first-order sentence β such that the following two conditions hold (we say that ($\alpha(\overline{x}, x)$, β) is a witness of the stability of $\mathfrak{C}$):

1. for all frames $\mathbb{F} = (W, R) \in \mathfrak{C}$, for all $\overline{s} \in W$, if a frame $\mathbb{F}'$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$, then $\mathbb{F}' \in \mathfrak{C}$; that is to say, $\mathfrak{C}$ is closed under taking relativized reducts;
2. for all frames $\mathbb{F}_0 \in \mathfrak{C}$, there exist $\mathbb{F}, \mathbb{F}' \in \mathfrak{C}$ and a list of worlds $\overline{s} \in \mathbb{F}$ such that $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$, $\mathbb{F} \vDash \beta$, $\mathbb{F}' \nvDash \beta$ and $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$.

The definition above is defined for the language of modal logic, and it can be adapted to other extended modal languages by revising the index of $\preceq$.

Now we briefly recall the proof of Balbiani and Tinchev's reduction theorem:

Theorem 4 (Theorem 1 in [4])**.** *If a class of frames $\mathfrak{C}$ is stable, then the validity problem of first-order sentences in $\mathfrak{C}$ is reducible to the modal definability problem with respect to $\mathfrak{C}$.*

Proof. See [4, Theorem 1]. Here we repeat it for the sake of checking the details of the proof.

Suppose $\mathfrak{C}$ is stable and $(\alpha(\overline{x}, x), \beta)$ witnesses the stability of $\mathfrak{C}$. Let γ be a sentence and $\delta := \exists\overline{x}(\exists x\alpha(\overline{x}, x) \wedge \neg(\gamma)_x^{\alpha(\overline{x},x)}) \wedge \beta$, then it can be shown that $\mathfrak{C} \vDash \gamma$ iff δ is modally definable with respect to $\mathfrak{C}$.

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$\Rightarrow$: Assume $\mathfrak{C} \vDash \gamma$. If δ is not modally definable with respect to $\mathfrak{C}$, then there is a frame $\mathbb{F} = (W, R) \in \mathfrak{C}$ such that $\mathbb{F} \vDash \delta$ (otherwise $\bot$ defines δ with respect to $\mathfrak{C}$). Then there is a list $\overline{s} \in W$ such that $\mathbb{F} \vDash \exists x\alpha(\overline{x}, x) \wedge \neg(\gamma)_x^{\alpha(\overline{x},x)}[\overline{s}]$, so we can consider the relativized reduct $\mathbb{F}'$ of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$. By condition 1 of stable class, since $\mathfrak{C}$ is stable, from $\mathbb{F} \in \mathfrak{C}$ we get $\mathbb{F}' \in \mathfrak{C}$. However, $\mathbb{F}'$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$, by Theorem 3, $\mathbb{F} \vDash (\gamma)_x^{\alpha(\overline{x},x)}[\overline{s}]$ iff $\mathbb{F}' \vDash \gamma$.Since $\mathbb{F} \nvDash (\gamma)_x^{\alpha(\overline{x},x)}[\overline{s}]$, we have $\mathbb{F}' \nvDash \gamma$, which contradicts $\mathfrak{C} \vDash \gamma$ and $\mathbb{F}' \in \mathfrak{C}$.

$\Leftarrow$: Assume that δ is modally definable with respect to $\mathfrak{C}$. Suppose φ modally defines δ with respect to $\mathfrak{C}$. If $\mathfrak{C} \nvDash \gamma$, then there is a frame $\mathbb{F}_0 \in \mathfrak{C}$ such that $\mathbb{F}_0 \nvDash \gamma$. Since $\mathfrak{C}$ is stable witnessed by $(\alpha(\overline{x}, x), \beta)$, there are frames $\mathbb{F} = (W, R)$, $\mathbb{F}' \in \mathfrak{C}$ and a list of worlds $\overline{s} \in W$ such that $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$, $\mathbb{F} \vDash \beta$, $\mathbb{F}' \nvDash \beta$ and $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$. Since $\mathbb{F}' \nvDash \beta$, $\mathbb{F}' \nvDash \delta$ as well. Since φ modally defines δ with respect to $\mathfrak{C}$, we have $\mathbb{F}' \nVdash \varphi$. By $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$, we have $\mathbb{F} \nVdash \varphi$. Since $\mathbb{F} \in \mathfrak{C}$ and φ modally defines δ with respect to $\mathfrak{C}$, we have $\mathbb{F} \nvDash \delta$. By $\mathbb{F} \vDash \beta$, we have $\mathbb{F} \nvDash \exists\overline{x}(\exists x\alpha(\overline{x}, x) \wedge \neg(\gamma)_x^{\alpha(\overline{x},x)})$. Since $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and $\overline{s}$, by Theorem 3, $\mathbb{F} \vDash (\gamma)_x^{\alpha(\overline{x},x)}[\overline{s}]$ iff $\mathbb{F}_0 \vDash \gamma$. By $\mathbb{F}_0 \nvDash \gamma$, we have $\mathbb{F} \vDash \neg(\gamma)_x^{\alpha(\overline{x},x)}[\overline{s}]$. Moreover, $\mathbb{F} \vDash \exists x\alpha(\overline{x}, x)[\overline{s}]$. Therefore, $\mathbb{F} \vDash \exists\overline{x}(\exists x\alpha(\overline{x}, x) \wedge \neg(\gamma)_x^{\alpha(\overline{x},x)})$, a contradiction. □

It is easy to see that the only two places that uses the properties of the modal language are the following:

- The modal language contains a formula like $\bot$ such that it is valid on no Kripke frames in $\mathfrak{C}$;
- The relation $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$ that $\mathbb{F}'$ validates more (or the same) modal formulas than $\mathbb{F}$.

Therefore, when considering an extended modal language $\mathcal{L}$, once it contains $\bot$ and we consider the relation $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ instead of $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$ when defining the stable class and proving the theorem, we can obtain the definition of $\mathcal{L}$-stable class by replacing $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$ with $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$, and obtain the analogue of the theorem above by the following theorem:

Theorem 5. *If a class of frames $\mathfrak{C}$ is $\mathcal{L}$-stable, then the validity problem of first-order sentences in $\mathfrak{C}$ is reducible to the $\mathcal{L}$-definability problem with respect to $\mathfrak{C}$.*

3.2 Example of showing undecidability of modal definability within certain frame class

We can give the following example that modal definability problem is undecidable in the class of serial frames, i.e. the frames satisfying $\forall x\exists yRxy$, by showing that

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the class of serial frames is stable. To the author's knowledge, this result is original.

Theorem 6. *The class $\mathfrak{C}_{Ser}$ of serial frames is stable. Therefore, the modal definability problem in $\mathfrak{C}_{Ser}$ is undecidable.*

Proof. For the validity problem of $Th(\mathfrak{C}_{Ser})$, since the first-order theory of lattice is a finite extension of $Th(\mathfrak{C}_{Ser})$, the undecidability of $Th(\mathfrak{C}_{Ser})$ follows from the undecidability of the first-order theory of lattice [17]. Therefore, it suffices to show that $\mathfrak{C}_{Ser}$ is stable.

Now define $\alpha(x) := \exists y Ryx$, $\beta := \neg\forall x Rxx$, then we can show that conditions 1 and 2 hold for $\mathfrak{C}_{Ser}$ witnessed by $(\alpha(x), \beta)$:

- For condition 1, take any frame $\mathbb{F} = (W, R) \in \mathfrak{C}_{Ser}$, $\mathbb{F}$ is serial. Consider a frame $\mathbb{F}' = (W', R')$ which is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(x)$, then it is easy to see that $W' \neq \varnothing$, since for a serial frame $\mathbb{F} = (W, R)$, $R \neq \varnothing$, so there exists a $w \in W$ such that w has an R-predecessor. We can show that R' is a serial relation on W': suppose otherwise, w has R-successors but no R'-successors, then the worlds in the set $R[w] = \{v \in W \mid Rwv\}$ are all deleted when taking the relativized reduct, so those vs have no R-predecessor, a contradiction to Rwv.
- For condition 2, for any serial frame $\mathbb{F}_0 = (W_0, R_0) \in \mathfrak{C}_{Ser}$, we can construct $\mathbb{F}$ and $\mathbb{F}'$ as follows:
  $\mathbb{F} := (W, R)$, where $W = W_0 \cup \{s, t\}$, $R = R_0 \cup (\{s, t\} \times W_0)$;
  $\mathbb{F}' := (W', R')$, where $W' = \{r\}$, $R' = \{(r, r)\}$;
  It is easy to see that $\mathbb{F}$ and $\mathbb{F}'$ are serial. Since in $\mathbb{F}$, the worlds with immediate predecessors are exactly the ones in W_0, so $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $\exists y Ryx$.
  It is easy to see that $\mathbb{F} \vDash \neg\forall x Rxx$, $\mathbb{F}' \nvDash \neg\forall x Rxx$.
  Finally, define $f : \mathbb{F} \to \mathbb{F}'$ such that every world is mapped to r, it is easy to see that f is a surjective bounded morphic morphism, so $\mathbb{F} \preceq_{\mathcal{L}_M} \mathbb{F}'$.

Therefore, $\mathfrak{C}_{Ser}$ is stable. □

4 Undecidability Results

In this section, we will make use of the stable class technique to show that certain $\mathcal{L}$-definability problems with respect to certain frame classes $\mathfrak{C}$ are undecidable by showing that $\mathfrak{C}$ is $\mathcal{L}$-stable and that the validity problem of first-order sentences in $\mathfrak{C}$ is undecidable.

Theorem 7. *The class $\mathfrak{C}$ of all Kripke frames is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable.*

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Proof. Define $\alpha(\overline{x}, x) := Rx_1x$, $\beta := \exists x \exists y(x \neq y \wedge \neg\exists zRzx \wedge \neg\exists zRzy)$, then we can show that for $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stability, conditions 1 and 2 hold for $\mathfrak{C}$ witnessed by $(\alpha(\overline{x}, x), \beta)$:

- For condition 1, since the class of all Kripke frames is closed under taking subframes, this condition is automatically satisfied;
- For condition 2, for any Kripke frame $\mathbb{F}_0 = (W_0, R_0) \in \mathfrak{C}$, we can construct $\mathbb{F}$ and $\mathbb{F}'$ as follows:

  $\mathbb{F}' := (W', R')$, where $W' = W_0 \cup \{s\}$, $R' = R_0 \cup (\{s\} \times W_0)$;

  We take an isomorphic copy $\mathbb{F}''$ of $\mathbb{F}'$, and define $\mathbb{F} := \mathbb{F}' \uplus \mathbb{F}''$, where the isomorphic copy of s in the $\mathbb{F}''$ part is denoted as s';

  It is trivial that $\mathbb{F}$ and $\mathbb{F}' \in \mathfrak{C}$.

  Since in $\mathbb{F}$, the worlds who has s as an immediate predecessor are exactly the ones in the original copy (i.e., $\mathbb{F}'$ part) W_0, so $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to Rx_1x and s.

  It is easy to see that $\mathbb{F} \vDash \exists x \exists y(x \neq y \wedge \neg\exists zRzx \wedge \neg\exists zRzy)$, $\mathbb{F}' \nvDash \exists x \exists y(x \neq y \wedge \neg\exists zRzx \wedge \neg\exists zRzy)$, since in $\mathbb{F}$ there are two worlds without immediate predecessor, but in $\mathbb{F}'$ there is only one world without immediate predecessor.

  Finally, define $f : \mathbb{F} \to \mathbb{F}'$ such that both the $\mathbb{F}'$ part and the $\mathbb{F}''$ part are mapped to $\mathbb{F}'$ in an isomorphic way. Then it is easy to check that f is a surjective tense bounded morphic morphism, a surjective bounded morphic morphism, $\mathbb{F}'$ is a generated subframe of $\mathbb{F}$, so $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$.

Therefore, $\mathfrak{C}$ is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable. □

Corollary 1. *The $\mathcal{L}$-definability problem in $\mathfrak{C}$ is undecidable for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$.*

Proof. By Theorem 5 and Theorem 7, it suffices to show that the validity problem of first-order sentences in $\mathfrak{C}$ is undecidable, which is already shown in [4, Corollary 1]. □

Theorem 8. *The class $\mathfrak{C}_{Ref}$ of all reflexive Kripke frames is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable.*

Proof. Define $\alpha(\overline{x}, x) := Rx_1x \wedge \neg x_1 = x$, $\beta := \exists x \exists y(x \neq y \wedge \neg\exists z(Rzx \wedge \neg z = x) \wedge \neg\exists z(Rzy \wedge \neg z = y))$, then we can show that for $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stability, conditions 1 and 2 hold for $\mathfrak{C}_{Ref}$ witnessed by $(\alpha(\overline{x}, x), \beta)$:

- For condition 1, since the class of all reflexive Kripke frames is closed under taking subframes, this condition is automatically satisfied;
- For condition 2, for any Kripke frame $\mathbb{F}_0 = (W_0, R_0) \in \mathfrak{C}_{Ref}$, we can construct $\mathbb{F}$ and $\mathbb{F}'$ as follows:

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$\mathbb{F}' := (W', R')$, where $W' = W_0 \cup \{s\}$, $R' = R_0 \cup \{(s,s)\} \cup (\{s\} \times W_0)$;
We take an isomorphic copy $\mathbb{F}''$ of $\mathbb{F}'$, and define $\mathbb{F} := \mathbb{F}' \uplus \mathbb{F}''$, where the isomorphic copy of s in the $\mathbb{F}''$ part is denoted as s';
It is trivial that $\mathbb{F}$ and $\mathbb{F}' \in \mathfrak{C}_{Ref}$.
Since in $\mathbb{F}$, the worlds who has s as a strict immediate predecessor are exactly the ones in the original copy (i.e., $\mathbb{F}'$ part) W_0, so $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $Rx_1x \wedge \neg x_1 = x$ and s.
It is easy to see that $\mathbb{F} \vDash \exists x \exists y(x \neq y \wedge \neg\exists z(Rzx \wedge \neg z = x) \wedge \neg\exists z(Rzy \wedge \neg z = y))$, $\mathbb{F}' \nvDash \exists x \exists y(x \neq y \wedge \neg\exists z(Rzx \wedge \neg z = x) \wedge \neg\exists z(Rzy \wedge \neg z = y))$, since in $\mathbb{F}$ there are two worlds without strict immediate predecessor, but in $\mathbb{F}'$ there is only one world without strict immediate predecessor.
Finally, define $f : \mathbb{F} \to \mathbb{F}'$ such that both the $\mathbb{F}'$ part and the $\mathbb{F}''$ part are mapped to $\mathbb{F}'$ in an isomorphic way. Then it is easy to check that f is a surjective tense bounded morphic morphism, a surjective bounded morphic morphism, $\mathbb{F}'$ is a generated subframe of $\mathbb{F}$, so $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$.

Therefore, $\mathfrak{C}_{Ref}$ is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable. □

Corollary 2. *The $\mathcal{L}$-definability problem in $\mathfrak{C}_{Ref}$ is undecidable for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$.*

Proof. By Theorem 5 and Theorem 8, it suffices to show that the validity problem of first-order sentences in $\mathfrak{C}_{Ref}$ is undecidable, which is already shown in [4, Corollary 3]. □

Theorem 9. *The class $\mathfrak{C}_{Tra}$ of all transitive Kripke frames, $\mathfrak{C}_{Ref,Tra}$ of all reflexive and transitive Kripke frames, $\mathfrak{C}_{Poset}$ of all partial orders are $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable.*

Proof. We define the same $\alpha(\overline{x}, x)$ and β as in Theorem 8. Then we can show that $(\alpha(\overline{x}, x), \beta)$ witnesses the $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stability of $\mathfrak{C}_{Tra}$, $\mathfrak{C}_{Ref,Tra}$ and $\mathfrak{C}_{Poset}$:

- For condition 1, these three classes are all closed under taking subframes, so this condition is automatically satisfied;
- For condition 2, for any Kripke frame $\mathbb{F}_0 = (W_0, R_0)$ in one of the three classes, we can use the same construction of $\mathbb{F}$ and $\mathbb{F}'$ as in Theorem 8. Then it is the case that $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $\alpha(\overline{x}, x)$ and s, $\mathbb{F} \vDash \beta$, $\mathbb{F}' \nvDash \beta$, $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$. It is easy to check that if $\mathbb{F}_0 \in \mathfrak{C}$, then $\mathbb{F}$ and $\mathbb{F}' \in \mathfrak{C}$ holds for $\mathfrak{C} \in \{\mathfrak{C}_{Tra}, \mathfrak{C}_{Ref,Tra}, \mathfrak{C}_{Poset}\}$.

Therefore, $\mathfrak{C}_{Tra}, \mathfrak{C}_{Ref,Tra}, \mathfrak{C}_{Poset}$ are $\mathcal{L}$-stable for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$. □

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Corollary 3. *The $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-definability problems in $\mathfrak{C}_{Tra}$, $\mathfrak{C}_{Ref,Tra}$, $\mathfrak{C}_{Poset}$ are undecidable.*

Proof. By Theorem 5 and Theorem 9, it suffices to show that the validity problems of first-order sentences in $\mathfrak{C}_{Tra}, \mathfrak{C}_{Ref,Tra}, \mathfrak{C}_{Poset}$ are undecidable, which is already shown in [4, Corollary 3, 5]. □

Theorem 10. *The class $\mathfrak{C}_{Sym}$ of all symmetric Kripke frames is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable.*

Proof. Define $\alpha(\overline{x}, x) := Rx_1x$, $\beta := \neg\exists x\forall y(x = y \vee Rxy)$, then we can show that for $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stability, conditions 1 and 2 hold for $\mathfrak{C}_{Sym}$ witnessed by $(\alpha(\overline{x}, x), \beta)$:

- For condition 1, since the class of all symmetric Kripke frames is closed under taking subframes, this condition is automatically satisfied;
- For condition 2, for any Kripke frame $\mathbb{F}_0 = (W_0, R_0) \in \mathfrak{C}_{Sym}$, we can construct $\mathbb{F}$ and $\mathbb{F}'$ as follows:

  $\mathbb{F}' := (W', R')$, where $W' = W_0 \cup \{s\}$, $R' = R_0 \cup (\{s\} \times W_0) \cup (W_0 \times \{s\})$;

  We take an isomorphic copy $\mathbb{F}''$ of $\mathbb{F}'$, and define $\mathbb{F} := \mathbb{F}' \uplus \mathbb{F}''$, where the isomorphic copy of s in the $\mathbb{F}''$ part is denoted as s';

  It is trivial that $\mathbb{F}$ and $\mathbb{F}' \in \mathfrak{C}_{Sym}$.

  Since in $\mathbb{F}$, the worlds who has s as an immediate predecessor are exactly the ones in the original copy (i.e., $\mathbb{F}'$ part) W_0, so $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to Rx_1x and s.

  It is easy to see that $\mathbb{F} \vDash \neg\exists x\forall y(x = y \vee Rxy)$, $\mathbb{F}' \nvDash \neg\exists x\forall y(x = y \vee Rxy)$, since in $\mathbb{F}'$, any non s point is an R'-successor of s, while in $\mathbb{F}$, each point is not connected with a point in the other isomorphic copy.

  Finally, define $f : \mathbb{F} \to \mathbb{F}'$ such that both the $\mathbb{F}'$ part and the $\mathbb{F}''$ part are mapped to $\mathbb{F}'$ in an isomorphic way. Then it is easy to check that f is a surjective tense bounded morphic morphism, a surjective bounded morphic morphism, $\mathbb{F}'$ is a generated subframe of $\mathbb{F}$, so $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$.

Therefore, $\mathfrak{C}_{Sym}$ is $\mathcal{L}_T$, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable. □

Corollary 4. *The $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-definability problem in $\mathfrak{C}_{Sym}$ is undecidable.*

Proof. By Theorem 5 and Theorem 10, it suffices to show that the validity problem of first-order sentences in $\mathfrak{C}_{Sym}$ is undecidable, which is already shown in [4, Corollary 3]. □

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Theorem 11. *The class $\mathfrak{C}_{Ref,Sym}$ of all reflexive and symmetric Kripke frames is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable.*

Proof. Define $\alpha(\overline{x}, x) := Rx_1x \wedge \neg x_1 = x$, $\beta := \neg\exists x\forall yRxy$, then we can show that for $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stability, conditions 1 and 2 hold for $\mathfrak{C}_{Ref,Sym}$ witnessed by $(\alpha(\overline{x}, x), \beta)$:

- For condition 1, since the class of all reflexive symmetric Kripke frames is closed under taking subframes, this condition is automatically satisfied;
- For condition 2, for any Kripke frame $\mathbb{F}_0 = (W_0, R_0) \in \mathfrak{C}_{Ref,Sym}$, we can construct $\mathbb{F}$ and $\mathbb{F}'$ as follows:

  $\mathbb{F}' := (W', R')$, where $W' = W_0 \cup \{s\}$, $R' = R_0 \cup (\{s\} \times W_0) \cup (W_0 \times \{s\}) \cup \{(s, s)\}$;

  We take an isomorphic copy $\mathbb{F}''$ of $\mathbb{F}'$, and define $\mathbb{F} := \mathbb{F}' \uplus \mathbb{F}''$, where the isomorphic copy of s in the $\mathbb{F}''$ part is denoted as s';

  It is trivial that $\mathbb{F}$ and $\mathbb{F}' \in \mathfrak{C}_{Ref,Sym}$.

  Since in $\mathbb{F}$, the worlds who has s as a strict immediate predecessor are exactly the ones in the original copy (i.e., $\mathbb{F}'$ part) W_0, so $\mathbb{F}_0$ is the relativized reduct of $\mathbb{F}$ with respect to $Rx_1x \wedge \neg x_1 = x$ and s.

  It is easy to see that $\mathbb{F} \vDash \neg\exists x\forall yRxy$, $\mathbb{F}' \nvDash \neg\exists x\forall yRxy$, since any point is an R'-successor of s in $\mathbb{F}'$, while in $\mathbb{F}$, points in different isomorphic copies are not connected.

  Finally, define $f : \mathbb{F} \to \mathbb{F}'$ such that both the $\mathbb{F}'$ part and the $\mathbb{F}''$ part are mapped to $\mathbb{F}'$ in an isomorphic way. Then it is easy to check that f is a surjective tense bounded morphic morphism, a surjective bounded morphic morphism, $\mathbb{F}'$ is a generated subframe of $\mathbb{F}$, so $\mathbb{F} \preceq_{\mathcal{L}} \mathbb{F}'$ for $\mathcal{L} \in \{\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}\}$.

Therefore, $\mathfrak{C}_{Ref,Sym}$ is $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-stable. □

Corollary 5. *The $\mathcal{L}_T$-, $\mathcal{L}_U$-, $\mathcal{L}_H$-, $\mathcal{L}_{H(@)}$-definability problem in $\mathfrak{C}_{Ref,Sym}$ is undecidable.*

Proof. By Theorem 5 and Theorem 11, it suffices to show that the validity problem of first-order sentences in $\mathfrak{C}_{Ref,Sym}$ is undecidable, which is already shown in [4, Corollary 3]. □

5 Conclusions and Further Directions

In this paper, we use the stable class technique in [4] to show that certain extended modal definability in certain frame classes are undecidable. Here we use a frame construction of $\mathbb{F}$ and $\mathbb{F}'$ from $\mathbb{F}_0$ which satisfies that $\mathbb{F}'$ is a tense bounded morphic image, a bounded morphic image, a generated subframe of $\mathbb{F}$ at the same time, so we can treat $\mathcal{L}_T, \mathcal{L}_U, \mathcal{L}_H, \mathcal{L}_{H(@)}$ uniformly.

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As we know, the more expressive the extended modal language is, the less kinds of frame constructions its validities are preserved under. Therefore, if we consider the hybrid language $\mathcal{L}_{H(E)}$ which has both the nominals and the universal modality, its validity is only preserved under taking ultrafilter morphic images (e.g., see [6]), which makes it harder to construct $\mathbb{F}$ and $\mathbb{F}'$. While for very expressive hybrid language like $\mathcal{L}_{H(E)}$ extended with the downarrow binder, each first-order formula is $\mathcal{L}_{H(E,\downarrow)}$-definable. Therefore, it is an interesting question that for which position $\mathcal{L}$ of the extended modal language hierarchy, the $\mathcal{L}$-definability problem for first-order sentences becomes decidable.

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扩展模态语言中模态可定义性的不可判定性结果

赵之光

摘　要

在本文中，我们使用 Balbiani 和 Tinchev 的稳定类方法证明在加全称模态词的模态语言 $\mathcal{L}_U$、时态语言 $\mathcal{L}_T$、混合语言 $\mathcal{L}_H$，$\mathcal{L}_{H(@)}$ 中，Chagrova 定理成立，即一阶公式相对于特定框架类的模态/时态/混合可定义性问题是不可判定的。

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